# 10

# Renormalization Group

The renormalization procedure in the last chapter has eliminated all UV-divergences from the Feynman integrals arising from large momenta in $D = 4 - \varepsilon$ dimensions. This was necessary to obtain finite correlation functions in the limit $\varepsilon \to 0$. We have seen in Chapter 7 that the dependence on the cutoff or any other mass scale, introduced in the regularization process, changes the Ward identities derived from scale transformations by an additional term—the anomaly of scale invariance. The precise consequences of this term for the renormalized proper vertex functions were first investigated independently by Callan and Symanzik [1].

## 10.1 Callan-Symanzik Equation

The original derivation of the scaling properties of interacting theories did not quite proceed along the lines discussed in Section 7.4. Callan and Symanzik wanted to find the behavior of the renormalized proper vertex functions $\bar{\Gamma}^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n)$ for $n \geq 1$ under a change of the renormalized mass $m$. They did this in the context of cutoff regularization and renormalization conditions at a subtraction point as in Eqs. (9.23)–(9.33). To keep track of the parameters of the theory, we shall from now on enter these explicitly into the list of arguments, and write the proper vertex functions as $\bar{\Gamma}^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n; m, g)$. To save space, we abbreviate the list of momenta $\mathbf{k}_1, \ldots, \mathbf{k}_n$ by a momentum symbol $\mathbf{k}_i$, and use the notation $\bar{\Gamma}^{(n)}(\mathbf{k}_i; m, g)$. In addition, the bare proper vertex functions depend on the cutoff, or the deviation $\varepsilon = 4 - D$ from the dimension $D = 4$, and will be denoted by $\bar{\Gamma}_B^{(n)}(\mathbf{k}_i; m_B, \lambda_B, \Lambda)$.

Differentiating the bare proper vertex function $\bar{\Gamma}_B^{(n)}(\mathbf{k}_i; m_B, \lambda_B, \Lambda)$ with respect to the renormalized mass $m$ at a fixed bare coupling constant and cutoff gives

$$m \frac{\partial}{\partial m} \bar{\Gamma}_B^{(n)}(\mathbf{k}_i; m_B, \lambda_B, \Lambda) \bigg|_{\lambda_B, \Lambda} = m \frac{\partial}{\partial m} m_B^2 \bigg|_{\lambda_B, \Lambda} \bar{\Gamma}_B^{(1,n)}(\mathbf{0}, \mathbf{k}_i; m_B, \lambda_B, \Lambda), \tag{10.1}$$

where

$$\bar{\Gamma}_B^{(1,n)}(\mathbf{0}, \mathbf{k}_i; m_B, \lambda_B, \Lambda) = \frac{\partial}{\partial m_B^2} \bar{\Gamma}_B^{(n)}(\mathbf{k}_i; m_B, \lambda_B, \Lambda) \tag{10.2}$$

is the proper vertex function associated with the correlation function containing an extra term $-\phi^2(\mathbf{x})/2$ inside the expectation value [recall Eqs. (5.98) and (5.93)]:

$$G^{(1,n)}(\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_n) = -\frac{1}{2} \langle \phi^2(\mathbf{x}) \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_n) \rangle. \tag{10.3}$$

With the help of the renormalization constants $Z_\phi$ we go over to renormalized correlation functions as in Eq. (9.31):

$$\bar{\Gamma}_B^{(n)}(\mathbf{k}_i; m_B, \lambda_B, \Lambda) \quad = \quad Z_\phi^{-n/2} \bar{\Gamma}^{(n)}(\mathbf{k}_i; m, g), \qquad n \geq 1 \; . \tag{10.4}$$

We also introduce a renormalization constant $Z_2$ which makes the composite vertex function finite in the limit $\lambda_B \to \infty$ via

$$\bar{\Gamma}_B^{(1,n)}(\mathbf{0}, \mathbf{k}_i; m_B, \lambda_B, \Lambda) \;=\; Z_\phi^{-n/2} Z_2 \, \bar{\Gamma}^{(1,n)}(\mathbf{0}, \mathbf{k}_i; m, g), \quad n \geq 1 \; . \tag{10.5}$$

The constant $Z_2$ is fixed by the normalization condition

$$\bar{\Gamma}^{(1,n)}(\mathbf{0}, \mathbf{0}; m, g) = 1. \tag{10.6}$$

Then we define auxiliary functions

$$\beta \;=\; m \frac{\partial g}{\partial m}\bigg|_{\lambda_B, \Lambda}, \tag{10.7}$$

$$\gamma \;=\; \frac{1}{2} Z_\phi^{-1} \, m \, \frac{\partial Z_\phi}{\partial m}\bigg|_{\lambda_B, \Lambda}, \tag{10.8}$$

and rewrite the differential equation (10.1) in the form

$$\left( m \frac{\partial}{\partial m} + \beta \frac{\partial}{\partial g} - n\gamma \right) \bar{\Gamma}^{(n)}(\mathbf{k}_i; m, g) = Z_2 \, m \, \frac{\partial m_B^2}{\partial m}\bigg|_{\lambda_B, \Lambda} \bar{\Gamma}^{(1,n)}(\mathbf{0}, \mathbf{k}; m, g), \qquad n \geq 1 \; . \tag{10.9}$$

The normalization conditions (9.23) requires that $\bar{\Gamma}^{(2)}(\mathbf{0}; m, g) = m^2$. Inserting this together with (10.6) into Eq. (10.9) for $n = 2$ gives

$$(2 - 2\gamma) m^2 = Z_2 m \, \frac{\partial m_B^2}{\partial m}\bigg|_{\lambda_B, \Lambda}. \tag{10.10}$$

This permits us to express the right-hand side of (10.9) in terms of renormalized quantities, yielding the *Callan-Symanzik equation*:

$$\left( m \frac{\partial}{\partial m} + \beta \frac{\partial}{\partial g} - n\gamma \right) \bar{\Gamma}^{(n)}(\mathbf{k}_i; m, g) = (2 - 2\gamma) m^2 \bar{\Gamma}^{(1,n)}(\mathbf{0}, \mathbf{k}; m, g), \qquad n \geq 1 \; . \tag{10.11}$$

In general, the dimensionless functions $\beta$ and $\gamma$ depend on $g$ and $m/\Lambda$. But since they govern differential equations for the renormalized proper vertex functions, they do not depend on $\Lambda$ after all, being functions of $g$ and $m$. Their properties will be studied in detail in the next section.

The Callan-Symanzik equation makes statements on the scaling properties of correlation functions by going to small masses or, equivalently, to large momenta. In this limit, one may invoke results by Weinberg [2], according to which the right-hand side becomes small compared with the left-hand side. From the resulting approximate homogeneous equation one may deduce the critical behavior of the theory, provided the $\beta$-function is zero at some coupling strength $g = g^*$. In the critical regime, the proper vertex functions therefore satisfy the differential equation

$$\left( m \frac{\partial}{\partial m} - n\gamma \right) \bar{\Gamma}^{(n)}(\mathbf{k}_i; m, g) \underset{m \approx 0}{\approx} 0, \qquad n \geq 1 \; . \tag{10.12}$$

By combining this with the original scaling relation (7.54) which is valid for renormalized quantities on the basis of naive scaling arguments, we find

$$\left[ \sum_{i=1}^{n} \mathbf{k}_i \partial_{\mathbf{k}_i} + n(d_\phi^0 + \gamma) - D \right] \bar{\Gamma}^{(n)}(\mathbf{k}_i; m, g) \underset{m \approx 0}{\approx} 0, \qquad n \geq 1 \; . \tag{10.13}$$

Recall that our renormalized coupling $g$ is dimensionless by definition, just as the quantity $\hat{\lambda}$ in (7.28).

The scaling equation (10.13) was discussed earlier in Eq. (7.74). It has the same form as the original scaling relation for a massless theory (7.20), except that the free-field dimension $d_\phi^0$ is replaced by the modified dimension $d_\phi = d_\phi^0 + \gamma$. By comparison with (7.75), we identify the critical exponent $\eta$ as

$$\eta = 2\gamma. \tag{10.14}$$

The interacting theory is invariant under scale transformations (7.76) of the renormalized interacting field. The critical scaling equation (10.13) is the origin of the anomalous dimensions observed in Eq. (7.66)–(7.68).

We shall not explore the consequences of the Callan-Symanzik equation further but derive a more powerful equation for studying the critical behavior of the theory.

## 10.2 Renormalization Group Equation

In Chapter 8 we have decided to regularize the theory by analytic extension of all Feynman integrals from integer values of the dimension $D$ into the complex $D$-plane, and by subtracting the singularities in $\varepsilon = 4 - D$ in a certain minimal way referred to as minimal subtraction. Adding the dimensional parameter $\varepsilon$ to the list of arguments, we shall write the bare correlation functions as

$$G_B^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n; m_B, \lambda_B, \varepsilon) \equiv \langle \phi_B(\mathbf{x}_1) \cdots \phi_B(\mathbf{x}_n) \rangle. \tag{10.15}$$

They are calculated from a generating functional (2.13), whose Boltzmann factor contains the bare energy functional $E_B[\phi]$ of Eq. (9.73).

In the subtracted terms, one has the freedom of introducing an arbitrary mass parameter $\mu$. The renormalization constants of the theory will therefore depend on $\mu$ rather than on a cutoff $\Lambda$. The renormalized correlation functions have the form

$$G^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n; m, g, \mu, \varepsilon) \equiv \langle \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_n) \rangle, \tag{10.16}$$

and are related to the bare quantities (10.15) by a multiplicative renormalization:

$$G_B^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n; m_B, \lambda_B, \varepsilon) = Z_\phi^{n/2}(g(\mu), \varepsilon) G^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n; m, g, \mu, \varepsilon), \qquad n \geq 1, \tag{10.17}$$

where $Z_\phi(g(\mu), \varepsilon)$ is the field normalization constant defined by $\phi_B = Z_\phi^{1/2} \phi$. For the propagator with $n = 2$, this implies the relation

$$G_B^{(2)}(\mathbf{x}_1, \mathbf{x}_2; m_B, \lambda_B, \varepsilon) = Z_\phi(g(\mu), \varepsilon)\, G^{(2)}(\mathbf{x}_1, \mathbf{x}_2; m, g, \mu, \varepsilon). \tag{10.18}$$

After a Fourier transform according to Eq. (4.13), the same factors $Z_\phi(g(\mu), \varepsilon)$ renormalize the momentum space $n$-point functions $G_B^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n; m_B, \lambda_B, \varepsilon)$ to $G^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n; m, g, \mu, \varepsilon)$. For the proper vertex functions $\bar{\Gamma}_B^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n; m_B, \lambda_B, \varepsilon)$, which are obtained from the 1PI parts of the connected $n$-point functions $G_B^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n; m_B, \lambda_B, \varepsilon)$ by amputating the external lines, i.e., by dividing out $n$ external propagators $G_B^{(2)}(\mathbf{k}_i; m_B, \lambda_B, \varepsilon)$, the renormalized quantities are given by

$$\bar{\Gamma}^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n; m, g, \mu, \varepsilon) = Z_\phi^{n/2}(g(\mu), \varepsilon)\, \bar{\Gamma}_B^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n; m_B, \lambda_B, \varepsilon), \qquad n \geq 1. \tag{10.19}$$

These expressions remain finite in the limit $\varepsilon \to 0$. In the following discussion we shall suppress the obvious $\varepsilon$-dependence in the arguments of all quantities, for brevity, unless it is helpful for a better understanding.

The renormalized parameters $g$, $m$, and $\phi$ defined in Eqs. (9.72) depend on the bare quantities, and on the mass parameter $\mu$:

$$\phi^2 = Z_\phi^{-1}(g(\mu),\varepsilon)\,\phi_B^2, \quad m^2 = m^2(\mu) \equiv \frac{Z_\phi(g(\mu),\varepsilon)}{Z_{m^2}(g(\mu),\varepsilon)}\, m_{\mathrm{B}}^2, \quad g = g(\mu) \equiv \mu^{-\varepsilon}\frac{Z_\phi^2(g(\mu),\varepsilon)}{Z_g(g(\mu),\varepsilon)}\,\lambda_{\mathrm{B}}\,. \tag{10.20}$$

The renormalized proper vertex functions $\bar{\Gamma}^{(n)}(\mathbf{k}_1,\dots,\mathbf{k}_n;m,g,\mu,\varepsilon)$ depend on $\mu$ in two ways: once explicitly, and once via $g(\mu)$ and $m(\mu)$. The explicit dependence comes from factors $\mu^\varepsilon$ which are generated when replacing $\lambda$ by $\mu^\varepsilon\, g$ in (8.58). By contrast, the unrenormalized proper vertex functions $\bar{\Gamma}_B^{(n)}(\mathbf{k}_1,\dots,\mathbf{k}_n;m_B,\lambda_B,\varepsilon)$ do not depend on $\mu$. On the right-hand side of Eq. (10.19), only $Z_\phi$ depends on $\mu$ via $g(\mu)$.

The bare proper vertex functions are certainly independent of the artificially introduced arbitrary mass parameter $\mu$. When rewriting them via Eq. (10.19) as $Z_\phi^{-n/2}\,\bar{\Gamma}^{(n)}(\mathbf{k}_1,\dots,\mathbf{k}_n;m,g,\mu,\varepsilon)$, this implies a nontrivial behavior of the renormalized vertex functions under changes of $\mu$. The associated changes of the renormalized proper vertex functions, and the other renormalized parameters, must be related to each other in a specified way. It is this relation which ensures that the physical information in the renormalized functions remains invariant under changes of $\mu$.

Let us calculate these changes. We apply the dimensionless operator $\mu\,\partial/\partial\mu$ to Eq. (10.19) with fixed bare parameters, and obtain for $n \geq 1$:

$$\left[-n\mu\frac{\partial}{\partial\mu}\log Z_\phi^{1/2}\Big|_B + \mu\,\frac{\partial g}{\partial\mu}\Big|_B\frac{\partial}{\partial g} + \mu\,\frac{\partial m}{\partial\mu}\Big|_B\frac{\partial}{\partial m} + \mu\frac{\partial}{\partial\mu}\right]\bar{\Gamma}^{(n)}(\mathbf{k}_1,\dots,\mathbf{k}_n;m,g,\mu) = 0. \tag{10.21}$$

The symbol $|_B$ indicates that the bare parameters $m_B, \lambda_B$ are kept fixed. This equation expresses the invariance of $\bar{\Gamma}^{(n)}(\mathbf{k}_1,\dots,\mathbf{k}_n;m,g,\mu)$ under a transformation $(\mu, m(\mu), g(\mu)) \to (\mu', m(\mu'), g(\mu'))$. The observables of the field system are invariant under a change of the mass scale $\mu \to \mu'$ if coupling constant $g(\mu)$ and mass $m(\mu)$ are changed appropriately. The mass scale $\mu$ is not an independent parameter.

The appropriate dependence of $g$, $m$ and $Z_\phi$ on $\mu$ is described by the *renormalization group functions* (RG functions):

$$\gamma(m,g,\mu) = \mu\frac{\partial}{\partial\mu}\log Z_\phi^{1/2}\Big|_B\,, \tag{10.22}$$

$$\gamma_m(m,g,\mu) = \frac{\mu}{m}\frac{\partial m}{\partial\mu}\Big|_B\,, \tag{10.23}$$

$$\beta(m,g,\mu) = \mu\frac{\partial g}{\partial\mu}\Big|_B\,. \tag{10.24}$$

They allow us to rewrite Eq. (10.21) as the *renormalization group equation* (RGE) for the proper vertex functions with $n \geq 1$:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(m,g,\mu)\frac{\partial}{\partial g} - n\gamma(m,g,\mu) + \gamma_m(m,g,\mu)\,m\frac{\partial}{\partial m}\right]\bar{\Gamma}^{(n)}(\mathbf{k}_1,\dots,\mathbf{k}_n;m,g,\mu) = 0\,. \tag{10.25}$$

The solution of a partial differential equation like (10.25) is generally awkward, since $\beta, \gamma, \gamma_m$ may depend on $m, g$ and $\mu$. It is an important property of 't Hooft's minimal subtraction scheme [3, 4] that the counterterms happen to be independent of the mass $m$, and that they depend only on the coupling constant $g$, apart from $\varepsilon$. The renormalization group functions (10.22)–(10.24) are therefore independent of $m$ and $\mu$, and depend only on $g$:

$$\gamma(g) \overset{\text{MS}}{=} \mu \frac{\partial}{\partial\mu} \log Z_\phi^{1/2}\Big|_B , \tag{10.26}$$

$$\gamma_m(g) \overset{\text{MS}}{=} \frac{\mu}{m}\frac{\partial m}{\partial\mu}\Big|_B , \tag{10.27}$$

$$\beta(g) \overset{\text{MS}}{=} \mu \frac{\partial g}{\partial\mu}\Big|_B . \tag{10.28}$$

With these, the renormalization group equation (10.25) becomes

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - n\gamma(g) + \gamma_m(g)\, m\frac{\partial}{\partial m}\right]\bar{\Gamma}^{(n)}(\mathbf{k}_1,\ldots,\mathbf{k}_n; m, g, \mu) = 0\,, \qquad n \geq 1\ , \tag{10.29}$$

which is much easier to solve than the general form (10.25).

## 10.3 Calculation of Coefficient Functions from Counterterms

We now calculate the renormalization group functions taking advantage of the fact that the renormalization constants depend, with minimal subtractions, only on $\mu$ via the renormalized coupling constant $g(\mu)$. Consider first the function $\beta(g)$. Inserting the renormalization equation $\lambda_B = \mu^\varepsilon Z_g Z_\phi^{-2}\, g$ of (10.20) into (10.28), we find

$$\beta(g) = -\mu\frac{(\partial_\mu\lambda_B)_g}{(\partial_g\lambda_B)_\mu} = -\varepsilon\left[\frac{d}{dg}\log(gZ_gZ_\phi^{-2})\right]^{-1} = -\varepsilon g\left[\frac{d\log g_B(g)}{d\log g}\right]^{-1}. \tag{10.30}$$

By the chain rule of differentiation, we rewrite (10.26) as

$$\gamma(g) = \mu\frac{\partial g}{\partial\mu}\Big|_B \frac{d}{dg}\log Z_\phi^{1/2} = \beta(g)\frac{d}{dg}\log Z_\phi^{1/2}. \tag{10.31}$$

With this, Eq. (10.30) takes the form

$$\beta(g) = \frac{-\varepsilon + 4\gamma(g)}{d\log[gZ_g(g)]/dg} . \tag{10.32}$$

Finally, we find from the relation $m_{\rm B}^2 = m^2\, Z_{m^2}/Z_\phi$ of (10.20) the renormalization group function

$$\gamma_m(g) = -\frac{\beta(g)}{2}\left[\frac{d}{dg}\log Z_{m^2} - \frac{d}{dg}\log Z_\phi\right] = -\frac{\beta(g)}{2}\frac{d}{dg}\log Z_{m^2} + \gamma(g). \tag{10.33}$$

In principle, the right-hand sides still depend on $\varepsilon$, so that we should really write the RG functions as

$$\beta = \beta(g,\varepsilon), \quad \gamma = \gamma(g,\varepsilon), \quad \gamma_m = \gamma_m(g,\varepsilon). \tag{10.34}$$

However, the $\varepsilon$-dependence turns out to be extremely simple. Due to the renormalizability of the theory, the functions $\beta(g,\varepsilon)$, $\gamma(g,\varepsilon)$, $\gamma_m(g,\varepsilon)$ have to remain finite in the limit $\varepsilon \to 0$, and thus free of poles in $\varepsilon$. In fact, an explicit evaluation of the right-hand side of Eqs. (10.30), (10.31), and (10.33) demonstrates the cancellation of all poles in $\varepsilon$. Thus we can expand these functions in a power series in $\varepsilon$ with nonnegative powers $\varepsilon^n$.

Moreover, we can easily convince ourselves that, of the nonnegative powers $\varepsilon^n$, only the first is really present, and this only in the function $\beta(g,\varepsilon)$. In order to show this, we make use of the explicit general form of the $1/\varepsilon$-expansions of the renormalization constants in minimal subtraction, which is

$$Z_\phi(g,\varepsilon) = 1+\sum_{n=1}^{\infty} Z_{\phi,n}(g)\frac{1}{\varepsilon^n}, \tag{10.35}$$

$$Z_{m^2}(g,\varepsilon) = 1+\sum_{n=1}^{\infty} Z_{m^2,n}(g)\frac{1}{\varepsilon^n}, \tag{10.36}$$

$$Z_g(g,\varepsilon) = 1+\sum_{n=1}^{\infty} Z_{g,n}(g)\frac{1}{\varepsilon^n}. \tag{10.37}$$

Then Eqs. (10.31)–(10.33) can be rewritten as

$$\gamma(g,\varepsilon)\left[1+\sum_{n=1}^{\infty} Z_{\phi,n}(g)\varepsilon^{-n}\right] = \frac{1}{2}\beta(g,\varepsilon)\sum_{n=1}^{\infty} Z'_{\phi,n}(g)\varepsilon^{-n}, \tag{10.38}$$

$$\beta(g,\varepsilon)\left\{1+\sum_{n=1}^{\infty}[gZ_{g,n}(g)]'\varepsilon^{-n}\right\} = [-\varepsilon+4\gamma(g,\varepsilon)]\,g\left[1+\sum_{n=1}^{\infty} Z_{g,n}(g)\varepsilon^{-n}\right], \tag{10.39}$$

$$[-\gamma_m(g,\varepsilon)+\gamma(g,\varepsilon)]\left[1+\sum_{n=1}^{\infty} Z_{m^2,n}(g)\varepsilon^{-n}\right] = \frac{\beta(g,\varepsilon)}{2}\sum_{n=1}^{\infty} Z'_{m^2,n}(g)\varepsilon^{-n}. \tag{10.40}$$

By inserting (10.38) into (10.39), we see that $\beta(g,\varepsilon)$ can at most contain the following powers of $\varepsilon$:

$$\beta(g,\varepsilon) = \beta_0(g)+\varepsilon\beta_1(g). \tag{10.41}$$

Using this to eliminate $\beta(g,\varepsilon)$ from Eqs. (10.38) and (10.40), we find $\gamma(g,\varepsilon)$ and $\gamma_m(g,\varepsilon)$ as functions of $\varepsilon$. By equating the regular terms in the three equations, we find

$$\begin{aligned}\beta_0+\varepsilon\beta_1+\beta_1(Z_{g,1}+gZ'_{g,1}) &= (-\varepsilon+4\gamma)g-gZ_{g,1}\,,\\ \gamma &= \tfrac{1}{2}\beta_1 Z_{\phi,1}\,,\\ \gamma_m-\gamma &= -\tfrac{1}{2}\beta_1 Z'_{m^2,1}\,.\end{aligned} \tag{10.42}$$

The solutions are

$$\beta_1(g) = -g, \tag{10.43}$$

$$\beta_0(g) = gZ'_{g,1}(g)+4g\gamma(g), \tag{10.44}$$

$$\gamma(g) = \tfrac{1}{2}Z'_{\phi,1}(g)\,\beta_1(g), \tag{10.45}$$

$$\gamma_m(g) = \tfrac{1}{2}gZ'_{m,1}(g)+\gamma(g). \tag{10.46}$$

Thus, amazingly, the three functions $\beta(g),\gamma(g),\gamma_m(g)$ have all been expressed in terms of the derivatives of the three residues $Z_{g,1}(g)$, $Z_{\phi,1}(g)$, $Z_{m^2,1}(g)$ of the simple $1/\varepsilon$ -pole in the counterterms. The dimensional parameter $\varepsilon = 4 - D$ enters the renormalization group function only at a single place: in the $-\varepsilon g$-term of $\beta(g)$:

$$\beta(g) = -\varepsilon g + g^2 Z'_{g,1}(g) + 4g\gamma(g) . \tag{10.47}$$

The finiteness of the observables $\beta, \gamma, \gamma_m$ at $\varepsilon = 0$ requires that none of the higher residues of Eqs. (10.38)–(10.40) can contribute. Indeed, we can easily verify in the available expansions that there exists an infinite set of relations among the expansion coefficients, useful for checking calculations:

$$\beta_0(gZ_{g,n})' - g(gZ_{g,n+1})' = 4\gamma\, gZ_{g,n} - Z_{g,n+1}\, g, \tag{10.48}$$

$$\gamma Z_{\phi,n} = \tfrac{1}{2}\beta_0 Z'_{\phi,n} - \tfrac{1}{2} g Z'_{\phi,n+1}, \tag{10.49}$$

$$(\gamma_m - \gamma) Z_{m^2,n} = -\tfrac{1}{2}\beta_0 Z'_{m^2,n} + \tfrac{1}{2} g Z'_{m^2,n+1}. \tag{10.50}$$

From the two-loop renormalization constants in Eqs. (9.115)–(9.119), we extract the residues of $1/\varepsilon$:

$$\begin{aligned} Z_{g,1} &= \frac{N+8}{3}\frac{g}{(4\pi)^2} - \frac{5N+22}{9}\frac{g^2}{(4\pi)^4}\,, \\ Z_{\phi,1} &= -\frac{N+2}{36}\frac{g^2}{(4\pi)^4}\,, \\ Z_{m^2,1} &= \frac{N+2}{3}\frac{g}{(4\pi)^2} - \frac{N+2}{6}\frac{g^2}{(4\pi)^4}\,, \end{aligned} \tag{10.51}$$

so that we obtain:

$$\beta_1(g) = -g\,, \qquad \beta_0(g) = g^2 Z'_{g,1}(g) + 4g\gamma(g) = \frac{N+8}{3}\frac{g^2}{(4\pi)^2} - \frac{3N+14}{3}\frac{g^3}{(4\pi)^4}\,, \tag{10.52}$$

$$\gamma(g) = \tfrac{1}{2} Z'_{\phi,1}(g)\,\beta_1(g) = \frac{N+2}{36}\frac{g^2}{(4\pi)^4}\,, \tag{10.53}$$

$$\gamma_m(g) = \tfrac{1}{2} g Z'_{m^2,1}(g) + \gamma(g) = \frac{N+2}{6}\frac{g}{(4\pi)^2} - \frac{5(N+2)}{36}\frac{g^2}{(4\pi)^4}\,. \tag{10.54}$$

The coupling constant always appears with a factor $1/(4\pi)^2$, which is generated by the loop integrations. We therefore introduce a modified coupling constant

$$\bar{g} \equiv \frac{g}{(4\pi)^2}, \tag{10.55}$$

which brings the renormalization group functions to the shorter form:

$$\beta_{\bar{g}}(\bar{g}) = \bar{g}\left(-\varepsilon + \frac{N+8}{3}\bar{g} - \frac{3N+14}{3}\bar{g}^2\right), \tag{10.56}$$

$$\gamma(\bar{g}) = \frac{N+2}{36}\bar{g}^2, \tag{10.57}$$

$$\gamma_m(\bar{g}) = \frac{N+2}{6}\bar{g} - \frac{5(N+2)}{36}\bar{g}^2. \tag{10.58}$$

It is always possible to introduce a further modified coupling constant defined by an expansion of the generic type $g_H = G(g) = g + a_2 g^2 + \ldots$ with unit coefficient of the first term, which has the property that the function $\beta(g_H)$ consists only of the first three terms $\beta_H(g_H) = -\varepsilon\bar{g} + b_2 g_H^2 + b_3 g_H^3$ [5]. Since we shall not use this fact, we refer the reader to the original work for a proof.

It is instructive to verify explicitly the cancellation of $1/\varepsilon^n$-singularities in the calculation of the renormalization group functions from Eqs. (10.30)–(10.33). Take, for instance, $\gamma(g)$ of Eq. (10.31). For a more impressive verification, let us anticipate here the five-loop results (15.11) for the renormalization constant $Z_\phi(\bar{g})$ and Eq. (17.5) for the $\beta$-function, extending our two-loop expansions (9.115) and (10.56). Selecting the case of $N = 1$ for brevity of the formulas, the five-loop extension of the expansion (10.56) reads

$$\begin{aligned} \beta_{\bar{g}}(\bar{g}) &= -\varepsilon\,\bar{g} + 3\,\bar{g}^2 - \frac{17}{3}\,\bar{g}^3 + \left[\frac{145}{8} + 12\,\zeta(3)\right]\bar{g}^4 \\ &+ \left[-\frac{3499}{48} + \frac{\pi^4}{5} - 78\,\zeta(3) - 120\,\zeta(5)\right]\bar{g}^5 \\ &+ \left[\frac{764621}{2304} - \frac{1189\,\pi^4}{720} - \frac{5\,\pi^6}{14} + \frac{7965\,\zeta(3)}{16} + 45\,\zeta^2(3) + 987\,\zeta(5) + 1323\,\zeta(7)\right]\bar{g}^6. \end{aligned} \tag{10.59}$$

From $Z_\phi(\bar{g})$ of Eq. (17.5) for $N = 1$, we find the five-loop expansion of the logarithmic derivative on the right-hand side of Eq. (10.31):

$$\begin{aligned} [\log Z_\phi(\bar{g})]' &= -\frac{1}{6\varepsilon}\,\bar{g} + \left(\frac{-1}{2\varepsilon^2} + \frac{1}{8\varepsilon}\right)\bar{g}^2 + \left(\frac{-3}{2\varepsilon^3} + \frac{95}{72\varepsilon^2} - \frac{65}{96\varepsilon}\right)\bar{g}^3 \\ &+ \left[\frac{-9}{2\varepsilon^4} + \frac{163}{24\varepsilon^3} - \frac{553}{96\varepsilon^2} + \frac{3709}{1152\varepsilon} + \frac{\pi^4}{90\varepsilon} - \frac{2\,\zeta(3)}{\varepsilon^2} - \frac{3\,\zeta(3)}{8\varepsilon}\right]\bar{g}^4 \\ &+ \left(\frac{-13}{48\varepsilon^4} + \frac{179}{864\varepsilon^3} - \frac{23}{256\varepsilon^2}\right)\bar{g}^5. \end{aligned} \tag{10.60}$$

When forming the product $\beta_{\bar{g}}(\bar{g}) \times [\log Z_\phi(\bar{g})]'$ in (10.31) to obtain $\gamma(\bar{g})$, the contribution to $\gamma(g)$ comes from the product of the $\varepsilon$-term in $\beta_{\bar{g}}(\bar{g})$ with the $1/\varepsilon$ terms in $[\log Z_\phi(\bar{g})]'$. The higher singularities $1/\varepsilon^n$, $1/\varepsilon^{n-1}$, ... in the $\bar{g}^n$-term of $[\log Z_\phi(\bar{g})]'$ are reduced by one power of $1/\varepsilon$ when multiplied with the $\varepsilon$-term of $\beta_{\bar{g}}(\bar{g})$. For $n \geq 2$ the resulting terms are canceled by products of the other terms in $\beta_{\bar{g}}(\bar{g})$ with the singular terms associated with the lower powers $\bar{g}^{n-1}$, $\bar{g}^{n-2}$, ... in $[\log Z_\phi(\bar{g})]'$.

Having determined the renormalization group functions, we shall now solve the renormalization group equations (10.29). In order to avoid rewriting these equations in terms of the new reduced coupling constant $\bar{g}$, we shall rename $\bar{g}$ as $g$ and drop the subscript $\bar{g}$ on the $\beta$-function.

## 10.4 Solution of the Renormalization Group Equation

The renormalization group equation (10.29) is a partial differential equation. Its coefficients depend only on $g$. Such an equation is solved by the method of characteristics. We introduce a *dimensionless scale parameter* $\sigma$ and replace $\mu$ by $\sigma\mu$, so that variations of the mass scale $\mu$ are turned into variations of $\sigma$ at a fixed mass scale $\mu$. Then we introduce auxiliary functions $g(\sigma)$, $m(\sigma)$, called *running coupling constant* and *running mass*, which satisfy the first-order differential equations:

$$\beta(g(\sigma)) = \sigma\,\frac{dg(\sigma)}{d\,\sigma}, \qquad g(1) = g, \tag{10.61}$$

$$\gamma_m(g(\sigma)) = \frac{\sigma}{m(\sigma)}\,\frac{dm(\sigma)}{d\,\sigma}, \qquad m(1) = m\,, \tag{10.62}$$

$$\mu(\sigma) \;=\; \sigma\frac{d\mu(\sigma)}{d\sigma}\,, \qquad\qquad \mu(1)=\mu\;. \tag{10.63}$$

The solutions define trajectories in the $(\sigma, m, g)$-space which connect theories renormalized with different mass parameter $\sigma\mu$.

The temperature dependence of the theory is introduced via the mass parameter $m(1)$, i.e. via the renormalized mass $m(\sigma)$ at a fixed mass scale $\sigma\mu$ with $\sigma = 1$. Specifically we assume

$$m^2 = \mu^2 t, \qquad t = \frac{T}{T_c} - 1. \tag{10.64}$$

Recall that when setting up the energy density in Section 1.4 to serve as a starting point for the field-theoretic treatment of thermal fluctuations, we followed Landau by taking the squared *bare mass* to be proportional to the temperature deviation from the critical temperature:

$$m_B^2 \propto t, \qquad t = \frac{T}{T_c} - 1. \tag{10.65}$$

According to the renormalization equation (10.20), the renormalized mass is proportional to the bare mass, $m^2 = m_B^2 Z_\phi(g,\mu)/Z_{m^2}(g,\mu)$. Thus, at a fixed auxiliary mass scale $\mu$ and $\sigma = 1$, the renormalized mass is also proportional to $t$, as stated in Eq. (10.64). Note that this is a peculiarity of the present regularization procedure. In the earlier procedure used in the general qualitative discussion of scale behavior in Section 7.4, the mass was renormalized in Eq. (7.57) with $m$-dependent renormalization constants to $m^2 = m_B^2 Z_\phi(\lambda, m, \Lambda)/Z_{m^2}(\lambda, m, \Lambda)$. For small $m$, these change the linear behavior of the bare square mass $m_B^2 \propto t$ to the power behavior $m \propto t^\nu$ of the renormalized mass [see (7.70), (7.71)]. Such a behavior can be derived within the present scheme if we set the mass scale $\mu(\sigma)$ equal to the running renormalized mass $m(\sigma)$ for some $\sigma_m$. This will be done when fixing the parametrization in Eq. (10.78).

The solution of (10.61) is immediately found to be

$$\log\sigma = \int_g^{g(\sigma)} \frac{dg'}{\beta(g')}. \tag{10.66}$$

Inserting this into the second equation (10.62), we obtain

$$m(\sigma) = m\exp\left[\int_1^\sigma \frac{d\sigma'}{\sigma'}\,\gamma_m(g(\sigma'))\right]. \tag{10.67}$$

The last equation (10.63) is solved by

$$\mu(\sigma) = \mu\sigma\,. \tag{10.68}$$

With these functions, Eq. (10.29) becomes

$$\left[\sigma\frac{d}{d\sigma} - n\gamma(g(\sigma))\right]\bar\Gamma^{(n)}(\mathbf{k}_i; m(\sigma), g(\sigma), \mu(\sigma)) = 0\,, \qquad n \geq 1\,. \tag{10.69}$$

For brevity, we have written $\mathbf{k}_i$ for $\mathbf{k}_1, \ldots, \mathbf{k}_n$. The solution of Eq. (10.69) is

$$\bar\Gamma^{(n)}(\mathbf{k}_i; m, g, \mu) = \mathrm{e}^{-n\int_1^\sigma d\sigma'\,\gamma(g(\sigma'))/\sigma'}\;\bar\Gamma^{(n)}(\mathbf{k}_i; m(\sigma), g(\sigma), \mu\sigma), \qquad n \geq 1\,. \tag{10.70}$$

One set of proper vertex functions specified by the arguments $m, g, \mu$ represents an infinite family of vertex functions of the $\phi^4$-theory whose parameters are connected by a trajectory

$g(\sigma), m(\sigma), \mu\sigma$ in the parameter space. This trajectory is traced out when $\sigma$ runs from zero to infinity.

The renormalization group trajectory connects can be employed to study the behavior of the theory as the mass parameter approaches zero. For this purpose we take into account the trivial scaling behavior of the proper vertex function in all variables which follow from the dimensional analysis in Subsection 7.3.1. The $n$-point correlation function is the expectation value of $n$ fields of naive dimension $d_\phi^0 = D/2 - 1$. As such it has a mass dimension [compare (7.49)]

$$[G^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n)] = \mu^{n(D/2-1)}. \tag{10.71}$$

When going to momentum space [see (4.13)], each of the $n$ Fourier integrals adds a number $-D$ to the dimension, so that [compare (7.16)]

$$[G^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n)] = \mu^{n(-D/2-1)}. \tag{10.72}$$

For the two-point function, this implies

$$[G^{(2)}(\mathbf{k}_1, \mathbf{k}_2)] = \mu^{-D-2}. \tag{10.73}$$

The propagator $G(\mathbf{k})$ with a single momentum argument arises from this by removing the $D$-dimensional $\delta$-function which guarantees overall momentum conservation [recall (4.4)]. Its naive dimension is

$$[G(\mathbf{p})] = \mu^{-2}. \tag{10.74}$$

Using Eqs. (10.72), (10.73), and the fact that the dimension of the overall $\delta$-function is $\mu^{-D}$, we find the dimension for the connected $n$-point proper vertex functions

$$[\bar{\Gamma}^{(n)}(\mathbf{k}_i; m, g, \mu)] = \mu^{D-n(D/2-1)}\ . \tag{10.75}$$

If we now rescale all dimensional parameters by an appropriate power of $\sigma$, we obtain the trivial scaling relation [compare (7.53)]

$$\bar{\Gamma}^{(n)}(\mathbf{k}_i; m, g, \mu) \;=\; \sigma^{D-n(D/2-1)}\, \bar{\Gamma}^{(n)}\left(\mathbf{k}_i/\sigma; m/\sigma, g, \mu/\sigma\right). \tag{10.76}$$

Inserting (10.70) into the right-hand side, we find for $n \geq 1$:

$$\bar{\Gamma}^{(n)}(\mathbf{k}_i; m, g, \mu) = \sigma^{D-n(D/2-1)} \exp\left[-n \int_1^\sigma d\sigma' \frac{\gamma(g(\sigma'))}{\sigma'}\right] \bar{\Gamma}^{(n)}\left(\mathbf{k}_i/\sigma; m(\sigma)/\sigma, g(\sigma), \mu\right). \tag{10.77}$$

We now choose $\sigma = \sigma_m$ in such a way that the running mass $m(\sigma)$ equals the running additional mass scale $\mu(\sigma)$:

$$m^2(\sigma_m) = \mu^2(\sigma_m) = \mu^2\sigma_m^2. \tag{10.78}$$

For $m^2 > 0$, the rescaled mass $m(\sigma)/\sigma$ is now equal to the mass parameter $\mu$. Then (10.77) becomes

$$\bar{\Gamma}^{(n)}(\mathbf{k}_i; m, g, \mu) = \sigma_m^{D-n(D/2-1)} \exp\left[-n \int_1^{\sigma_m} d\sigma' \frac{\gamma(g(\sigma'))}{\sigma'}\right] \bar{\Gamma}^{(n)}\left(\mathbf{k}_i/\sigma_m; \mu, g(\sigma_m), \mu\right). \tag{10.79}$$

This equation relates the renormalized proper vertex functions $\bar{\Gamma}^{(n)}(\mathbf{k}_i, m, g, \mu)$ of an arbitrary mass to those of a fixed mass equal to the mass parameter $\mu$ at rescaled momenta $\mathbf{k}_i/\sigma_m$ and a running coupling constant $g(\sigma_m)$. Apart from a trivial overall rescaling factor $\sigma_m^{D-n(D/2-1)}$ due to the naive dimension, there is also a nontrivial exponential function.

Our goal is to study the behavior of the proper vertex functions on the left-hand side of (10.79) in the critical region where $m \to 0$. This is possible with the help of Eq. (10.79), whose right-hand side has a fixed mass equal to the mass parameter $\mu$. All mass dependence of the right-hand side resides in the rescaling parameter $\sigma_m$. The index on $\sigma_m$ indicates that it is related to $m$. The relation is found from Eq. (10.67), which yields the ratio

$$\begin{aligned}\frac{m^2}{\mu^2} &= \frac{m^2(\sigma_m)}{\mu^2}\exp\left\{\int_1^{\sigma_m}\frac{d\sigma'}{\sigma'}\left[-2\gamma_m(g(\sigma'))\right]\right\}\\ &= \frac{m^2(\sigma_m)}{\sigma_m^2\mu^2}\exp\left\{\int_1^{\sigma_m}\frac{d\sigma'}{\sigma'}\left[2-2\gamma_m(g(\sigma'))\right]\right\}.\end{aligned} \tag{10.80}$$

Inserting here Eq. (10.78), we obtain

$$\frac{m^2}{\mu^2} = \exp\left\{\int_1^{\sigma_m}\frac{d\sigma'}{\sigma'}\left[2-2\gamma_m(g(\sigma'))\right]\right\}. \tag{10.81}$$

Near the critical point, experimental correlation functions show the simple scaling behavior stated in Eq. (1.28). Such a behavior can be reproduced by Eqs. (10.81) and (10.79), if the coupling constant $g$ runs for $m \to 0$ into a *fixed point* $g^*$, for which the running coupling constant $g(\sigma)$ becomes independent of $\sigma$, satisfying

$$\left[\frac{d\,g(\sigma)}{d\,\sigma}\right]_{g=g^*} = 0. \tag{10.82}$$

Assuming that $\gamma_m^* \equiv \gamma_m(g^*) < 1$, which will be found to be true in the present field theory, the integrand in Eq. (10.81) is singular at $\sigma = 0$, and the asymptotic behavior of $\sigma_m$ for $m \to 0$ can immediately be found:

$$\frac{m^2}{\mu^2} = t \overset{m\approx 0}{\approx} \exp\left\{\int_1^{\sigma_m}\frac{d\sigma'}{\sigma'}\left[2-2\gamma_m(g^*)\right]\right\} = \sigma_m^{2-2\gamma_m^*}. \tag{10.83}$$

This shows that $\sigma_m$ goes to zero for $m \to 0$ with the power law

$$\sigma_m \approx \left(\frac{m^2}{\mu^2}\right)^{1/(2-2\gamma_m^*)} \equiv t^{1/(2-2\gamma_m^*)}. \tag{10.84}$$

The power behavior of $\sigma_m \propto t^{1/(2-2\gamma_m^*)}$ enters crucially into the critical behavior of all correlation functions for $T \to T_c$.

In the limit $\sigma_m \to 0$, the exponential prefactor in (10.79) becomes the following power of $t$:

$$\exp\left[-n\int_1^{\sigma_m} d\sigma'\,\frac{\gamma(g(\sigma'))}{\sigma'}\right] \overset{\sigma_m\approx 0}{\approx} \sigma_m^{-n\gamma^*} \propto t^{-n\gamma^*/(2-2\gamma_m^*)}, \tag{10.85}$$

where $\gamma^* \equiv \gamma(g^*)$. The $n$-point proper vertex function behaves therefore like

$$\bar\Gamma^{(n)}(\mathbf{k}_i; m, g, \mu) \overset{m\approx 0}{\approx} \sigma_m^{D-n(D/2-1)-n\gamma^*}\bar\Gamma^{(n)}(\mathbf{k}_i/\sigma_m; \mu, g^*, \mu), \qquad n \geq 1\,. \tag{10.86}$$

For the two-point proper vertex function this implies a scaling form

$$\bar\Gamma^{(2)}(\mathbf{k}_i; m, g, \mu) \overset{m\approx 0}{\approx} \mu^2\sigma_m^{2-2\gamma^*}\tilde g(\mathbf{k}/\mu\sigma_m), \tag{10.87}$$

with some function $\tilde{g}(x)$. Comparison of the argument of $\tilde{g}$ with the general scaling expression in Eq. (1.8), we identify $\mu\sigma_m$ with the inverse correlation length $\xi^{-1}$. Together with equation (10.78), this implies that the renormalized running mass is equal to the inverse coherence length: $m(\sigma_m) = \xi^{-1}$.

Comparing further the behavior (10.84) with that in (1.10), we can identify the critical exponent $\nu$ as

$$\nu = \frac{1}{2 - 2\gamma_m^*}. \tag{10.88}$$

With this, the relation (10.84) between $\sigma_m$ and $t = m/\mu^2$ reads simply

$$\sigma_m \approx \left(\frac{m^2}{\mu^2}\right)^\nu \equiv t^\nu. \tag{10.89}$$

The length scale $\xi$ characterizing the spatial behavior of the correlation function (10.87) diverges for $m^2 \to 0$ like

$$\xi(t) = \xi_0 \left|\frac{m^2}{\mu^2}\right|^{-\nu} = \xi_0 \,|t|^{-\nu}, \qquad t = \left|\frac{T}{T_c} - 1\right|. \tag{10.90}$$

If the expression (10.87) is nonzero for $m \to 0$, the limit must have the momentum dependence

$$\bar{\Gamma}^{(2)}(\mathbf{k}_i; m, g, \mu) \overset{m\to 0}{=} \text{const} \times \mu^{2\gamma^*} |\mathbf{k}|^{2-2\gamma^*}, \tag{10.91}$$

with some function $\tilde{f}(\kappa)$. When going over to the two-point function in $x$-space [recall (4.34)]

$$G^{(2)}(\mathbf{x}; m, g, \mu) = \int \frac{d^D k}{(2\pi)^D}\, e^{i\mathbf{kx}} \frac{1}{\bar{\Gamma}^{(2)}(\mathbf{k}_i; m, g, \mu)} \tag{10.92}$$

this amounts to an $x$-dependence

$$G^{(2)}(\mathbf{x}; m, g, \mu) \overset{m\approx 0}{\propto} \frac{1}{r^{D-2+2\gamma^*}}\, \tilde{G}(\mathbf{x}\,\mu\sigma_m) = \frac{1}{r^{D-2+2\gamma^*}}\, \tilde{G}\left(\mathbf{x}/\xi\right). \tag{10.93}$$

This expression exhibits precisely the scaling form (1.28) discovered by Kadanoff, such that we identify the critical exponent $\eta$ as

$$\eta = 2\gamma^*. \tag{10.94}$$

By analogy with this relation, we shall also introduce a critical exponent $\eta_m$ as

$$\eta_m = 2\gamma_m^*, \tag{10.95}$$

so that (10.88) becomes

$$\nu = \frac{1}{2 - \eta_m}. \tag{10.96}$$

## 10.5 Fixed Point

Let us now see how such a fixed point with the property Eq. (10.82) is derived from Eq. (10.66). In first-order perturbation theory, the $\beta$-function has, from Eqs. (10.41), (10.43), and (10.44), the general form

$$\beta(g) = -\varepsilon g + bg^2\,, \tag{10.97}$$

where the constant $b$ is, according to Eq. (10.52), equal to $(N+8)/3$ [recall that we have gone over to $\bar{g}$ via (10.55) and dropped the bar over $g$]. The $\beta$-function starts out with negative slope and has a zero at

$$g^* = \varepsilon/b, \tag{10.98}$$

as pictured in Fig. 10.1. For small $\varepsilon$, this statement is reliable even if we know $\beta(g)$ only to order $g^2$ in perturbation theory. Inserting (10.97) into equation (10.66), we calculate

$$\log\sigma = \int_g^{g(\sigma)} \frac{dg'}{-\varepsilon g' + bg'^2}. \tag{10.99}$$

This equation shows the important consequence of any zero in the $\beta$-function: If $g$ is sufficiently close to a zero at $g = g^*$ then the value $g(\sigma)$ always runs into $g^*$ in the limit $\sigma \to 0$, no matter whether $g = g(1)$ lies slightly above or below $g^*$. The point $g^*$ is the fixed point of the renormalization flow. Since $g^*$ is reached in the limit $\sigma \to 0$, which is the small-mass limit of the theory, one speaks of an *infrared-stable fixed point*. In Fig. 10.1, the flow of $g(\sigma)$ for $\sigma \to 0$ is illustrated by an arrow.

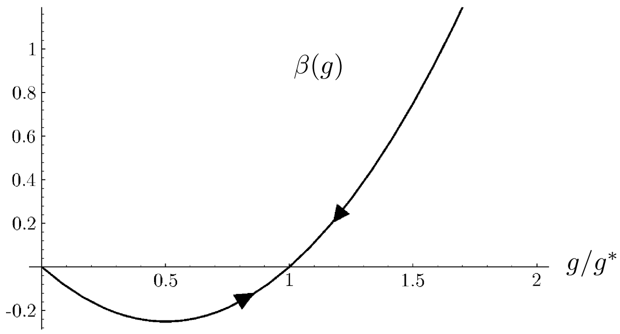


FIGURE 10.1 Flow of the coupling constant $g(\sigma)$ as the scale parameter $\sigma$ approaches zero, i.e., in the infrared limit. For the opposite scale change, i.e., in the ultraviolet limit, the arrows reverse and the origin is a stable fixed point.

Using the variable $1/g$ instead of $g$, Eq. (10.99) becomes

$$\log\sigma = -\frac{1}{\varepsilon}\int_{1/g}^{1/g(\sigma)} \frac{dx}{1/g^* - x}\,. \tag{10.100}$$

This can be integrated directly to

$$\sigma = \frac{|1/g^* - 1/g(\sigma)|^{1/\varepsilon}}{|1/g^* - 1/g|^{1/\varepsilon}}, \tag{10.101}$$

so that

$$g(\sigma) = \frac{g^*}{1 + \sigma^\varepsilon\,(g^*/g - 1)}. \tag{10.102}$$

Near the fixed point, the behavior of $g(\sigma_m)$ can be calculated more generally. For this we go back to Eq. (10.66) and expand the denominator around the zero of the $\beta$-function:

$$\beta(g) \sim \beta'(g^*)(g - g^*) + \ldots \;\; \equiv \omega(g - g^*) + \ldots\;, \tag{10.103}$$

where we have introduced the slope of the $\beta$-function at the fixed point $g^*$:

$$\omega \equiv \beta'(g^*). \tag{10.104}$$

This is another critical exponent, as we shall see in Section 10.8. The exponent $\omega$ governs the leading corrections to the scaling laws. The sign of $\omega$ controls the stability of the fixed point. For an infrared stable fixed point, $\omega$ must be positive. Then we obtain from (10.62) an equation for $\sigma = \sigma_m$:

$$\log \sigma_m = \int_g^{g(\sigma_m)} \frac{dg'}{\beta(g')} \sim \frac{1}{\omega} \log \left[ \frac{g(\sigma_m) - g^*}{g - g^*} \right], \tag{10.105}$$

implying the following $\sigma_m$-dependence of $g(\sigma_m)$, correct to lowest order in $g - g^*$:

$$\frac{g(\sigma_m) - g^*}{g - g^*} = \sigma_m^\omega. \tag{10.106}$$

This agrees with the specific solution (10.102) derived from the $\beta$-function (10.97) which has $\omega = \varepsilon$.

In general, the $\beta$-function may behave in many different ways for larger $g$. In particular, there may be more zeros to the right of $g^*$. We can see from Eq. (10.99) that, for positive $\beta$, the coupling constant $g(\sigma)$ will always run towards zero from the right. For negative $\beta$, it will run away from zero to the right.

Note that, in general, the initial coupling $g(\mu) = g(1)$ can flow only into the zero which lies in its *range of attraction*. In the present case this is guaranteed for small $\varepsilon$, if $g(1)$ is sufficiently small.

In the limit $\sigma \to \infty$, we see from Eq. (10.102) that $g(\sigma)$ tends to zero, which is the trivial zero of the $\beta$-function. This happens for any zero with a negative slope of $\beta(g)$. The limit $\sigma \to \infty$ corresponds to $m^2 \to \infty$, and for this reason such zeros are called *ultraviolet stable*. In this limit $g(\sigma) \to 0$, $\gamma(g(\sigma)) \to 0$, and $\gamma_m(g(\sigma)) \to 0$. Then scaling relation (10.84) implies that $\sigma_m = m/\mu$, and the correlation functions behave, by (10.86), like those of a free theory:

$$\bar{\Gamma}^{(n)}(\mathbf{k}_i; m, g, \mu) \overset{m \approx 0}{\approx} \left( \frac{m}{\mu} \right)^{D-n(D/2-1)} \bar{\Gamma}^{(n)}(\mathbf{k}_i \mu/m; 0, \mu, \mu). \tag{10.107}$$

This is the behavior of a free-field theory where the fields fluctuate in a trivial purely Gaussian way. The zero in $\beta(g)$ at $g = 0$ is therefore called the *Gaussian* or *trivial fixed point*. In the $\phi^4$-theory, the Gaussian fixed point is ultraviolet stable (UV-stable). Since the theory tends for $m \to \infty$ against a free theory, one also says that it is ultraviolet free. Note that this is true only in less than four dimensions.

In $D = 4$ dimensions, where $\varepsilon = 0$, the $\beta$-function has only one fixed point, the trivial Gaussian fixed point at the origin.

## 10.6 Effective Energy and Potential

The above considerations are useful for deriving the critical properties of a system only in the normal phase, where $T \geq T_c$. If we want to study the system in the phase with spontaneous symmetry breakdown, which exists for $T \leq T_c$, we have to perform a renormalization group analysis for the effective energy $\Gamma[\Phi]$ of the system, introduced in Section 5.6, and analyze its behavior as a function of the mass parameter $\mu$. For this purpose we expand the effective energy in a power series in the field expectations in momentum space $\Phi(\mathbf{p})$ as

$$\Gamma[\Phi; m, g, \mu] = \sum_{n=1}^{\infty} \frac{1}{n!} \int \frac{d^D p_1}{(2\pi)^D} \cdots \frac{d^D p_n}{(2\pi)^D} \Phi(\mathbf{p}_1) \cdots \Phi(\mathbf{p}_n) \bar{\Gamma}^{(n)}(\mathbf{p}_1, \ldots, \mathbf{p}_n; m, g, \mu). \quad (10.108)$$

The coefficients are the proper vertex functions. We have omitted for a moment the term $n = 0$ in this expansion, since it will require extra treatment. Actually, the omission calls for a new notation for the effective action, but since the reduced sum (10.108) will appear frequently in what follows, while the full effective action appears only in a few equations, we prefer keeping the notation unchanged, and shall instead refer to the full effective action including the $n = 0$ - term as $\Gamma_{\rm tot}[\Phi; m, g, \mu]$.

Let us now apply the renormalization group equation (10.29) to each coefficient. Then we observe that the factor $n$ in front of $\gamma(g)$ in Eq. (10.29) can be generated by a functional derivative with respect to the field expectations $\Phi(\mathbf{p})$, replacing

$$n \to \int \frac{d^D p}{(2\pi)^D} \Phi(\mathbf{p}) \frac{\delta}{\delta \Phi(\mathbf{p})}. \quad (10.109)$$

Then we find immediately the renormalization group equation

$$\left[\mu \partial_\mu + \beta(g) \partial_g - \gamma(g) \int \frac{d^D p}{(2\pi)^D} \Phi(\mathbf{p}) \frac{\delta}{\delta \Phi(\mathbf{p})} + \gamma_m(g) m \partial_m \right] \Gamma[\Phi(\mathbf{p}); m, g, \mu] = 0. \quad (10.110)$$

The addition of the missing $n = 0$ term will modify this equation as we shall see in the next section.

A corresponding equation holds for the *effective potential* $v(\Phi)$. This is defined as the negative effective energy density at a constant average field $\Phi(x) \equiv \Phi$:

$$v(\Phi) = -L^{-D} \bar{\Gamma}[\Phi; m, g, \mu]|_{\Phi(x) \equiv \Phi}. \quad (10.111)$$

Here $L$ is the linear size of the $D$-dimensional box under consideration. The effective potential satisfies the differential equation

$$[\mu \partial_\mu + \beta(g) \partial_g - \gamma(g) \Phi \partial_\Phi + \gamma_m(g) m \partial_m] \, v(\Phi; m, g, \mu) = 0. \quad (10.112)$$

Due to its special relevance to physical applications, we solve here only the latter equation along the lines of Eqs. (10.61)–(10.70). We introduce a running field strength $\Phi(\sigma)$, satisfying the differential equation

$$\frac{1}{\Phi(\sigma)} \sigma \frac{d}{d\sigma} \Phi(\sigma) = -\gamma(g(\sigma)), \quad (10.113)$$

with the initial condition

$$\Phi(1) = \Phi. \quad (10.114)$$

The equation is solved by

$$\frac{\Phi(\sigma)}{\Phi} = \exp\left\{-\int_1^\sigma \frac{d\sigma'}{\sigma'} \gamma(g(\sigma'))\right\} = \exp\left\{-\int_g^{g(\sigma)} dg' \frac{\gamma(g')}{\beta(g')}\right\}. \quad (10.115)$$

Using this function $\Phi(\sigma)$, the effective potential satisfies the renormalization group equation [analogous to (10.70)]:

$$v(\Phi; m, g, \mu) = v(\Phi(\sigma); m(\sigma), g(\sigma), \mu\sigma). \tag{10.116}$$

Note that there is no prefactor as in (10.70).

Since $\Gamma[\Phi]$ is dimensionless and $v(\Phi)$ is related to $\Gamma[\Phi]$ by (10.111), there is a naive scaling relation analogous to (10.76):

$$v(\Phi; m, g, \mu) = \sigma^D v\left(\frac{\Phi}{\sigma^{D/2-1}}; \frac{m}{\sigma}, g, \frac{\mu}{\sigma}\right). \tag{10.117}$$

Together with (10.116), this gives

$$v(\Phi; m, g, \mu) = \sigma^D v\left(\frac{\Phi(\sigma)}{\sigma^{D/2-1}}; \frac{m(\sigma)}{\sigma}, g(\sigma), \mu\right). \tag{10.118}$$

At the mass dependent value $\sigma_m = m(\sigma_m)/\mu$, we obtain the analog of (10.79) for the effective potential

$$v(\Phi; m, g, \mu) = \sigma_m^D\, v\left(\frac{\Phi(\sigma_m)}{\sigma_m^{D/2-1}}; \mu, g(\sigma_m), \mu\right). \tag{10.119}$$

In the limit $m \to 0$, where $\sigma \to 0$, we see from Eq. (10.115) that the field behaves like

$$\frac{\Phi(\sigma_m)}{\Phi} \approx \sigma_m^{-\gamma^*}. \tag{10.120}$$

The effective potential has therefore the power behavior

$$v(\Phi; m, g, \mu) \overset{m\approx 0}{\approx} \sigma_m^D\, v(\Phi/\sigma_m^{\gamma^*+D/2-1}; \mu, g^*, \mu), \tag{10.121}$$

where $\sigma_m$ is related to $t = m^2/\mu^2$ by (10.84).

For applications to many-body systems below $T_c$, it is most convenient to consider, instead of $\bar{\Gamma}[\Phi; m, g, \mu]$, the proper vertex functions in the presence of an external magnetization. They are obtained by expanding $\bar{\Gamma}[\Phi; m, g, \mu]$ functionally around $\Phi(x) \equiv \Phi_0$:

$$\bar{\Gamma}^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n; \Phi; m, g, \mu) \equiv \left.\frac{\delta^n \bar{\Gamma}[\Phi; m, g, \mu]}{\delta\Phi(\mathbf{x}_1)\ldots\delta\Phi(\mathbf{x}_n)}\right|_{\Phi\equiv\Phi_0}. \tag{10.122}$$

In momentum space, this gives

$$\bar{\Gamma}^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n; \Phi_0; m, g, \mu) = \sum_{n'=0}^{\infty} \frac{\Phi_0^{n'}}{n'!} \bar{\Gamma}^{(n+n')}(\mathbf{k}_1, \ldots \mathbf{k}_n, \mathbf{0}, \ldots, \mathbf{0}; m, g, \mu), \tag{10.123}$$

where the zeros after the arguments $\mathbf{k}_1, \ldots, \mathbf{k}_n$ indicate that there are $n'$ more momentum arguments $\mathbf{k}_{n+1}, \ldots, \mathbf{k}_{n+n'}$ which have been set zero since a constant field $\Phi(\mathbf{x}) \equiv \Phi_0$ has a Fourier transform $\Phi(\mathbf{k}) \equiv \Phi_0 \delta^{(D)}(\mathbf{k})$. Thus the renormalization group equation for the proper vertex function at a nonzero field, $\bar{\Gamma}^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n; \Phi_0; m, g, \mu)$, can be obtained from those at zero field $\bar{\Gamma}^{(n+n')}(\mathbf{k}_1, \ldots, \mathbf{k}_n, \mathbf{k}_{n+1}, \ldots, \mathbf{k}_{n+n'}; m, g, \mu)$ with the last $n'$ momenta set equal to zero, i.e., from

$$\begin{aligned}&[\mu\partial_\mu + \beta(g)\partial_g - (n+n')\gamma(g) + \gamma_m(g) m\partial_m] \\ &\qquad \times \bar{\Gamma}^{(n+n')}(\mathbf{k}_1, \ldots, \mathbf{k}_n, \mathbf{0}, \ldots, \mathbf{0}; m, g, \mu) = 0.\end{aligned} \tag{10.124}$$

Inserting this into (10.123), we obtain the renormalization group equation

$$\left[\mu\partial_\mu + \beta(g)\partial_g - \gamma(g)\left(n + \Phi_0\frac{\partial}{\partial\Phi_0}\right) + \gamma_m(g)m\partial_m\right] \times \bar{\Gamma}^{(n)}(\mathbf{k}_1,\dots,\mathbf{k}_n;\Phi_0,m,g,\mu) = 0. \quad (10.125)$$

When treated as above, this leads to the scaling relation

$$\bar{\Gamma}^{(n)}(\mathbf{k}_i;\Phi_0;m,g,\mu) = e^{-n\int_1^\sigma \frac{d\sigma'}{\sigma'}\gamma(g(\sigma'))}\bar{\Gamma}^{(n)}(\mathbf{k}_i;\Phi_0(\sigma);m(\sigma),g(\sigma),\mu\sigma). \quad (10.126)$$

Together with the trivial scaling relation

$$\bar{\Gamma}^{(n)}(\mathbf{k}_i;\Phi_0;m,g,\mu) = \sigma^{D-n(D/2-1)}\bar{\Gamma}^{(n)}(\mathbf{k}_i/\sigma;\Phi_0/\sigma^{D/2-1};m/\sigma,g,\mu/\sigma), \quad (10.127)$$

we find

$$\bar{\Gamma}^{(n)}(\mathbf{k}_i;\Phi_0;m,g,\mu) = \sigma^{D-n(D/2-1)}e^{-n\int_1^\sigma \frac{d\sigma'}{\sigma'}\gamma(g(\sigma'))} \times \bar{\Gamma}^{(n)}(\mathbf{k}_i/\sigma;\Phi_0(\sigma)/\sigma^{D/2-1};m(\sigma)/\sigma,g(\sigma),\mu). \quad (10.128)$$

This becomes, at $\sigma = \sigma_m$ of Eq. (10.78),

$$\bar{\Gamma}^{(n)}(\mathbf{k}_i;\Phi_0;m,g,\mu) = \sigma_m^{D-n(D/2-1)}e^{-n\int_1^{\sigma_m} \frac{d\sigma'}{\sigma'}\gamma(g(\sigma'))} \times \bar{\Gamma}^{(n)}(\mathbf{k}_i/\sigma_m;\Phi_0(\sigma_m)/\sigma_m^{D/2-1};\mu,g(\sigma_m),\mu), \quad (10.129)$$

and thus, near the critical point,

$$\bar{\Gamma}^{(n)}(\mathbf{k}_i;\Phi_0;m,g,\mu) = n(\gamma^* + D/2 - 1)\bar{\Gamma}^{(n)}(\mathbf{k}_i/\sigma_m;\Phi_0/\sigma_m^{\gamma^*+D/2-1};\mu,g^*,\mu). \quad (10.130)$$

## 10.7 Special Properties of Ground State Energy

When deriving the behavior of the proper vertex functions $\bar{\Gamma}^{(n)}(\mathbf{k}_1,\dots,\mathbf{k}_n;m,g)$ under changes of the scale parameter $\mu$, the number $n$ was restricted to positive integer values $n \geq 1$. The vacuum energy contained in $\bar{\Gamma}^{(0)}(m,g)$ was omitted from the sum in Eq. (10.108). Indeed, the vacuum energy does not follow the regular renormalization pattern. For the above calculation of the critical exponents, this irregularity is irrelevant. But if we want to calculate amplitude ratios (recall the definition in Section 1.2), we have to know the full thermodynamic potential as the temperature approaches the critical point from above and from below. Then the special renormalization properties of the vacuum diagrams can no longer be ignored. The fundamental difference between the ground state energies above and below the transition was seen at the mean-field level in Eq. (1.43). While the vacuum energy is identically zero above the transition, it behaves like $-(T - T_c)^2$ below $T_c$, which is the condensation energy in mean-field approximation.

More subtleties appear when calculating loop corrections. The lowest-order vacuum diagram shows a peculiar feature: with the help of (8.116) and (8.117), we find the full semiclassical effective potential at zero average field $\Phi$ and coupling constant:

$$v_{\text{tot}}(0;m,0,\mu) = \frac{N}{2}\int\frac{d^Dp}{(2\pi)^D}\log(\mathbf{p}^2+m^2)$$
$$= \frac{N}{2}\frac{2}{D}\frac{(m^2)^{D/2}}{(4\pi)^{D/2}}\Gamma(1-D/2) = \frac{N}{2}\frac{m^4}{\mu^\varepsilon}\frac{1}{(4\pi)^2}\left[-\frac{1}{\varepsilon}+\frac{1}{2}\left(\log\frac{m^2}{4\pi\mu^2e^{-\gamma}}-\frac{3}{2}\right)\right]+\mathcal{O}(\varepsilon). \quad (10.131)$$

The pole term at $\varepsilon = 0$ can be removed by adding to the potential a counterterm

$$v^{\rm sg} \equiv \frac{N}{2}\frac{m^4}{\mu^\varepsilon}\frac{1}{(4\pi)^2}\frac{1}{\varepsilon}. \tag{10.132}$$

To find a finite effective potential order by order in perturbation theory, we must perform the perturbation expansion with an additional term in the initial energy functional, which carries an additional set of pole terms, to be written collectively as [6, 7]

$$\Delta E \equiv -L^D \frac{m^4(\mu)}{(4\pi)^2 g(\mu)\mu^\epsilon} Z_v(g(\mu), \varepsilon), \tag{10.133}$$

where $Z_v(g, \varepsilon)$ is the renormalization constant of the vacuum which has an expansion in powers of $1/\varepsilon$ analogous to the other renormalization constants in (10.35)–(10.37):

$$Z_v(g, \varepsilon) \equiv \sum_{n=1}^{\infty} Z_{v,n}(g)\frac{1}{\varepsilon^n}. \tag{10.134}$$

The expansion coefficients of $Z_v$ up to five loops will be given in Eq. (15.35).

The analogy of $Z_v$ with the other renormalization constants is not perfect: there is no constant zero-loop term in $Z_v$. Such a term would be there if we had added to the bare energy functional a term $-L^D m_B^4 h_B/\lambda_B$ with an arbitrary constant $h_B$. Such a term would have a temperature dependence $\propto m_B^4 \propto t^2$ contributing a constant background term to the specific heat near $T_c$ which is needed to describe experiments. Indeed, the specific heat [see the curves in Fig. 1.1 and their best fit (1.22)] shows a critical power behavior of $t$ superimposed upon a smooth background term. The latter can be fitted by an appropriate constant $h_B$. The sum over the pole terms in $Z_v$, on the other hand, diverges for $t \to 0$ and generates the critical power behavior proportional to $|t|^{D\nu}$ which, after two derivatives with respect to $t$, produces the observed peak in the specific heat $C \propto t^{D\nu-2} \propto |t|^{-\alpha}$. This will be seen explicitly on the next page.

The effective energy of the vacuum is obtained from the sum of all loop diagrams $\Gamma_B^{(0)}$, plus the additional term $\Delta\, E$. The total sum is the renormalized effective energy of the vacuum:

$$\Gamma^{(0)} = \Gamma_B^{(0)} + \Delta\, E. \tag{10.135}$$

Now, the bare effective energy at fixed bare quantities is certainly independent of the regularization parameter $\mu$, and therefore satisfies trivially the differential equation

$$\mu\frac{d}{d\mu}\Gamma_B^{(0)}\bigg|_B = 0. \tag{10.136}$$

For the renormalized effective energy (10.135), this implies that

$$\mu\frac{d}{d\mu}\Gamma^{(0)}\bigg|_B = \mu\frac{d}{d\mu}\Delta\, E\bigg|_B. \tag{10.137}$$

Inserting here the right-hand side of (10.133), this equation can be written as

$$\mu\frac{d}{d\mu}\Gamma^{(0)}\bigg|_B = -L^D \frac{m^4(\mu)}{(4\pi)^2 g(\mu)\mu^\varepsilon}\gamma_v(g(\mu)), \tag{10.138}$$

with the *renormalization group function of the vacuum*

$$\gamma_v(g) \equiv -\left[\varepsilon + \beta(g,\varepsilon)/g - 4\gamma_m\right] Z_v(g,\varepsilon) + \beta(g,\varepsilon)\partial_g Z_v(g,\varepsilon). \tag{10.139}$$

This function depends only on the renormalized coupling constant $g$ as a consequence of the renormalizability of the theory and the minimal subtraction scheme. The derivative on the left-hand side of (10.137) is converted, via the chain rule, into a sum of differentiations with respect to the renormalized parameters, as in Eq. (10.21). Thus we obtain for $\Gamma^{(0)}(m,g,\mu)$ the renormalization group equation

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(m,g,\mu)\frac{\partial}{\partial g} + \gamma_m(m,g,\mu)\, m\frac{\partial}{\partial m}\right]\Gamma^{(0)}(m,g,\mu) = -\frac{L^D}{(4\pi)^2}\frac{m^4(\mu)}{\mu^\varepsilon g(\mu)}\gamma_v(g(\mu))\,. \tag{10.140}$$

On the right-hand side we have emphasized the $\mu$-dependence of $m$ and $g$, to avoid confusion with $m = m(1)$ and $g = g(1)$ defined in (10.61), (10.62). Inserting the expansion (10.134) of $Z_v$ into the right-hand side of (10.140), we find that the differentiations isolate from $Z_v$ precisely the residue of the simple pole term $1/\varepsilon$, yielding

$$\gamma_v = g Z'_{v,1}(g). \tag{10.141}$$

All higher pole terms cancel, since the functions $Z_{v,n}$ satisfy recursion relations similar to (10.48)–(10.50):

$$g Z'_{v,n+1}(g) = \left[4\gamma_m - \beta_0(g)/g\right] Z_{v,n} + \beta_0(g) Z'_{v,n}(g), \tag{10.142}$$

with $\beta_0(g)$ of Eq. (10.41).

The renormalization group equation (10.140) can now be solved to find the renormalized effective energy of the vacuum [recall (10.119)]:

$$\Gamma^{(0)}(m,g,\mu) = L^D\sigma_m^D\, v\left(\frac{\Phi(\sigma_m)}{\sigma_m^{D/2-1}};\mu, g(\sigma_m),\mu\right)_{\min} - \frac{L^D}{(4\pi)^2}\frac{m^4 h}{g\mu^\varepsilon}\int_1^{\sigma_m}\frac{d\sigma}{\sigma^{1+\varepsilon}}\gamma_v(g(\sigma))m^4(\sigma). \tag{10.143}$$

In the scaling regime, where $\sigma_m$ is small and the mass goes to zero like

$$m(\sigma_m) \approx m\,\sigma_m^{\gamma_m^*}, \tag{10.144}$$

the additional term $\Delta E$ in the effective energy of the vacuum in (10.135) is proportional to

$$\Delta E \propto m^4\sigma_m^{4\gamma_m^*-\varepsilon} \propto t^{\nu(4\gamma_m^*-\varepsilon)+2} = t^{D\nu}. \tag{10.145}$$

Thus it has the same scaling behavior as the *incomplete* effective potential $v(\Phi)$ at $\Phi = 0$ [i.e., the effective potential without the $n = 0$ -term in the sum (10.108)], which according to (10.89) and (10.121) behaves like $\sigma_m^D \propto t^{D\nu}$. It is also the same as that for the effective potential at a nontrivial minimum $\Phi = \Phi_0$ in the ordered state, as we shall see from (10.167). This will be important later in Subsection 10.10.3 when calculating the critical exponent of the specific heat defined in (1.16). The calculation of the universal ratios of the amplitudes of the specific heat and other quantities depend crucially on the renormalized vacuum energy $\Gamma^{(0)}(m,g,\mu)$ [8].

## 10.8 Approach to Scaling

In Eq. (10.93), we derived Kadanoff's scaling law (1.28) from the scaling relation (10.86) for the two-point proper vertex function. From this, we extracted the critical exponents $\nu =$

$1/(2-2\gamma_m^*)$ governing the temperature behavior of the correlation length, and the exponent $\eta = 2\gamma^*$ determining the critical power behavior of the Green function.

In Eq. (10.104), we introduced a further important critical exponent which governs the approach to the scaling law (10.93) for $g \to g^*$. In order to find this, we expand the right-hand side of Eq. (10.79) around $g^*$ and write

$$\bar{\Gamma}^{(n)}(\mathbf{k}_i; m, g, \mu) = \sigma_m^{D-n(D/2-1)} \exp\left[-n \int_1^{\sigma_m} d\sigma' \frac{\gamma(g(\sigma'))}{\sigma'}\right] \bar{\Gamma}^{(n)}(\mathbf{k}_i/\sigma_m, \mu, g^*, \mu) \times C^{(n)}(\mathbf{k}_i/\sigma_m; \mu, g(\sigma_m), \mu), \tag{10.146}$$

with the correction factor $C^{(n)}$ given by

$$C^{(n)}(\mathbf{k}_i/\sigma_m; \mu, g(\sigma_m), \mu) = 1 + [g(\sigma_m) - g^*] \frac{\partial}{\partial g} \log \bar{\Gamma}^{(n)}(\mathbf{k}_i/\sigma_m; \mu, g, \mu)\Big|_{g=g^*} + \dots. \tag{10.147}$$

Using Eq. (10.106), the correction factor is rewritten as

$$C^{(n)}(\mathbf{k}_i/\sigma_m; \mu, g(\sigma_m), \mu) = 1 + (g - g^*)\sigma_m^{\omega} \frac{\partial}{\partial g} \log \bar{\Gamma}^{(n)}(\mathbf{k}_i/\sigma_m; \mu, g, \mu)\Big|_{g=g^*} + \dots, \tag{10.148}$$

where $\omega$ is the slope of the $\beta$-function at $g = g^*$, as defined in Eq. (10.104).

When approaching the critical point $\sigma_m \to 0$, a finite correction to scaling is observed if $\partial \log \bar{\Gamma}^{(n)}/\partial g$ is at $g = g^*$ homogenous of degree $\omega$ in the variables $\mathbf{k}_i/\sigma_m$. For the two-point proper vertex function such a behavior implies the following form of the correction factor

$$C^{(2)}(\mathbf{k}/\sigma_m, 1, g^*, 1) \overset{m\approx 0}{\approx} 1 + \text{const} \times (g - g^*)\sigma_m^{\omega} \times (|\mathbf{k}|/\mu\sigma_m)^{-\omega} + \dots. \tag{10.149}$$

Then

$$\bar{\Gamma}^{(2)}(\mathbf{k}; m, g, \mu) = \sigma_m^2 \exp\left[-2 \int_1^{\sigma_m} d\sigma' \frac{\gamma(g(\sigma'))}{\sigma'}\right] \bar{\Gamma}^{(2)}(\mathbf{k}/\sigma_m, \mu, g^*, \mu)\, C^{(2)}(\mathbf{k}/\sigma_m; \mu, g(\sigma_m), \mu) \tag{10.150}$$

behaves for $t \approx 0$ like

$$\bar{\Gamma}^{(2)}(\mathbf{k}; m, g, \mu) \overset{m\approx 0}{\approx} |\mathbf{k}|^{2-\eta} f(|\mathbf{k}|/\mu t^{\nu}) \left[1 + (g - g^*) \times \text{const} \times \left(\frac{|\mathbf{k}|}{\mu}\right)^{-\omega} + \dots\right]. \tag{10.151}$$

Thus the correction to scaling is described by the exponent $\omega$ which is the slope of the $\beta$-function at the fixed point $g^*$. From the above discussion it is obvious that $\omega$ is positive for an infrared stable fixed point.

The most accurately measured approach to scaling comes from space shuttle experiments on the specific heat in superfluid helium, plotted in Fig. 1.2. The correction factor for this approach is obtained from Eq. (10.148) for $n = 0$ to have the general scaling form

$$C^{(0)} = 1 + \text{const} \times \sigma_m^{\omega} = 1 + \text{const} \times t^{\nu\omega}, \tag{10.152}$$

where we have used (10.84) to express $\sigma_m$ in terms of $t = T/T_c - 1$. The exponent $\nu\omega$ is usually called $\Delta$ [compare Eq. (1.22)].

At this point one may wonder about the universality of this result since, in principle, other corrections to scaling might arise from neglected higher powers of the field of higher gradient terms in the energy functional, for example $\phi^6$ or $\phi(\partial\phi)^2$. Fortunately, all such terms can be shown to be irrelevant for the value of $\omega$. This is suggested roughly by dimensional considerations, and proved by studying the flow of these terms towards the critical limit with the help of the renormalization group [9].

## 10.9 Further Critical Exponents

The critical exponents $\nu, \eta, \omega$ determine the critical behavior of all observables and the approach to this behavior. Let us derive the scaling relations for several important thermodynamic quantities and correlation functions.

### 10.9.1 Specific Heat

Consider the specific heat as a function of temperature. The ground state energy above $T_c$ is given by the effective potential at zero average field $v_h(\Phi = 0)$ which, according to (10.89), (10.121), and (10.145), has the scaling behavior

$$v_h(\Phi = 0) \overset{m\approx 0}{\approx} \sigma_m^D \times \text{const.} + m^4 \sigma_m^{4\gamma_m^*-\varepsilon} \times \text{const.} \approx t^{D\nu}, \tag{10.153}$$

the second term coming from the sum of the vacuum diagrams in Eq. (10.145). Forming the second derivative with respect to $t$, we find for the specific heat at constant volume

$$C \overset{t\approx 0}{\approx} t^{D/(2-2\gamma_m^*)-2} = t^{D\nu-2}, \qquad t > 0. \tag{10.154}$$

This behavior has been observed experimentally, and the critical exponent has been named $\alpha$ [recall (1.16)]:

$$C \overset{t\approx 0}{\approx} t^{-\alpha}; \qquad t > 0. \tag{10.155}$$

Thus we can identify

$$\alpha = 2 - D\nu = 2 - D/(2 - 2\gamma_m^*), \tag{10.156}$$

showing that the exponent $\alpha$ is directly related to $\nu$, as stated before in the scaling relation (1.32).

### 10.9.2 Susceptibility

Suppose the system at $T > T_c$ is coupled to a nonzero external source $j$, which is the generalization of an external magnetic field in magnetic systems [recall Eq. (1.44) and (1.45)]. The equilibrium value of the magnetization $M \equiv \Phi$ is no longer zero but $M_B \equiv \Phi(j)$. It is determined by the equation of state [the generalization of (1.46)]

$$j = \left.\frac{\partial v(\Phi)}{\partial \Phi}\right|_{\Phi(j)}. \tag{10.157}$$

From (10.121) we see that in the vicinity of the critical point

$$j \overset{t\approx 0}{\approx} {\sigma_m}^{D-\gamma^*-(D/2-1)} v'(\Phi(j)/\sigma_m^{\gamma^*+D/2-1}; \mu, g^*, \mu), \qquad t > 0. \tag{10.158}$$

The *susceptibility* is obtained by an additional differentiation with respect to $\Phi$ [see Eq. (1.47)]:

$$\chi^{-1} \equiv \left.\frac{\partial^2 v}{\partial \Phi^2}\right|_{\Phi=\Phi(j)} \overset{t\approx 0}{\approx} {\sigma_m}^{D-(D-2)} \sigma_m^{-2\gamma^*} v''(\Phi(j)\sigma_m^{-(D-2)/2-\gamma^*}; \mu, g^*, \mu), \qquad t > 0. \tag{10.159}$$

For O($N$)-symmetric systems above $T_c$, this equation applies to the invariant part of the susceptibility matrix defined by Eq. (1.12). Below $T_c$, we must distinguish between longitudinal and transverse susceptibilities. This will be done in Subsection 10.10.5.

At $t = 0$ and zero field, one has $\Phi(j) = 0$ and finds

$$\chi^{-1} \propto \sigma_m^{2-2\gamma^*} = t^{(2-2\gamma^*)/(2-2\gamma_m^*)}. \qquad (10.160)$$

Experimentally, this critical exponent is called $\gamma$ [recall (1.17)]:

$$\chi^{-1} \overset{t\approx 0}{\approx} t^{\gamma}, \qquad t > 0. \qquad (10.161)$$

The critical exponent $\gamma$ should not be confused with the renormalization group function $\gamma(g)$ of Eq. (10.26). Comparing (10.161) with (10.160), we identify

$$\gamma = 2\frac{1-\gamma^*}{2-2\gamma_m^*} = \nu(2-\eta), \qquad (10.162)$$

thus reproducing the scaling relation (1.34).

### 10.9.3 Critical Magnetization

At the critical point, the proportionality of $j$ and $\Phi$ (or of $H$ and $M$) is destroyed by fluctuations. Experimentally, one observes a scaling relation [recall (1.20)]

$$M \approx B^{1/\delta}, \quad t = 0. \qquad (10.163)$$

This can be derived from Eq. (10.158) which shows that a finite effective potential at the critical point, where $\sigma_m = 0$, requires the derivative $v'$ to behave like some power for small $\sigma_m = 0$:

$$v' \overset{t\approx 0}{\approx} \text{const.} \times \left(\Phi/\sigma_m^{\gamma^*+D/2-1}\right)^{\delta}, \qquad t > 0. \qquad (10.164)$$

From the proportionality $j \propto \sigma_m^{D-\gamma^*-(D/2-1)} v'$, the power $\delta$ which makes $j$ finite in the limit $\sigma_m \to 0$ must satisfy

$$D - \gamma^* - (D/2-1) - [\gamma^* + (D/2-1)]\,\delta = 0. \qquad (10.165)$$

From this we obtain

$$\delta = \frac{D+2-2\gamma^*}{D-2+2\gamma^*} = \frac{D+2-\eta}{D-2+\eta}, \qquad (10.166)$$

which is the scaling relation (1.35).

## 10.10 Scaling Relations Below $T_c$

Let us now turn to scaling results *below* $T_c$. Since all individual vertex functions in the expansion of the effective energy (10.108) can be calculated for $m^2 < 0$ just as well as for $m^2 > 0$, the main difference lies in $v(\Phi)$ not having a minimum at $\Phi = 0$ but at $\Phi = \Phi_0 \neq 0$ for vanishing external fields.

### 10.10.1 Spontaneous Magnetization

Consider first the behavior of the spontaneous magnetization $M_0 \equiv \Phi_0$ as the temperature approaches $T_c$ from below. The equilibrium value of $\Phi$ is determined by the minimum of the effective potential $v(\Phi)$. According to (10.121), the minimum must have a constant ratio:

$$\Phi_0/\sigma_m^{\gamma^*+D/2-1} = \text{const}\,. \tag{10.167}$$

Hence, $\Phi_0$ depends on $m^2$ and thus on the reduced temperature $t$ as follows:

$$M_0 \equiv \Phi_0 \propto \sigma_m^{\gamma^*+D/2-1} = t^{(\gamma^*+D/2-1)/(2-2\gamma_m^*)}. \tag{10.168}$$

Thus we derive the experimentally observable relation [compare (1.19)]

$$M_0 \equiv \Phi_0 \propto (-t)^{\beta}, \tag{10.169}$$

with the critical exponent

$$\beta = \frac{\gamma^* + D/2 - 1}{2 - 2\gamma_m^*} = \frac{\nu}{2}(D - 2 + \eta). \tag{10.170}$$

This relation agrees with (1.33).

### 10.10.2 Correlation Length

Consider now the temperature dependent correlation length below $T_c$. From (10.130) we read off that the two-point function satisfies

$$\bar{\Gamma}^{(2)}(\mathbf{k};\Phi_0;m,g,\mu) \overset{t\approx 0}{\approx} \sigma_m^{2-2\gamma^*}\bar{\Gamma}^{(2)}\left(\mathbf{k}/\sigma_m;\Phi_0/\sigma_m^{\gamma^*+D/2-1};\mu,g^*,\mu\right), \quad t<0. \tag{10.171}$$

As in the previous case above $T_c$ [recall (10.86), (10.87)], this is a function of

$$\mathbf{k}/\sigma_m = \mu\,\xi(t)\mathbf{k}, \tag{10.172}$$

with the same temperature behavior as in (10.90). Thus the same critical exponent governs the divergence of the correlation length below and above $T_c$.

This *above-below equality* will now also be derived for the critical exponents $\alpha$ and $\gamma$ of specific heat and susceptibility, respectively. The derivation of their scaling behaviors for $t < 0$ requires keeping track of the change of the average field $\Phi_0$ with temperature.

### 10.10.3 Specific Heat

The exponent $\alpha$ of the specific heat below $T_c$ follows from the $\Phi_0 \neq 0$ -version of Eq. (10.153) for the effective potential:

$$v_h(\Phi_0) \overset{t\approx 0}{\approx} \sigma_m^D v\left(\Phi_0/\sigma_m^{\gamma^*+D/2-1},\mu,g^*,\mu\right) + \text{const.}\times m^4\sigma_m^{4\gamma_m^*-\varepsilon}. \tag{10.173}$$

Since the temperature change of $\Phi_0$ takes place at a constant combination $\Phi_0/\sigma_m^{\gamma^*+D/2-1}$ [see Eq. (10.167)], the presence of $\Phi_0 \neq 0$ can be ignored and we obtain the same result as in (10.153), implying a temperature behavior

$$v(\Phi_0) \propto \sigma_m^D \approx t^{D\nu}. \tag{10.174}$$

This agrees with the $T > T_c$ -behavior (10.153), leading to the same critical exponent of the specific heat as in (10.156).

### 10.10.4 Susceptibility

Suppose now that an external magnetic field is switched on in the ordered phase. It will cause a deviation $\delta\Phi \equiv \Phi(j) - \Phi_0$ from $\Phi_0$. From Eqs. (10.157) and (10.158) we obtain for $\delta\Phi$ the scaling relation

$$j \to \sigma_m^{D-(\gamma^*+D/2-1)} v'((\Phi_0 + \delta\Phi)/\sigma_m^{\gamma^*+D/2-1}; \mu, g^*, \mu). \tag{10.175}$$

Expanding this to first order in $\delta\Phi$ gives

$$\delta j \to \delta\Phi\, \sigma_m^{2-2\gamma^*} v''(\Phi_0/\sigma_m^{\gamma^*+D/2-1}; \mu, g^*, \mu). \tag{10.176}$$

Since $\Phi_0$ changes with $t$ according to (10.169), the last factor is independent of temperature, and the susceptibility $\chi(t)$ has the same functional form as in (10.159), exhibiting the same critical exponent $\gamma$ as in (10.161).

In O($N$)-symmetric systems, this result holds for the longitudinal susceptibility only. The transverse susceptibility requires the following separate discussion.

### 10.10.5 Transverse Susceptibility and Bending Stiffness

Suppose now that the ground state breaks spontaneously an O($N$) symmetry of the system. Then the susceptibility decomposes into a longitudinal part and a transverse part, as shown in Eq. (1.14), and these two parts have completely different scaling properties. Since susceptibilities are proportional to correlation functions according to (1.15), we extract their scaling properties from the lowest gradient term in the effective energy $\bar{\Gamma}[\mathbf{\Phi}; m, g, \mu]$ in the deviation of the average field $\mathbf{\Phi}(\mathbf{x})$ from the equilibrium value $\mathbf{\Phi}_0$. We write bold-face letters for vectors in O($N$) field space. The quadratic term in the deviation $\delta\mathbf{\Phi}(\mathbf{x}) \equiv \mathbf{\Phi}(\mathbf{x}) - \mathbf{\Phi}_0$ has the general form

$$\bar{\Gamma}[\mathbf{\Phi}; m, g, \mu] \approx \int d^D x\, \delta\mathbf{\Phi}(\mathbf{x}) \bar{\Gamma}^{(2)}(-i\partial_{\mathbf{x}}; \mathbf{\Phi}_0; m, g, \mu)) \delta\mathbf{\Phi}(\mathbf{x}). \tag{10.177}$$

For smooth field configurations $\delta\mathbf{\Phi}(\mathbf{x})$, we expand

$$\Gamma^{(2)}(-i\partial_{\mathbf{x}}; \mathbf{\Phi}_0; m, g, \mu) \approx c_1(\mathbf{\Phi}_0; m, g, \mu) - c_2(\mathbf{\Phi}_0; m, g, \mu)\partial_{\mathbf{x}}^2. \tag{10.178}$$

The temperature dependence of the expansion coefficients $c_{1,2}(\mathbf{\Phi}_0; m, g, \mu)$ can be extracted from Eq. (10.171) and (10.167). To ensure the existence of a nontrivial term proportional to $\mathbf{k}^2$ on the right-hand side of (10.171), we see that

$$\bar{\Gamma}^{(2)}(\mathbf{k}_i; \mathbf{\Phi}_0; m, g, \mu) \overset{t\approx 0}{\propto} \sigma_m^{2-2\gamma^*} + \text{const} \times \sigma_m^{-2\gamma^*} \mathbf{k}^2. \tag{10.179}$$

Recalling the dependence (10.89) of $\sigma_m$ on $t = m^2/\mu^2$, and the relation (10.94) for the critical exponent $\eta$, we obtain, for smooth field configurations, the temperature dependence of the leading terms in the effective energy

$$\bar{\Gamma}^{(2)}[\mathbf{\Phi}; m, g, \mu] \overset{t\approx 0}{\propto} \int d^D x \left\{ t^{(2-\eta)\nu} \mathbf{\Phi}^2(\mathbf{x}) + \text{const} \times t^{-\eta\nu} \left[\partial_{\mathbf{x}} \mathbf{\Phi}(\mathbf{x})\right]^2 \right\}. \tag{10.180}$$

For an O($N$)-symmetric system, the order field can be decomposed into size and direction as

$$\mathbf{\Phi}(\mathbf{x}) = \Phi(\mathbf{x})\, \mathbf{n}(\mathbf{x}) \tag{10.181}$$

which brings the effective energy for small and smooth deviations $\delta\Phi(\mathbf{x})$ and $\delta\mathbf{n}(\mathbf{x}) \equiv \mathbf{n}(\mathbf{x}) - \mathbf{n}_0$ from the average ordered configurations to the form as

$$\bar{\Gamma}^{(2)}[\mathbf{\Phi}; m, g, \mu] \overset{t\approx 0}{\propto} \int d^D x \left( t^{(2-\eta)\nu} \left\{ \Phi_0^2 + [\delta\Phi(\mathbf{x})]^2 \right\} + \text{const} \times t^{-\eta\nu} \left\{ [\partial_{\mathbf{x}}\delta\Phi(\mathbf{x})]^2 + \Phi_0^2 \left[ \partial_{\mathbf{x}}\delta\mathbf{n}(\mathbf{x}) \right]^2 \right\} \right). \tag{10.182}$$

In O(2)-symmetric systems, the order field has two components, and can be replaced by a complex field $\Phi(\mathbf{x}) = e^{i\theta(\mathbf{x})}\Phi_0(\mathbf{x})$ with a real $\Phi_0(\mathbf{x})$. Then the last term in (10.182) has the form

$$\text{const} \times \int d^D x t^{-\eta\nu} \Phi_0^2 \left[ \partial_{\mathbf{x}}\delta\theta(\mathbf{x}) \right]^2. \tag{10.183}$$

In both gradient terms we can, of course, omit the deviation symbols $\delta$. The directional deviation field $\delta\mathbf{n}(\mathbf{x})$ possesses only a gradient term, and describes long-range (massless) excitations whose existence is ensured by the Nambu-Goldstone theorem stated after Eq. (1.50).

From the coefficients of the quadratic terms we extract the scaling behavior of the longitudinal and tranverse correlation functions in momentum space:

$$G_{\mathrm{c}\,L}(\mathbf{k}) \propto \left[ t^{(2-\eta)\nu} + \text{const} \times t^{-\eta\nu}\mathbf{k}^2 \right]^{-1}, \tag{10.184}$$

$$G_{\mathrm{c}\,T}(\mathbf{k}) \propto \left[ \text{const} \times t^{-\eta\nu}\Phi_0^2\mathbf{k}^2 \right]^{-1}. \tag{10.185}$$

The longitudinal and tranverse susceptibilities are proportional to these [recall Eq. (1.15)]. Their critical behavior is given by

$$\chi_{\mathrm{c}\,L}^{-1}(\mathbf{0}) \propto t^{(2-\eta)\nu} \tag{10.186}$$

$$\mathbf{k}^{-2}\,\chi_{\mathrm{c}\,T}^{-1}(\mathbf{k}) \propto \frac{\partial}{\partial\mathbf{k}^2}\,\chi_{\mathrm{c}\,L}^{-1}(\mathbf{k}) \propto t^{-\eta\nu}\Phi_0^2. \tag{10.187}$$

Recalling the temperature dependence (10.169) of $\Phi_0$, and using the scaling relation for the average field $\Phi_0$ in Eq. (10.169), the second relation becomes

$$\mathbf{k}^{-2}\,\chi_{\mathrm{c}\,T}^{-1}(\mathbf{k}) \propto \frac{\partial}{\partial\mathbf{k}^2}\,\chi_{\mathrm{c}\,L}^{-1}(\mathbf{k}) \propto t^{(D-2)\nu}. \tag{10.188}$$

Comparison of the last term in Eq. (10.182) with (1.110) shows that the prefactor supplies us with the temperature behavior of the bending stiffness of the directional field $\mathbf{n}(\mathbf{x})$ near the critical point. In superfluid helium, this is by definition proportional to the experimentally measured superfluid density $\rho_s$ [recall Eq. (1.122)]. The bending stiffness, or the superfluid density, are therefore proportional to those in (10.188), and we obtain the temperature behavior of the superfluid density

$$\rho_s \propto t^{(D-2)\nu}. \tag{10.189}$$

The experimental verification of this scaling behavior was described in Chapter 1, the crucial plots being shown in Fig. **??**.

### 10.10.6 Widom's Relation

Finally it is worth noticing that Eq. (10.158) corresponds exactly to Widom's scaling relation (1.26). That relation can be differentiated with respect to $M$ to yield the magnetic equation of state

$$B = t^{3-\alpha} M^{-1-1/\beta} \psi'(t/M^{1/\beta}), \tag{10.190}$$

which may also be written as

$$\frac{B}{M^{\delta}} = f\left(\frac{t}{M^{1/\beta}}\right) \tag{10.191}$$

with some function $f(x)$. This is easily proven with the help of Griffith's scaling relation $\delta = -1 + (2-\alpha)/\beta$ [recall (1.29)]. In terms of the variables of our field theory, the equation of state (10.191) may be rewritten as

$$j = t^{\delta\beta}\left(\frac{t}{\Phi^{1/\beta}}\right)^{-\delta\beta} f\left(\frac{t}{\Phi^{1/\beta}}\right) = t^{\delta\beta} g\left(\frac{\Phi}{t^{\beta}}\right), \tag{10.192}$$

where $g(x)$ is some other function. By comparing this with (10.158), we see that

$$\delta\beta = [D - \gamma^* - (D/2-1)]\frac{1}{2-2\gamma_m^*} = \frac{\nu}{2}(D+2-\eta), \tag{10.193}$$

which is in agreement with (10.166), (10.170), or (1.33) and (1.35).

## 10.11 Comparison of Scaling Relations with Experiment

For a comparison with experiment, we may pick three sets of critical data and extract the values of $\eta$, $\nu$, and $\omega$. The remaining critical exponents can then be found from the scaling relations (1.32)–(1.35).

As an example take the magnetic system $CrBr_3$ where one measures

$$\beta \approx 0.368, \quad \delta \approx 4.3, \quad \gamma \approx 1.215. \tag{10.194}$$

Inserting these into Widom's scaling relation [recall (1.30)]

$$\beta = \gamma/(\delta-1), \tag{10.195}$$

we see that the relation is satisfied excellently. Inserting $\delta$ into the relation

$$\eta = \frac{D+2-(D-2)\delta}{\delta+1}, \tag{10.196}$$

we find for $D = 3$

$$\eta \approx 0.132, \tag{10.197}$$

and from the relation $\nu = \gamma/(2-\eta)$ [recall (1.34)]:

$$\nu \approx 0.65. \tag{10.198}$$

## 10.12 Critical Values $g^*$, $\eta$, $\nu$, and $\omega$ in Powers of $\varepsilon$

Let us now calculate explicitly the critical properties of the O($N$)-symmetric $\phi^4$-theory in the two-loop approximation. In Eq. (10.56) we gave the $\beta$-function

$$\beta(\bar{g}) = -\varepsilon\bar{g} + \frac{N+8}{3}\bar{g}^2 - \frac{3N+14}{3}\bar{g}^3. \tag{10.199}$$

In $D = 4$ dimensions, $\beta(\bar{g})$ starts with $\bar{g}^2$, and the only IR-stable fixed point lies at $\bar{g}^* = 0$. Thus the massless $\phi^4$-theory behaves asymptotically as a free theory. From Eqs. (10.52) and

(10.53) we see that the anomalous dimensions $\gamma$ and $\gamma_m$ are zero for $\bar{g}^* = 0$. Hence the critical exponents in Eqs. (10.88), (10.94), and (10.104) possess the mean field values for $D = 4$:

$$\eta = 0\,, \qquad \nu = 1/2\,, \qquad \omega = 0\,. \tag{10.200}$$

In $D = 4 - \varepsilon$ dimensions, the equation $\beta(\bar{g}^*) = 0$ for the fixed point has the nontrivial solution

$$\bar{g}^* = \frac{3}{N+8}\varepsilon + \frac{9(3N+14)}{(N+8)^3}\varepsilon^2 + \dots\,. \tag{10.201}$$

If this expansion is inserted into the $\bar{g}$-expansions (10.57)–(10.58), we obtain for the critical exponents $\nu$ and $\eta$ the $\varepsilon$-expansions:

$$\eta \;=\; 2\gamma^*(\varepsilon, N) \;=\; \frac{N+2}{2(N+8)^2}\varepsilon^2 + \dots\,, \tag{10.202}$$

$$\nu \;=\; \frac{1}{2 - 2\gamma_m^*(\varepsilon, N)} \;=\; \frac{1}{2} + \frac{N+2}{4(N+8)}\varepsilon + \frac{(N+2)(N^2+23N+60)}{8(N+8)^3}\varepsilon^2 + \dots\,. \tag{10.203}$$

The critical exponent $\omega$ governing the approach to scaling is found from the derivative of the $\beta$-function (10.56) at $\bar{g} = \bar{g}^*$ [recall (10.104)]:

$$\omega = \beta'^*(\varepsilon, N) = \varepsilon - 3\frac{3N+14}{(N+8)^2}\varepsilon^2 + \dots\,. \tag{10.204}$$

All $\varepsilon$-expansions are independent of the choice of the coupling constant. The critical exponents depend via $\varepsilon$ and $N$ only on the dimension of space and order parameter space. This is a manifestation of the universality of phase transitions, which states that the critical behavior depends only on the type of interaction, its symmetry, and the space dimensionality.

Let us compare the above $\varepsilon$-expansion with the experimental critical exponents in Section 10.11. The expansion can be used only for infinitesimal $\varepsilon$. For applications to three dimensions we have to evaluate them at $\varepsilon = 1$, which cannot be done by simply inserting this large $\varepsilon$-value, since the series diverge. Let us ignore this problem for the moment, deferring a proper resummation until Chapters 16, 19, and 20. Inserting $\varepsilon = 1$, and estimating the reliability of the result from the size of the last term in each series, we calculate for $N = 0, 1, 2, 3, \infty$ :

$$\begin{aligned}
\nu &= \tfrac{1}{2} + \tfrac{1}{16}\varepsilon + \tfrac{15}{512}\varepsilon^2 + \dots &&= \tfrac{303}{512} + \dots &&\approx 0.5918 \pm 0.0293, && N = 0,\\
\nu &= \tfrac{1}{2} + \tfrac{1}{12}\varepsilon + \tfrac{7}{162}\varepsilon^2 + \dots &&= \tfrac{203}{324} + \dots &&\approx 0.6265 \pm 0.0432, && N = 1,\\
\nu &= \tfrac{1}{2} + \tfrac{1}{10}\varepsilon + \tfrac{11}{200}\varepsilon^2 + \dots &&= \tfrac{131}{200} + \dots &&\approx 0.6550 \pm 0.0550, && N = 2,\\
\nu &= \tfrac{1}{2} + \tfrac{5}{44}\varepsilon + \tfrac{345}{5324}\varepsilon^2 + \dots &&= \tfrac{903}{1331} + \dots &&\approx 0.6874 \pm 0.0648, && N = 3,\\
\nu &= \tfrac{1}{2} + \tfrac{1}{4}\varepsilon + \tfrac{1}{8}\varepsilon^2 + \dots &&= \tfrac{7}{8} + \dots &&\approx 0.8750 \pm 0.1250, && N = \infty.
\end{aligned} \tag{10.205}$$

The other critical exponents are

$$\begin{aligned}
\eta &= \tfrac{1}{64} &&\approx 0.016, && N = 0,\\
\eta &= \tfrac{1}{54} &&\approx 0.019, && N = 1,\\
\eta &= \tfrac{1}{50} &&\approx 0.02, && N = 2,\\
\eta &= \tfrac{5}{242} &&\approx 0.021, && N = 3,\\
\eta &= 0, && && N = \infty,
\end{aligned} \tag{10.206}$$

and

$$\begin{aligned}
\omega &= \varepsilon - \tfrac{17}{27}\varepsilon^2 &= \tfrac{10}{27} &\approx 0.3704 \pm 0.6296, && N=0,\\
\omega &= \varepsilon - \tfrac{17}{27}\varepsilon^2 &= \tfrac{10}{27} &\approx 0.3704 \pm 0.6296, && N=1,\\
\omega &= \varepsilon - \tfrac{3}{5}\varepsilon^2 &= \tfrac{2}{5} &\approx 0.4 \pm 0.6, && N=2,\\
\omega &= \varepsilon - \tfrac{69}{121}\varepsilon^2 &= \tfrac{52}{121} &\approx 0.4298 \pm 0.5702, && N=3,\\
\omega &= \varepsilon &= 1, & && N=\infty.
\end{aligned} \tag{10.207}$$

The $\varepsilon$-expansion for $\nu$ has decreasing contributions from higher orders. The value up to order $\varepsilon^2$ is $\nu \approx 0.627$, and agrees reasonably with the experimental value $\nu \approx 0.65$ of Eq. (10.198).

The expansions for $\eta$ contain only one term, so no convergence can be judged. The agreement with experiment is nevertheless reasonable. The value to order $\varepsilon^2$ at $\varepsilon = 1$ is $\eta \approx 0.019$ which, via the scaling relation $\nu = \gamma/(2-\eta)$, leads to the exponent $\gamma \approx 1.287$, quite close to the experimental value $\gamma = 1.215$ in Eq. (10.194).

The expansion for $\omega$ are obviously useless since the errors are too large.

If we attempt to calculate critical exponents to higher order than $\varepsilon^2$ by inserting $\varepsilon = 1$ into the expansions, we observe that the agreement becomes worse since the series diverge. The rough agreement for $\nu$ and $\epsilon$ up to order $\varepsilon^2$ is a consequence of the *asymptotic convergence* of the series. In Chapter 16, we shall see how high-precision estimates can still be extracted from asymptotic series. The reader who is curious to see how the direct evaluation of the series becomes worse with higher orders in $\varepsilon$ may anticipate the five-loop expansions from Eqs. (17.13)–(17.15) and insert $\varepsilon = 1$ into these.

For a judgment of the reliability of all numbers (10.205)–(10.207), we refer the reader to the most accurate currently available critical exponents in Tables 20.2 and **??**.

## 10.13 Several Coupling Constants

For fields with more than one component, several $\phi^4$-couplings are possible which may all become simultaneously relevant in four dimensions. This was discussed in detail in Chapter 6. For each coupling constant, there exists a $\beta$-function and there may be two or more fixed points. The stability of the fixed points depends on $N$ and channels the flow in the space of the coupling constants. It can be shown in general [10] that the O($N$)-symmetric fixed point is the only stable one for $N \leq 4 - \mathcal{O}(\varepsilon)$.

In order to have only a single wave function renormalization constant for the $N$ field components $\phi_\alpha$, the following condition has to be fulfilled:

$$\bar{\Gamma}^{(2)}_{\alpha\beta}(k) \sim \bar{\Gamma}^{(2)}(k)\ \delta_{\alpha\beta}\ . \tag{10.208}$$

This property is guaranteed for all theories which are symmetric under reflection $\phi_\alpha \to -\phi_\alpha$ and under permutations of the $N$ field indices $\alpha$. The same symmetry ensures that $\bar{\Gamma}^{(4)}$ is, to all orders in perturbation theory, a linear combination of the tensors specifying the $\phi^4$-couplings. For two tensors $T^{(1)}_{\alpha\beta\gamma\delta}$ and $T^{(2)}_{\alpha\beta\gamma\delta}$, this condition reads

$$\bar{\Gamma}^{(4)}_{\alpha\beta\gamma\delta} \sim \bar{\Gamma}^{(4)}_1\, T^{(1)}_{\alpha\beta\gamma\delta} + \bar{\Gamma}^{(4)}_2\, T^{(2)}_{\alpha\beta\gamma\delta}\,. \tag{10.209}$$

If the conditions (10.208) and (10.209) are satisfied, we can find four scalar renormalization constants $Z_A$ $(A = \phi, m^2, g_1, g_2)$ relating the bare mass $m_B$ and the two coupling constants $g_{iB}$ to the corresponding physical parameters by

$$m_B^2 = \frac{Z_{m^2}}{Z_\phi} m^2; \qquad g_{iB} = \mu^\varepsilon \frac{Z_{g_i}}{(Z_\phi)^2} g_i \qquad \text{for } i = 1, 2\ . \tag{10.210}$$

The renormalization group functions are introduced in the usual way:

$$\beta_i(g_1, g_2) = \mu\partial_\mu g_i|_{g_{1B}, g_{2B}, m_B, \varepsilon} = \mu\partial_\mu g_i\Big|_B, \tag{10.211}$$

$$\gamma(g_1, g_2) = \mu\partial_\mu \log Z_\phi^{1/2}|_{g_{1B}, g_{2B}, m_B, \varepsilon} = \mu\partial_\mu \log Z_\phi^{1/2}\Big|_B, \tag{10.212}$$

$$\gamma_m(g_1, g_2) = \mu\partial_\mu \log m|_{g_{1B}, g_{2B}, m_B, \varepsilon} = \mu\partial_\mu \log m\Big|_B. \tag{10.213}$$

We have written Eqs. (10.211)–(10.213) by analogy with Eqs. (10.30)–(10.33). Since the renormalization constants depend on $g_1$ and $g_2$, the functions $g_1$ and $g_2$ are implicitly given by

$$g_{iB} = \mu^\varepsilon Z_{g_i} Z_\phi^{-2} g_i = g_{iB}(\mu, g_1(\mu), g_2(\mu)). \tag{10.214}$$

The derivatives $\partial_\mu g_1$ and $\partial_\mu g_2$ follow therefore from the two equations

$$\frac{\partial g_{iB}}{\partial \mu} + \frac{\partial g_{iB}}{\partial g_1}\frac{\partial g_1}{\partial \mu} + \frac{\partial g_{iB}}{\partial g_2}\frac{\partial g_2}{\partial \mu} = 0 \quad \text{for } i = 1, 2. \tag{10.215}$$

Using $\partial_\mu g_{iB} = \varepsilon\, g_{iB}/\mu$, we find

$$\frac{\partial \log g_{iB}}{\partial g_1}\beta_1 + \frac{\partial \log g_{iB}}{\partial g_2}\beta_2 = -\varepsilon \quad \text{for } i = 1, 2. \tag{10.216}$$

The renormalization group function $\gamma(g_1, g_2)$ is given by

$$\gamma(g_1, g_2) = \frac{\beta_1(g_1, g_2)}{2}\frac{\partial \log Z_\phi}{\partial g_1} + \frac{\beta_2(g_1, g_2)}{2}\frac{\partial \log Z_\phi}{\partial g_2}, \tag{10.217}$$

while $\gamma_m$ is obtained from the equation

$$\gamma_m(g_1, g_2) = -\frac{\beta_1(g_1, g_2)}{2}\frac{\partial \log Z_{m^2}}{\partial g_1} - \frac{\beta_2(g_1, g_2)}{2}\frac{\partial \log Z_{m^2}}{\partial g_2} + \gamma(g_1, g_2). \tag{10.218}$$

Extracting the regular terms of Eqs. (10.216)–(10.218), we find the analog of Eqs. (10.43)–(10.46) for the case with two coupling constants:

$$\begin{aligned} \beta_1 &= -\varepsilon g_1 + g_1\left(g_1\partial_{g_1} Z_{g_1,1} + g_2\partial_{g_2} Z_{g_1,1} + 4\,\gamma\right), \\ \beta_2 &= -\varepsilon g_2 + g_2\left(g_2\partial_{g_2} Z_{g_2,1} + g_1\partial_{g_1} Z_{g_2,1} + 4\,\gamma\right), \\ \gamma &= -\tfrac{1}{2} g_1\partial_{g_1} Z_{\phi,1} - \tfrac{1}{2} g_2\partial_{g_2} Z_{\phi,1}, \\ \gamma_m &= \tfrac{1}{2} g_1\partial_{g_1} Z_{m^2,1} + \tfrac{1}{2} g_2\partial_{g_2} Z_{m^2,1} + \gamma. \end{aligned} \tag{10.219}$$

The stability of the fixed points can be examined using the critical exponents $\omega_1$, $\omega_2$, which are the eigenvalues of the matrix $\partial\beta_i/\partial g_j$. They should be positive for an infrared stable fixed point. An example for a system with two coupling constants will be treated in Chapter 18.

## 10.14 Ultraviolet versus Infrared Properties

Some remarks may be useful concerning the special role of ultraviolet divergences in critical phenomena. In three dimensions, $\phi^4$-theories are superrenormalizable and possess finite correlation functions after only a few subtractions. So one may wonder about the relevance of ultraviolet divergences to critical phenomena, in particular, since the system at short distances is not supposed to be represented by the field theory. The explanation of this apparent paradox is the following. Consider some real physical system with a microstructure, such as a lattice, at a temperature very close to the critical temperature at which the correlation length $\xi$ extends over many lattice spacings. There the correlation functions have three regimes. At very long distances $x \gg \xi$, they fall off exponentially like $e^{-x/\xi}$. For distances much larger than the lattice spacing but much smaller than the correlation length, they behave like a power in $x$. At the critical temperature, this power behavior extends all the way out to infinite distances. In the third regime, where distances are of the order of the lattice spacing, the behavior is nonuniversal and depends crucially on the composition of the material. Nothing can be said about this regime on the basis of field-theoretic studies.

Let us compare these behaviors of correlation functions of real systems with the behaviors found in the present $\phi^4$ field theories. Here we can also distinguish three regimes. The third, unphysical regime, lies now at distances which are shorter than the inverse cutoff $\Lambda$ of the theory. In this regime, the perturbation theory has unphysical singularities, first discussed by Landau, that are completely irrelevant to the critical phenomena to be explained. At length scales much shorter than the correlation length, but much longer than $1/\Lambda$, the correlation functions show power behavior, from which we can extract the critical exponents of the field theory and compare them with experiments made in the above lattice system. In field theory, this is the so-called short-distance behavior. Its properties are governed by the ultraviolet divergences. At the critical point, the short-distance behavior extends all the way to infinity. This is the reason why ultraviolet divergences are relevant for the understanding of long-distance phenomena observed in many-body systems near the critical point, that are independent of the microstructure.

## Notes and References

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