Make not my paths offensive to the Gods
AESCHYLOS (524 BC–455 BC), Agamemnon

10

Spaces with Curvature and Torsion

The path integral of a free particle in spherical coordinates has taught us an important lesson: In a Euclidean space, we were able to obtain the correct time-sliced amplitude in curvilinear coordinates by setting up the sliced action in Cartesian coordinates x^i and transforming them to the spherical coordinates $q^\mu = (r, \theta, \phi)$. It was crucial to do the transformation at the level of the *finite* coordinate differences, $\Delta x^i \to \Delta q^\mu$. This produced higher-order terms in the differences Δq^μ which had to be included up to the order $(\Delta q)^4/\epsilon$. They all contributed to the relevant order ϵ. It is obvious that as long as the space is Euclidean, the same procedure can be used to find the path integral in an arbitrary curvilinear coordinate system q^μ, if we ignore subtleties arising near coordinate singularities which are present in centrifugal barriers, angular barriers, or Coulomb potentials. For these, a special treatment will be developed in Chapters 12–14.

We are now going to develop an entirely nontrivial but quite natural extension of this procedure and define a path integral in an arbitrary metric-affine space with curvature and torsion. It must be emphasized that the quantum theory in such spaces is not uniquely defined by the formalism developed so far. The reason is that also the original Schrödinger theory which was used in Chapter 2 to justify the introduction of path integrals is not uniquely defined in such spaces. In classical physics, the equivalence principle postulated by Einstein is a powerful tool for deducing equations of motion in curved space from those in flat space. At the quantum level, this principle becomes insufficient since it does not forbid the appearance of arbitrary coordinate-independent terms proportional to Planck's quantum $\hbar^2$ and the scalar curvature R to appear in the Schödinger equation. We shall set up a simple extension of Einstein's equivalence principle which will allow us to carry quantum theories from flat to curved spaces which are, moreover, permitted to carry certain classes of torsion.

In such spaces, not only the time-sliced action but also the measure of path integration requires a special treatment. To be valid in general it will be necessary to find construction rules for the time evolution amplitude which do not involve the crutch of Cartesian coordinates. The final formula will be purely intrinsic to the general metric-affine space [1].

A crucial test of the validity of the resulting path integral formula will come from applications to systems whose correct operator quantum mechanics is known on the basis of symmetries and group commutation rules rather than canonical commutation rules. In contrast to earlier approaches, our path integral formula will always yield the same quantum mechanics as operator quantum mechanics quantized via group commutation rules.

Our formula can, of course, also be used for an alternative approach to the path integrals solved before in Chapter 8, where a Euclidean space was parametrized in terms of curvilinear coordinates. There it gives rise to a more satisfactory treatment than before, since it involves only the intrinsic variables of the coordinate systems.

10.1 Einstein's Equivalence Principle

To motivate the present study we invoke *Einstein's equivalence principle*, according to which gravitational forces upon a spinless mass point are indistinguishable from those felt in an accelerating local reference.[1] They are independent of the atomic composition of the particle and strictly proportional to the value of the mass, the same mass that appears in the relation between force and acceleration, in Newton's second law. The strict equality between the two masses, gravitational and inertial, is fundamental to Einstein's equivalence principle. Experimentally, the equality holds to an extremely high degree of accuracy. Any possible small deviation can presently be attributed to extra non-gravitational forces. Einstein realized that as a consequence of this equality, all spinless point particles move in a gravitational field along the same orbits which are independent of their composition and mass. This *universality* of orbital motion permits the gravitational field to be attributed to geometric properties of spacetime.

In Newton's theory of gravity, the gravitational forces between mass points are inversely proportional to their distances in a Euclidean space. In Einstein's geometric theory the forces are explained entirely by a curvature of spacetime. In general the spacetime of general relativity may also carry another geometric property, called torsion. Torsion is supposed to be generated by the spin densities of material bodies. Quantitatively, this may have only extremely small effects, too small to be detected by present-day experiments. But this is only due to the small intrinsic spin of ordinary gravitational matter. In exceptional states of matter such as polarized neutron stars or black holes, torsion can become relevant. It is now generally accepted that spacetime should carry a nonvanishing torsion at least locally at those points which are occupied by spinning elementary particles [55]. This follows from rather general symmetry considerations. The precise equations of motion for the torsion field, on the other hand, are still a matter of speculation. Thus it is an open question whether

[1]Quotation from his original paper *Über das Relativitätsprinzip und die aus demselben gezogenen Folgerungen*, Jahrbuch der Relativität und Elektonik *4*, 411 (1907): "Wir ... wollen daher im folgenden die völlige physikalische Gleichwertigkeit von Gravitationsfeld und entsprechender Beschleunigung des Bezugssystems annehmen".

or not the torsion field is able to propagate into the empty space away from spinning matter.

Even though the effects of torsion are small we shall keep the discussion as general as possible and study the motion of a particle in a metric-affine space with both curvature and torsion. To prepare the grounds let us first recapitulate a few basic facts about classical orbits of particles in a gravitational field. For simplicity, we assume here only the three-dimensional space to have a nontrivial geometry.[2] Then there is a natural choice of a time variable t which is conveniently used to parametrize the particle orbits.

Starting from the free-particle action we shall then introduce a path integral for the time evolution amplitude in any metric-affine space which determines the quantum mechanics via the quantum fluctuations of the particle orbits.

10.2 Classical Motion of Mass Point in General Metric-Affine Space

On the basis of the equivalence principle, Einstein formulated the rules for finding the classical laws of motion in a gravitational field as a consequence of the geometry of spacetime. Let us recapitulate his reasoning adapted to the present problem of a nonrelativistic point particle in a non-Euclidean geometry.

10.2.1 Equations of Motion

Consider first the action of the particle along the orbit $\mathbf{x}(t)$ in a flat space parametrized with rectilinear, Cartesian coordinates:

$$\mathcal{A} = \int_{t_a}^{t_b} dt \frac{M}{2}(\dot{x}^i)^2, \quad i = 1, 2, 3. \tag{10.1}$$

It is transformed to curvilinear coordinates q^μ, $\mu = 1, 2, 3$, via some functions

$$x^i = x^i(q), \tag{10.2}$$

leading to

$$\mathcal{A} = \int_{t_a}^{t_b} dt \frac{M}{2} g_{\mu\nu}(q)\dot{q}^\mu \dot{q}^\nu, \tag{10.3}$$

where

$$g_{\mu\nu}(q) = \partial_\mu x^i(q)\partial_\nu x^i(q) \tag{10.4}$$

is the *induced metric* for the curvilinear coordinates. Repeated indices are understood to be summed over, as usual.

[2]The generalization to non-Euclidean spacetime will be obvious after the development in Chapter 19.

The length of the orbit in the flat space is given by

$$l = \int_{t_a}^{t_b} dt \sqrt{g_{\mu\nu}(q)\dot{q}^\mu \dot{q}^\nu}. \tag{10.5}$$

Both the action (10.3) and the length (10.5) are invariant under arbitrary *reparametrizations of space* $q^\mu \to q'^\mu$.

Einstein's equivalence principle amounts to the postulate that the transformed action (10.3) describes directly the motion of the particle in the presence of a gravitational field caused by other masses. The forces caused by the field are all a result of the geometric properties of the metric tensor.

The equations of motion are obtained by extremizing the action in Eq. (10.3) with the result

$$\partial_t(g_{\mu\nu}\dot{q}^\nu) - \frac{1}{2}\partial_\mu g_{\lambda\nu}\dot{q}^\lambda \dot{q}^\nu = g_{\mu\nu}\ddot{q}^\nu + \bar{\Gamma}_{\lambda\nu\mu}\dot{q}^\lambda \dot{q}^\nu = 0. \tag{10.6}$$

Here

$$\bar{\Gamma}_{\lambda\nu\mu} \equiv \frac{1}{2}(\partial_\lambda g_{\nu\mu} + \partial_\nu g_{\lambda\mu} - \partial_\mu g_{\lambda\nu}) \tag{10.7}$$

is the Riemann connection or Christoffel symbol of the first kind [recall (1.70)]. With the help of the Christoffel symbol of the second kind [recall (1.71)]

$$\bar{\Gamma}_{\lambda\nu}{}^{\mu} \equiv g^{\mu\sigma}\bar{\Gamma}_{\lambda\nu\sigma}, \tag{10.8}$$

we can write

$$\ddot{q}^\mu + \bar{\Gamma}_{\lambda\nu}{}^{\mu}\dot{q}^\lambda \dot{q}^\nu = 0. \tag{10.9}$$

The solutions of these equations are the classical orbits. They coincide with the extrema of the length of a line l in (10.5). Thus, in a curved space, classical orbits are the shortest lines, the *geodesics* [recall (1.72)].

The same equations can also be obtained directly by transforming the equation of motion from

$$\ddot{x}^i = 0 \tag{10.10}$$

to curvilinear coordinates q^μ, which gives

$$\ddot{x}^i = \frac{\partial x^i}{\partial q^\mu}\ddot{q}^\mu + \frac{\partial^2 x^i}{\partial q^\lambda \partial q^\nu}\dot{q}^\lambda \dot{q}^\nu = 0. \tag{10.11}$$

At this place it is again useful to employ the quantities defined in Eq. (1.361), the basis triads and their reciprocals

$$e^i{}_\mu(q) \equiv \frac{\partial x^i}{\partial q^\mu}, \quad e_i{}^\mu(q) \equiv \frac{\partial q^\mu}{\partial x^i}, \tag{10.12}$$

which satisfy the orthogonality and completeness relations (1.362):

$$e_i{}^\mu e^i{}_\nu = \delta^\mu{}_\nu, \qquad e_i{}^\mu e^j{}_\mu = \delta_i{}^j. \tag{10.13}$$

The induced metric can then be written as

$$g_{\mu\nu}(q) = e^i{}_\mu(q) e^i{}_\nu(q). \tag{10.14}$$

Labeling Cartesian coordinates, upper and lower indices i are the same. The indices μ, ν of the curvilinear coordinates, on the other hand, can be lowered only by contraction with the metric $g_{\mu\nu}$ or raised with the inverse metric $g^{\mu\nu} \equiv (g_{\mu\nu})^{-1}$. Using the basis triads, Eq. (10.11) can be rewritten as

$$\frac{d}{dt}(e^i{}_\mu \dot{q}^\mu) = e^i{}_\mu \ddot{q}^\mu + \partial_\nu e^i{}_\mu \dot{q}^\mu \dot{q}^\nu = 0,$$

or as

$$\ddot{q}^\mu + e_i{}^\mu \partial_\lambda e^i{}_\kappa \dot{q}^\kappa \dot{q}^\lambda = 0. \tag{10.15}$$

The quantity in front of $\dot{q}^\kappa \dot{q}^\lambda$ is called the *affine connection*:

$$\Gamma_{\lambda\kappa}{}^\mu = e_i{}^\mu \partial_\lambda e^i{}_\kappa. \tag{10.16}$$

Due to (10.13), it can also be written as [compare (1.370)]

$$\Gamma_{\lambda\kappa}{}^\mu = -e^i{}_\kappa \partial_\lambda e_i{}^\mu. \tag{10.17}$$

Thus we arrive at the transformed flat-space equation of motion

$$\ddot{q}^\mu + \Gamma_{\kappa\lambda}{}^\mu \dot{q}^\kappa \dot{q}^\lambda = 0. \tag{10.18}$$

The solutions of this equation are called the *straightest lines* or *autoparallels*.

If the coordinate transformation functions $x^i(q)$ are smooth and single-valued, their derivatives commute as required by Schwarz's integrability condition

$$(\partial_\lambda \partial_\kappa - \partial_\kappa \partial_\lambda) x^i(q) = 0. \tag{10.19}$$

Then the triads satisfy the identity

$$\partial_\lambda e^i_\kappa - \partial_\kappa e^i_\lambda = 0, \tag{10.20}$$

implying that the connection $\Gamma_{\lambda\kappa}{}^\mu$ is symmetric in the lower indices. In fact, it coincides with the Riemann connection, the Christoffel symbol $\bar{\Gamma}_{\lambda\kappa}{}^\mu$. This follows immediately after inserting $g_{\mu\nu}(q) = e^i{}_\mu(q) e^i{}_\nu(q)$ into (10.7) and working out all derivatives using (10.20). Thus, for a space with curvilinear coordinates q^μ which can be reached by an integrable coordinate transformation from a flat space, the autoparallels coincide with the geodesics.

10.2.2 Nonholonomic Mapping to Spaces with Torsion

It is possible to map the x-space locally into a q-space with torsion via an infinitesimal transformation

$$dx^i = e^i{}_\mu(q)dq^\mu. \tag{10.21}$$

We merely have to assume that the coefficient functions $e^i{}_\mu(q)$ do not satisfy the property (10.20) which follows from the Schwarz integrability condition (10.19):

$$\partial_\lambda e^i{}_\kappa(q) - \partial_\kappa e^i{}_\lambda(q) \neq 0, \tag{10.22}$$

implying that second derivatives in front of $x^i(q)$ do not commute as in Eq. (10.19):

$$(\partial_\lambda\partial_\kappa - \partial_\kappa\partial_\lambda)x^i(q) \neq 0. \tag{10.23}$$

In this case we shall call the differential mapping (10.21) *nonholonomic*, in analogy with the nomenclature for nonintegrable constraints in classical mechanics. The property (10.23) implies that $x^i(q)$ is a multivalued function $x^i(q)$, of which we shall give typical examples below in Eqs. (10.44) and (10.55).

Educated readers in mathematics have been wondering whether such nonholonomic coordinate transformations make any sense. They will understand this concept better if they compare the situation with the quite similar but much simpler creation of magnetic field in a field-free space by *nonholonomic gauge transformations*. More details are explained in Appendix 10A.

From Eq. (10.22) we see that the image space of a nonholonomic mapping carries torsion. The connection $\Gamma_{\lambda\kappa}{}^\mu = e_i{}^\mu e^i{}_{\kappa,\lambda}$ has a nonzero antisymmetric part, called the *torsion tensor*:[3]

$$S_{\lambda\kappa}{}^\mu = \frac{1}{2}(\Gamma_{\lambda\kappa}{}^\mu - \Gamma_{\kappa\lambda}{}^\mu) = \frac{1}{2}e_i{}^\mu\left(\partial_\lambda e^i{}_\kappa - \partial_\kappa e^i{}_\lambda\right). \tag{10.24}$$

In contrast to $\Gamma_{\lambda\kappa}{}^\mu$, the antisymmetric part $S_{\lambda\kappa}{}^\mu$ is a proper tensor under general coordinate transformations. The contracted tensor

$$S_\mu \equiv S_{\mu\lambda}{}^\lambda \tag{10.25}$$

transforms like a vector, whereas the contracted connection $\Gamma_\mu \equiv \Gamma_{\mu\nu}{}^\nu$ does not. Even though $\Gamma_{\mu\nu}{}^\lambda$ is not a tensor, we shall freely lower and raise its indices using contractions with the metric or the inverse metric, respectively: $\Gamma^\mu{}_\nu{}^\lambda \equiv g^{\mu\kappa}\Gamma_{\kappa\nu}{}^\lambda$, $\Gamma_\mu{}^{\nu\lambda} \equiv g^{\nu\kappa}\Gamma_{\mu\kappa}{}^\lambda$, $\Gamma_{\mu\nu\lambda} \equiv g_{\lambda\kappa}\Gamma_{\mu\nu}{}^\kappa$. The same thing will be done with $\bar\Gamma_{\mu\nu}{}^\lambda$.

In the presence of torsion, the affine connection (10.16) is no longer equal to the Christoffel symbol. In fact, by rewriting $\Gamma_{\mu\nu\lambda} = e_{i\lambda}\partial_\mu e^i{}_\nu$ trivially as

$$\begin{aligned}\Gamma_{\mu\nu\lambda} = {} & \frac{1}{2}\left\{\left[e_{i\lambda}\partial_\mu e^i{}_\nu + \partial_\mu e_{i\lambda}e^i{}_\nu\right] + \left[e_{i\mu}\partial_\nu e^i{}_\lambda + \partial_\nu e_{i\mu}e^i{}_\lambda\right] - \left[e_{i\mu}\partial_\lambda e^i{}_\nu + \partial_\lambda e_{i\mu}e^i{}_\nu\right]\right\} \\ & + \frac{1}{2}\left\{\left[e_{i\lambda}\partial_\mu e^i{}_\nu - e_{i\lambda}\partial_\nu e^i{}_\mu\right] - \left[e_{i\mu}\partial_\nu e^i{}_\lambda - e_{i\mu}\partial_\lambda e^i{}_\nu\right] + \left[e_{i\nu}\partial_\lambda e^i{}_\mu - e_{i\nu}\partial_\mu e^i{}_\lambda\right]\right\}\end{aligned} \tag{10.26}$$

[3]Our notation for the geometric quantities in spaces with curvature and torsion is the same as in J.A. Schouten, *Ricci Calculus*, Springer, Berlin, 1954.

and using $e^i{}_\mu(q)e^i{}_\nu(q) = g_{\mu\nu}(q)$, we find the decomposition

$$\Gamma_{\mu\nu}{}^\lambda = \bar{\Gamma}_{\mu\nu}{}^\lambda + K_{\mu\nu}{}^\lambda, \tag{10.27}$$

where the combination of torsion tensors

$$K_{\mu\nu\lambda} \equiv S_{\mu\nu\lambda} - S_{\nu\lambda\mu} + S_{\lambda\mu\nu} \tag{10.28}$$

is called the *contortion tensor*. It is antisymmetric in the last two indices so that

$$\Gamma_{\mu\nu}{}^\nu = \bar{\Gamma}_{\mu\nu}{}^\nu. \tag{10.29}$$

In the presence of torsion, the shortest and straightest lines are no longer equal. Since the two types of lines play geometrically an equally favored role, the question arises as to which of them describes the correct classical particle orbits. Intuitively, we expect the straightest lines to be the correct trajectories since massive particles possess *inertia* which tend to minimize their deviations from a straight line in spacetime. It is hard to conceive how a particle should know which path to take at each instant in time in order to minimize the path length to a distant point. This would contradict the principle of locality which pervades all laws of physics. Only in a spacetime without torsion is this possible, since there the shortest lines happen to coincide with straightest ones for purely mathematical reasons. In Subsection 10.2.3, the straightest lines will be derived from an action principle.

In Einstein's theory of gravitation, matter produces curvature in four-dimensional Minkowski spacetime, thereby explaining the universal nature of gravitational forces. The flat spacetime metric is

$$\eta_{ab} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}_{ab}, \quad a, b = 0,\, 1,\, 2,\, 3. \tag{10.30}$$

The *Riemann-Cartan curvature tensor* is defined as the covariant curl of the affine connection:

$$R_{\mu\nu\lambda}{}^\kappa = \partial_\mu \Gamma_{\nu\lambda}{}^\kappa - \partial_\nu \Gamma_{\mu\lambda}{}^\kappa - [\Gamma_\mu, \Gamma_\nu]_\lambda{}^\kappa, \qquad \mu, \nu, \ldots = 0,\, 1,\, 2,\, 3. \tag{10.31}$$

The last term is written in a matrix notation for the connection, in which the tensor components $\Gamma_{\mu\lambda}{}^\kappa$ are viewed as matrix elements $(\Gamma_\mu)_\lambda{}^\kappa$. The matrix commutator in (10.31) is then equal to

$$[\Gamma_\mu, \Gamma_\nu]_\lambda{}^\kappa \equiv (\Gamma_\mu \Gamma_\nu - \Gamma_\nu \Gamma_\mu)_\lambda{}^\kappa = \Gamma_{\mu\lambda}{}^\sigma \Gamma_{\nu\sigma}{}^\kappa - \Gamma_{\nu\lambda}{}^\sigma \Gamma_{\mu\sigma}{}^\kappa. \tag{10.32}$$

Expressing the affine connection (10.16) in (10.31) with the help of Eqs. (10.16) in terms of the four-dimensional generalization of the triads (10.12) and their reciprocals (10.12), the tetrads $e^a{}_\mu$ and their reciprocals $e_a{}^\mu$, we obtain the compact formula

$$R_{\mu\nu\lambda}{}^\kappa = e_a{}^\kappa (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) e^a{}_\lambda. \tag{10.33}$$

For the mapping (10.21), this implies that not only the coordinate transformation $x^a(q)$, but also its first derivatives fail to satisfy Schwarz's integrability condition:

$$(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\partial_\lambda x^a(q) \neq 0. \tag{10.34}$$

Such general transformation matrices $e^a{}_\mu(q)$ will be referred to as *multivalued basis tetrads*.

A transformation for which $x^a(q)$ have commuting derivatives, while the first derivatives $\partial_\mu x^a(q) = e^i{}_\mu(q)$ do not, carries a flat-space region into a purely curved one.

Einstein's original theory of gravity assumes the absence of torsion. The space properties are completely specified by the *Riemann curvature tensor* formed from the Riemann connection (the Christoffel symbol)

$$\bar{R}_{\mu\nu\lambda}{}^\kappa = \partial_\mu \bar{\Gamma}_{\nu\lambda}{}^\kappa - \partial_\nu \bar{\Gamma}_{\mu\lambda}{}^\kappa - [\bar{\Gamma}_\mu, \bar{\Gamma}_\nu]_\lambda{}^\kappa. \tag{10.35}$$

The relation between the two curvature tensors is

$$R_{\mu\nu\lambda}{}^\kappa = \bar{R}_{\mu\nu\lambda}{}^\kappa + \bar{D}_\mu K_{\nu\lambda}{}^\kappa - \bar{D}_\nu K_{\mu\lambda}{}^\kappa - [K_\mu, K_\nu]_\lambda{}^\kappa. \tag{10.36}$$

In the last term, the $K_{\mu\lambda}{}^\kappa$'s are viewed as matrices $(K_\mu)_\lambda{}^\kappa$. The symbols $\bar{D}_\mu$ denote the *covariant derivatives* formed with the Christoffel symbol. Covariant derivatives act like ordinary derivatives if they are applied to a scalar field. When applied to a vector field, they act as follows:

$$\begin{aligned} \bar{D}_\mu v_\nu &\equiv \partial_\mu v_\nu - \bar{\Gamma}_{\mu\nu}{}^\lambda v_\lambda, \\ \bar{D}_\mu v^\nu &\equiv \partial_\mu v^\nu + \bar{\Gamma}_{\mu\lambda}{}^\nu v^\lambda. \end{aligned} \tag{10.37}$$

The effect upon a tensor field is the generalization of this; every index receives a corresponding additive $\bar{\Gamma}$ contribution.

Note that the Laplace-Beltrami operator (1.371) applied to a scalar field $\sigma(q)$ can be written as

$$\Delta\,\sigma = g^{\mu\nu}\bar{D}_\mu \bar{D}_\nu \sigma. \tag{10.38}$$

In the presence of torsion, there exists another covariant derivative formed with the affine connection $\Gamma_{\mu\nu}{}^\lambda$ rather than the Christoffel symbol which acts upon a vector field as

$$\begin{aligned} D_\mu v_\nu &\equiv \partial_\mu v_\nu - \Gamma_{\mu\nu}{}^\lambda v_\lambda, \\ D_\mu v^\nu &\equiv \partial_\mu v^\nu + \Gamma_{\mu\lambda}{}^\nu v^\lambda. \end{aligned} \tag{10.39}$$

Note by definition of $\Gamma_{\lambda\kappa}{}^\mu$ in (10.16) and (10.17), the covariant derivatives of $e^i{}_\mu$ and $e_i{}^\mu$ vanish:

$$D_\mu e^i{}_\nu \equiv \partial_\mu e^i{}_\nu - \Gamma_{\mu\nu}{}^\lambda e^i{}_\lambda = 0, \qquad D_\mu e_i{}^\nu \equiv \partial_\mu e_i{}^\nu + \Gamma_{\mu\lambda}{}^\nu e_i{}^\lambda = 0. \tag{10.40}$$

This will be of use later.

From either of the two curvature tensors, $R_{\mu\nu\lambda}{}^{\kappa}$ and $\bar{R}_{\mu\nu\lambda}{}^{\kappa}$, one can form the once-contracted tensors of rank two, the *Ricci tensor*

$$R_{\nu\lambda} = R_{\mu\nu\lambda}{}^{\mu}, \tag{10.41}$$

and the *curvature scalar*

$$R = g^{\nu\lambda} R_{\nu\lambda}. \tag{10.42}$$

The celebrated Einstein equation for the gravitational field postulates that the tensor

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{10.43}$$

the so-called *Einstein tensor*, is proportional to the symmetric energy-momentum tensor of all matter fields. This postulate was made only for spaces with no torsion, in which case $R_{\mu\nu} = \bar{R}_{\mu\nu}$ and $R_{\mu\nu}$, $G_{\mu\nu}$ are both symmetric. As mentioned before, it is not yet clear how Einstein's field equations should be generalized in the presence of torsion since the experimental consequences are as yet too small to be observed. In this text, we are not concerned with the generation of curvature and torsion but only with their consequences upon the motion of point particles.

It is useful to set up two simple examples for nonholonomic mappings which illustrate the way in which these are capable of generating curvature and torsion from a Euclidean space. The reader not familiar with this subject is advised to consult a textbook on the physics of defects [2]. where such mappings are standard and of great practical importance; every plastic deformation of a material can only be described in terms of such mappings.

As a first example consider the transformation in two dimensions

$$dx^i = \begin{cases} dq^1 & \text{for } i = 1, \\ dq^2 + \epsilon \partial_\mu \phi(q) dq^\mu & \text{for } i = 2, \end{cases} \tag{10.44}$$

with an infinitesimal parameter ϵ and the multi-valued function

$$\phi(q) \equiv \arctan(q^2/q^1). \tag{10.45}$$

The triads reduce to dyads, with the components

$$\begin{aligned} e^1{}_\mu &= \delta^1{}_\mu \ , \\ e^2{}_\mu &= \delta^2{}_\mu + \epsilon \partial_\mu \phi(q) \ , \end{aligned} \tag{10.46}$$

and the torsion tensor has the components

$$e^1{}_\lambda S_{\mu\nu}{}^{\lambda} = 0, \qquad e^2{}_\lambda S_{\mu\nu}{}^{\lambda} = \frac{\epsilon}{2}(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\phi. \tag{10.47}$$

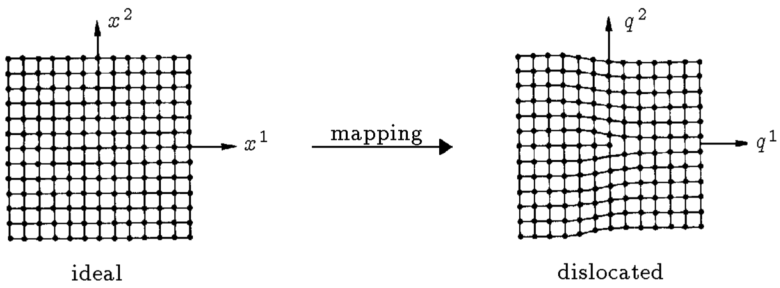


Figure 10.1 Edge dislocation in crystal associated with missing semi-infinite plane of atoms. The nonholonomic mapping from the ideal crystal to the crystal with the dislocation introduces a δ-function type torsion in the image space.

If we differentiate (10.45) formally, we find $(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\phi \equiv 0$. This, however, is incorrect at the origin. Using Stokes' theorem we see that

$$\int d^2q(\partial_1\partial_2 - \partial_2\partial_1)\phi = \oint dq^\mu \partial_\mu \phi = \oint d\phi = 2\pi \tag{10.48}$$

for any closed circuit around the origin, implying that there is a δ-function singularity at the origin with

$$e^2{}_\lambda S_{12}{}^\lambda = \frac{\epsilon}{2} 2\pi \delta^{(2)}(q). \tag{10.49}$$

By a linear superposition of such mappings we can generate an arbitrary torsion in the q-space. The mapping introduces no curvature.

In defect physics, the mapping (10.46) is associated with a dislocation caused by a missing or additional layer of atoms (see Fig. 10.1). When encircling a dislocation along a closed path C, its counter image C' in the ideal crystal does not form a closed path. The closure failure is called the *Burgers vector*

$$b^i \equiv \oint_{C'} dx^i = \oint_C dq^\mu e^i{}_\mu. \tag{10.50}$$

It specifies the direction and thickness of the layer of additional atoms. With the help of Stokes' theorem, it is seen to measure the torsion contained in any surface S spanned by C:

$$b^i \;=\; \oint_S d^2 s^{\mu\nu} \partial_\mu e^i{}_\nu = \oint_S d^2 s^{\mu\nu} e^i{}_\lambda S_{\mu\nu}{}^\lambda, \tag{10.51}$$

where $d^2 s^{\mu\nu} = -d^2 s^{\nu\mu}$ is the projection of an oriented infinitesimal area element onto the plane $\mu\nu$. The above example has the Burgers vector

$$b^i = (0, \epsilon). \tag{10.52}$$

A corresponding closure failure appears when mapping a closed contour C in the ideal crystal into a crystal containing a dislocation. This defines a Burgers vector:

$$b^\mu \equiv \oint_{C'} dq^\mu = \oint_C dx^i e_i{}^\mu. \tag{10.53}$$

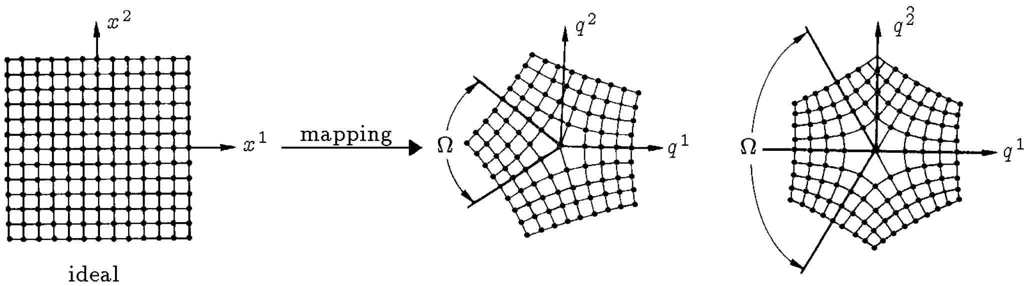


Figure 10.2 Edge disclination in crystal associated with missing semi-infinite section of atoms of angle Ω. The nonholonomic mapping from the ideal crystal to the crystal with the disclination introduces a δ-function type curvature in the image space.

By Stokes' theorem, this becomes a surface integral

$$\begin{aligned} b^\mu &= \oint_S d^2s^{ij}\partial_i e_j{}^\mu = \oint_S d^2s^{ij} e_i{}^\nu \partial_\nu e_j{}^\mu \\ &= -\oint_S d^2s^{ij} e_i{}^\nu e_j{}^\lambda S_{\nu\lambda}{}^\mu, \end{aligned} \tag{10.54}$$

the last step following from (10.17).

As a second example for a nonholonomic mapping, we generate curvature by the transformation

$$x^i = \delta^i{}_\mu \left[q^\mu + \frac{\Omega}{2\pi} \epsilon^\mu{}_\nu q^\nu \phi(q) \right], \tag{10.55}$$

with the multi-valued function (10.45). The symbol $\epsilon_{\mu\nu}$ denotes the antisymmetric Levi-Civita tensor. The transformed metric

$$g_{\mu\nu} = \delta_{\mu\nu} - \frac{\Omega}{\pi} \epsilon_{\mu\lambda}\epsilon_{\nu\kappa} \frac{q^\lambda q^\kappa}{q^\sigma q_\sigma} \tag{10.56}$$

is single-valued and has commuting derivatives. The torsion tensor vanishes since $(\partial_1\partial_2 - \partial_2\partial_1)x^{1,2}$ are both proportional to $q^{2,1}\delta^{(2)}(q)$, a distribution identical to zero. The local rotation field $\omega(q) \equiv \frac{1}{2}(\partial_1 x^2 - \partial_2 x^1)$, on the other hand, is equal to the multi-valued function $-\Omega\phi(q)/2\pi$, thus having the noncommuting derivatives:

$$(\partial_1\partial_2 - \partial_2\partial_1)\omega(q) = -\Omega\delta^{(2)}(q). \tag{10.57}$$

To lowest order in Ω, this determines the curvature tensor, which in two dimensions possesses only one independent component, for instance R_{1212}. Using the fact that $g_{\mu\nu}$ has commuting derivatives, R_{1212} can be written as

$$R_{1212} = (\partial_1\partial_2 - \partial_2\partial_1)\omega(q). \tag{10.58}$$

In defect physics, the mapping (10.55) is associated with a disclination which corresponds to an entire section of angle Ω missing in an ideal atomic array (see Fig. 10.2).

It is important to emphasize that our multivalued basis tetrads $e^a{}_\mu(q)$ are *not* related to the standard tetrads or *vierbein fields* $h^\alpha{}_\mu(q)$ used in the theory of gravitation with spinning particles. The difference is explained in Appendix 10B.

10.2.3 New Equivalence Principle

In classical mechanics, many dynamical problems are solved with the help of nonholonomic transformations. Equations of motion are differential equations which remain valid if transformed differentially to new coordinates, even if the transformation is not integrable in the Schwarz sense. Thus we *postulate that the correct equations of motion of point particles in a space with curvature and torsion are the images of the equation of motion in a flat space.* The equations (10.18) for the autoparallels yield therefore the correct trajectories of spinless point particles in a space with curvature and torsion.

This postulate is based on our knowledge of the motion of many physical systems. Important examples are the Coulomb system which will be discussed in detail in Chapter 13, and the spinning top in the body-fixed reference system [3]. Thus the postulate has a good chance of being true, and will henceforth be referred to as a *new equivalence principle*.

10.2.4 Classical Action Principle for Spaces with Curvature and Torsion

Before setting up a path integral for the time evolution amplitude we must find an action principle for the classical motion of a spinless point particle in a space with curvature and torsion, i.e., the movement along autoparallel trajectories. This is a nontrivial task since autoparallels must emerge as the extremals of an action (10.3) involving only the metric tensor $g_{\mu\nu}$. The action is independent of the torsion and carries only information on the Riemann part of the space geometry. Torsion can therefore enter the equations of motion only via some novel feature of the variation procedure. Since we know how to perform variations of an action in the Euclidean x-space, we deduce the correct procedure in the general metric-affine space by transferring the variations $\delta x^i(t)$ under the nonholonomic mapping

$$\dot{q}^\mu = e_i{}^\mu(q)\dot{x}^i \tag{10.59}$$

into the q^μ-space. Their images are quite different from ordinary variations as illustrated in Fig. 10.3(a). The variations of the Cartesian coordinates $\delta x^i(t)$ are done at fixed endpoints of the paths. Thus they form *closed paths* in the x-space. Their images, however, lie in a space with defects and thus possess a closure failure indicating the amount of torsion introduced by the mapping. This property will be emphasized by writing the images $\delta^S q^\mu(t)$ and calling them *nonholonomic variations*. The superscript indicates the special feature caused by torsion.

Let us calculate them explicitly. The paths in the two spaces are related by the integral equation

$$q^\mu(t) = q^\mu(t_a) + \int_{t_a}^{t} dt' e_i{}^\mu(q(t'))\dot{x}^i(t'). \tag{10.60}$$

For two neighboring paths in x-space differing from each other by a variation $\delta x^i(t)$, equation (10.60) determines the nonholonomic variation $\delta^S q^\mu(t)$:

$$\delta^S q^\mu(t) = \int_{t_a}^{t} dt' \delta^S [e_i{}^\mu(q(t'))\dot{x}^i(t')]. \tag{10.61}$$

A comparison with (10.59) shows that the variation δ^S and the time derivatives d/dt of $q^\mu(t)$ commute with each other:

$$\delta^S \dot{q}^\mu(t) = \frac{d}{dt}\delta^S q^\mu(t), \tag{10.62}$$

just as for ordinary variations δx^i:

$$\delta \dot{x}^i(t) = \frac{d}{dt}\delta x^i(t). \tag{10.63}$$

Let us also introduce *auxiliary nonholonomic variations* in q-space:

$$\bar{\delta} q^\mu \equiv e_i{}^\mu(q)\delta x^i. \tag{10.64}$$

In contrast to $\delta^S q^\mu(t)$, these vanish at the endpoints,

$$\bar{\delta} q(t_a) = \bar{\delta} q(t_b) = 0, \tag{10.65}$$

just as the usual variations $\delta x^i(t)$, i.e., they form *closed* paths with the unvaried orbits.

Using (10.62), (10.63), and the fact that $\delta^S x^i(t) \equiv \delta x^i(t)$, by definition, we derive from (10.61) the relation

$$\begin{aligned} \frac{d}{dt}\delta^S q^\mu(t) &= \delta^S e_i{}^\mu(q(t))\dot{x}^i(t) + e_i{}^\mu(q(t))\frac{d}{dt}\delta x^i(t) \\ &= \delta^S e_i{}^\mu(q(t))\dot{x}^i(t) + e_i{}^\mu(q(t))\frac{d}{dt}[e^i{}_\nu(t)\bar{\delta} q^\nu(t)]. \end{aligned} \tag{10.66}$$

After inserting

$$\delta^S e_i{}^\mu(q) = -\Gamma_{\lambda\nu}{}^\mu \delta^S q^\lambda e_i{}^\nu, \qquad \frac{d}{dt}e^i{}_\nu(q) = \Gamma_{\lambda\nu}{}^\mu \dot{q}^\lambda e^i{}_\mu, \tag{10.67}$$

this becomes

$$\frac{d}{dt}\delta^S q^\mu(t) = -\Gamma_{\lambda\nu}{}^\mu \delta^S q^\lambda \dot{q}^\nu + \Gamma_{\lambda\nu}{}^\mu \dot{q}^\lambda \bar{\delta} q^\nu + \frac{d}{dt}\bar{\delta} q^\mu. \tag{10.68}$$

It is useful to introduce the difference between the nonholonomic variation $\delta^S q^\mu$ and an auxiliary closed nonholonomic variation $\bar{\delta} q^\mu$:

$$\delta^S b^\mu \equiv \delta^S q^\mu - \bar{\delta} q^\mu. \tag{10.69}$$

Then we can rewrite (10.68) as a first-order differential equation for $\delta^S b^\mu$:

$$\frac{d}{dt}\delta^S b^\mu = -\Gamma_{\lambda\nu}{}^\mu \delta^S b^\lambda \dot{q}^\nu + 2S_{\lambda\nu}{}^\mu \dot{q}^\lambda \bar{\delta} q^\nu. \tag{10.70}$$

After introducing the matrices

$$G^\mu{}_\lambda(t) \equiv \Gamma_{\lambda\nu}{}^\mu(q(t))\dot{q}^\nu(t) \tag{10.71}$$

and

$$\Sigma^\mu{}_\nu(t) \equiv 2S_{\lambda\nu}{}^\mu(q(t))\dot{q}^\lambda(t), \tag{10.72}$$

equation (10.70) can be written as a vector differential equation:

$$\frac{d}{dt}\delta^S b = -G\delta^S b + \Sigma(t)\, \bar{\delta} q^\nu(t). \tag{10.73}$$

Although not necessary for the further development, we solve this equation by

$$\delta^S b(t) = \int_{t_a}^{t} dt' U(t,t')\ \Sigma(t')\ \bar{\delta} q(t'), \tag{10.74}$$

with the matrix

$$U(t,t') = T \exp\left[-\int_{t'}^{t} dt'' G(t'')\right]. \tag{10.75}$$

In the absence of torsion, $\Sigma(t)$ vanishes identically and $\delta^S b(t) \equiv 0$, and the variations $\delta^S q^\mu(t)$ coincide with the auxiliary closed nonholonomic variations $\delta q^\mu(t)$ [see Fig. 10.3(b)]. In a space with torsion, the variations $\delta^S q^\mu(t)$ and $\bar{\delta} q^\mu(t)$ are different from each other [see Fig. 10.3(c)].

Under an arbitrary nonholonomic variation $\delta^S q^\mu(t) = \delta q^\mu + \delta^S b^\mu$, the action (10.3) changes by

$$\delta^S \mathcal{A} = M\int_{t_a}^{t_b} dt \left(g_{\mu\nu}\dot{q}^\nu \delta^S \dot{q}^\mu + \frac{1}{2}\partial_\mu g_{\lambda\kappa}\delta^S q^\mu \dot{q}^\lambda \dot{q}^\kappa \right). \tag{10.76}$$

After a partial integration of the $\delta\dot{q}$-term we use (10.65), (10.62), and the identity $\partial_\mu g_{\nu\lambda} \equiv \Gamma_{\mu\nu\lambda} + \Gamma_{\mu\lambda\nu}$, which follows directly form the definitions $g_{\mu\nu} \equiv e^i{}_\mu e^i{}_\nu$ and $\Gamma_{\mu\nu}{}^\lambda \equiv e_i{}^\lambda \partial_\mu e^i{}_\nu$, and obtain

$$\delta^S \mathcal{A} = M\int_{t_a}^{t_b} dt \left[-g_{\mu\nu}\left(\ddot{q}^\nu + \bar{\Gamma}_{\lambda\kappa}{}^\nu \dot{q}^\lambda \dot{q}^\kappa\right) \bar{\delta} q^\mu + \left(g_{\mu\nu}\dot{q}^\nu \frac{d}{dt}\delta^S b^\mu + \Gamma_{\mu\lambda\kappa}\delta^S b^\mu \dot{q}^\lambda \dot{q}^\kappa \right)\right]. \tag{10.77}$$

To derive the equation of motion we first vary the action in a space without torsion. Then $\delta^S b^\mu(t) \equiv 0$, and (10.77) becomes

$$\delta^S \mathcal{A} = -M\int_{t_a}^{t_b} dt g_{\mu\nu}(\ddot{q}^\nu + \bar{\Gamma}_{\lambda\kappa}{}^\nu \dot{q}^\lambda \dot{q}^\kappa)\bar{\delta} q^\nu. \tag{10.78}$$

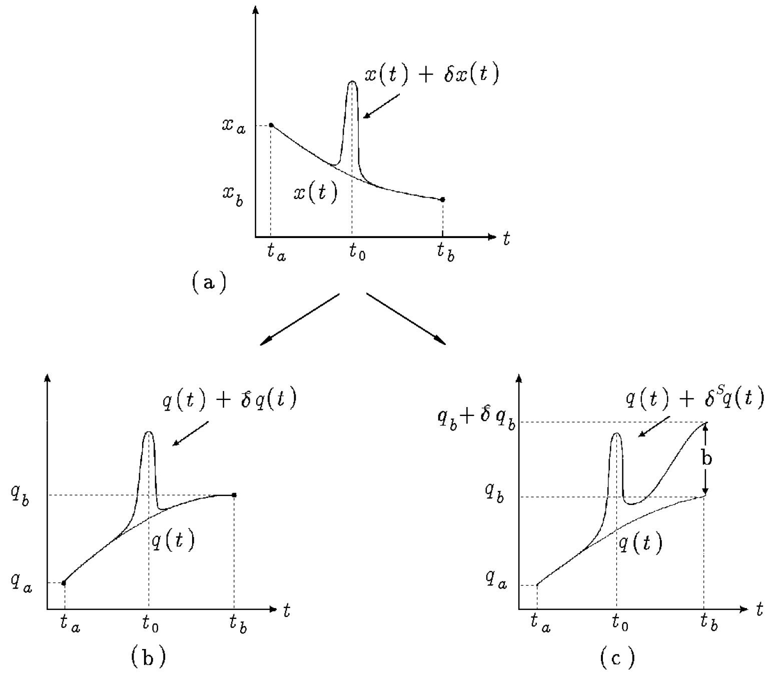


Figure 10.3 Images under holonomic and nonholonomic mapping of fundamental δ-function path variation. In the holonomic case, the paths $x(t)$ and $x(t)+\delta x(t)$ in (a) turn into the paths $q(t)$ and $q(t)+\bar{\delta} q(t)$ in (b). In the nonholonomic case with $S_{\mu\nu}^{\lambda} \neq 0$, they go over into $q(t)$ and $q(t)+\delta^S q(t)$ shown in (c) with a closure failure b^μ at t_b analogous to the Burgers vector b^μ in a solid with dislocations.

Thus, the action principle $\delta^S \mathcal{A}=0$ produces the equation for the geodesics (10.9), which are the correct particle trajectories in the absence of torsion.

In the presence of torsion, $\delta^S b^\mu$ is nonzero, and the equation of motion receives a contribution from the second parentheses in (10.77). After inserting (10.70), the nonlocal terms proportional to $\delta^S b^\mu$ cancel and the total nonholonomic variation of the action becomes

$$
\begin{aligned}
\delta^S \mathcal{A} &= -M \int_{t_a}^{t_b} dt g_{\mu\nu} \left[\ddot{q}^\nu + \left(\bar{\Gamma}_{\lambda\kappa}{}^{\nu} + 2S^{\nu}{}_{\lambda\kappa} \right) \dot{q}^\lambda \dot{q}^\kappa \right] \bar{\delta} q^\mu \\
&= -M \int_{t_a}^{t_b} dt g_{\mu\nu} \left(\ddot{q}^\nu + \Gamma_{\lambda\kappa}{}^{\nu} \dot{q}^\lambda \dot{q}^\kappa \right) \bar{\delta} q^\mu . \qquad (10.79)
\end{aligned}
$$

The second line follows from the first after using the identity $\Gamma_{\lambda\kappa}{}^{\nu} = \bar{\Gamma}_{\{\lambda\kappa\}}{}^{\nu} + 2S^{\nu}{}_{\{\lambda\kappa\}}$. The curly brackets indicate the symmetrization of the enclosed indices. Setting $\delta^S \mathcal{A}=0$ and inserting for $\bar{\delta} q(t)$ the image under (10.64) of an arbitrary δ-function variation $\delta x^i(t) \propto \epsilon^i \delta(t-t_0)$ gives the autoparallel equations of motions (10.18), which is what we wanted to show.

The above variational treatment of the action is still somewhat complicated and calls for a simpler procedure [4]. The extra term arising from the second parenthesis in the variation (10.77) can be traced to a simple property of the auxiliary closed nonholonomic variations (10.64). To find this we form the time derivative $d_t \equiv d/dt$ of the defining equation (10.64) and find

$$d_t \bar{\delta} q^\mu(t) = \partial_\nu e_i{}^\mu(q(t))\, \dot{q}^\nu(t) \delta x^i(t) + e_i{}^\mu(q(t)) d_t \delta x^i(t). \tag{10.80}$$

Let us now perform variation $\bar{\delta}$ and t-derivative in the opposite order and calculate $d_t \bar{\delta} q^\mu(t)$. From (10.59) and (10.13) we have the relation

$$d_t q^\lambda(t) \; = e_i{}^\lambda(q(t))\, d_t x^i(t)\,. \tag{10.81}$$

Varying this gives

$$\bar{\delta} d_t q^\mu(t) = \partial_\nu e_i{}^\mu(q(t))\, \bar{\delta} q^\nu d_t x^i(t) + e_i{}^\mu(q(t)) \bar{\delta} d_t x^i. \tag{10.82}$$

Since the variation in x^i-space commute with the t-derivatives [recall (10.63)], we obtain

$$\bar{\delta} d_t q^\mu(t) - d_t \bar{\delta} q^\mu(t) = \partial_\nu e_i{}^\mu(q(t))\, \bar{\delta} q^\nu d_t x^i(t) - \partial_\nu e_i{}^\mu(q(t))\, \dot{q}^\nu(t) \delta x^i(t). \tag{10.83}$$

After re-expressing $\delta x^i(t)$ and $d_t x^i(t)$ back in terms of $\bar{\delta} q^\mu(t)$ and $d_t q^\mu(t) = \dot{q}^\mu(t)$, and using (10.17), (10.24), this becomes

$$\bar{\delta} d_t q^\mu(t) - d_t \bar{\delta} q^\mu(t) = 2 S_{\nu\lambda}{}^\mu \dot{q}^\nu(t) \bar{\delta} q^\lambda(t). \tag{10.84}$$

Thus, due to the closure failure in spaces with torsion, the operations d_t and $\bar{\delta}$ do not commute in front of the path $q^\mu(t)$. In other words, in contrast to the open variations δ^S (and of course the usual ones δ), the auxiliary closed nonholonomic variations $\bar{\delta}$ of velocities $\dot{q}^\mu(t)$ no longer coincide with the velocities of variations. This property is responsible for shifting the trajectory from geodesics to autoparallels.

Indeed, let us vary an action

$$\mathcal{A} = \int_{t_1}^{t_2} dt L\left(q^\mu(t), \dot{q}^\mu(t)\right) \tag{10.85}$$

directly by $\bar{\delta} q^\mu(t)$ and impose (10.84), we find

$$\bar{\delta} \mathcal{A} = \int_{t_1}^{t_2} dt \left\{ \frac{\partial L}{\partial q^\mu} \bar{\delta} q^\mu + \frac{\partial L}{\partial \dot{q}^\mu} \frac{d}{dt} \bar{\delta} q^\mu \; + 2\, S^\mu{}_{\nu\lambda} \frac{\partial L}{\partial \dot{q}^\mu}\, \dot{q}^\nu\, \bar{\delta} q^\lambda \right\}. \tag{10.86}$$

After a partial integration of the second term using the vanishing $\bar{\delta} q^\mu(t)$ at the endpoints, we obtain the Euler-Lagrange equation

$$\frac{\partial L}{\partial q^\mu} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^\mu} = -2 S_{\mu\nu}{}^\lambda \dot{q}^\nu \frac{\partial L}{\partial \dot{q}^\lambda}. \tag{10.87}$$

This differs from the standard Euler-Lagrange equation by an additional contribution due to the torsion tensor. For the action (10.3), we thus obtain the equation of motion

$$M\left[\ddot{q}^\mu + g^{\mu\kappa}\Big(\partial_\nu g_{\lambda\kappa} - \frac{1}{2}\, \partial_\kappa g_{\nu\lambda}\Big) - 2 S^\mu{}_{\nu\lambda}\right] \dot{q}^\nu \dot{q}^\lambda = 0, \tag{10.88}$$

which is once more Eq. (10.18) for autoparallels.

10.3 Path Integral in Spaces with Curvature and Torsion

We now turn to the quantum mechanics of a point particle in a general metric-affine space. Proceeding in analogy with the earlier treatment in spherical coordinates, we first consider the path integral in a flat space with Cartesian coordinates

$$(\mathbf{x}\,t|\mathbf{x}'t') = \frac{1}{\sqrt{2\pi i\epsilon\hbar/M}^D} \prod_{n=1}^{N} \left[\int_{-\infty}^{\infty} dx_n\right] \prod_{n=1}^{N+1} K_0^\epsilon(\Delta\mathbf{x}_n), \tag{10.89}$$

where $K_0^\epsilon(\Delta\mathbf{x}_n)$ is an abbreviation for the short-time amplitude

$$K_0^\epsilon(\Delta\mathbf{x}_n) \equiv \langle\mathbf{x}_n|\exp\left(-\frac{i}{\hbar}\epsilon\hat{H}\right)|\mathbf{x}_{n-1}\rangle = \frac{1}{\sqrt{2\pi i\epsilon\hbar/M}^D}\exp\left[\frac{i}{\hbar}\frac{M}{2}\frac{(\Delta\mathbf{x}_n)^2}{\epsilon}\right], \tag{10.90}$$

with $\Delta\mathbf{x}_n \equiv \mathbf{x}_n - \mathbf{x}_{n-1}$, $\mathbf{x} \equiv \mathbf{x}_{N+1}$, $\mathbf{x}' \equiv \mathbf{x}_0$. A possible external potential has been omitted since this would contribute in an additive way, uninfluenced by the space geometry.

Our basic postulate is that the path integral in a general metric-affine space should be obtained by an appropriate nonholonomic transformation of the amplitude (10.89) to a space with curvature and torsion.

10.3.1 Nonholonomic Transformation of Action

The short-time action contains the square distance $(\Delta\mathbf{x}_n)^2$ which we have to transform to q-space. For an infinitesimal coordinate difference $\Delta\mathbf{x}_n \approx d\mathbf{x}_n$, the square distance is obviously given by $(d\mathbf{x})^2 = g_{\mu\nu}dq^\mu dq^\nu$. For a finite $\Delta\mathbf{x}_n$, however, we know from Chapter 8 that we must expand $(\Delta\mathbf{x}_n)^2$ up to the fourth order in $\Delta q_n{}^\mu = q_n{}^\mu - q_{n-1}{}^\mu$ to find all terms contributing to the relevant order ϵ.

It is important to realize that with the mapping from dx^i to dq^μ not being holonomic, the finite quantity Δq^μ is not uniquely determined by Δx^i. A unique relation can only be obtained by integrating the functional relation (10.60) along a specific path. The preferred path is the classical orbit, i.e., the autoparallel in the q-space. It is characterized by being the image of a straight line in x-space. There the velocity $\dot{x}^i(t)$ is constant, and the orbit has the linear time dependence

$$\Delta x^i(t) = \dot{x}^i(t_0)\Delta t, \tag{10.91}$$

where the time t_0 can lie anywhere on the t-axis. Let us choose for t_0 the final time in each interval (t_n, t_{n-1}). At that time, $\dot{x}_n^i \equiv \dot{x}^i(t_n)$ is related to $\dot{q}_n^\mu \equiv \dot{q}^\mu(t_n)$ by

$$\dot{x}_n^i = e^i{}_\mu(q_n)\dot{q}_n^\mu. \tag{10.92}$$

It is easy to express $\dot{q}_n^\mu$ in terms of $\Delta q_n^\mu = q_n^\mu - q_{n-1}^\mu$ along the classical orbit. First we expand $q^\mu(t_{n-1})$ into a Taylor series around t_n. Dropping the time arguments, for brevity, we have

$$\Delta q \equiv q^\lambda - q'^\lambda = \epsilon\dot{q}^\lambda - \frac{\epsilon^2}{2!}\ddot{q}^\lambda + \frac{\epsilon^3}{3!}\dddot{q}^\lambda + \dots\ , \tag{10.93}$$

where $\epsilon = t_n - t_{n-1}$ and $\dot{q}^\lambda, \ddot{q}^\lambda, \ldots$ are the time derivatives at the final time t_n. An expansion of this type is referred to as a *postpoint expansion*. Due to the arbitrariness of the choice of the time t_0 in Eq. (10.92), the expansion can be performed around any other point just as well, such as t_{n-1} and $\bar{t}_n = (t_n + t_{n-1})/2$, giving rise to the so-called *prepoint* or *midpoint* expansions of Δq.

Now, the term $\ddot{q}^\lambda$ in (10.93) is given by the equation of motion (10.18) for the autoparallel

$$\ddot{q}^\lambda = -\Gamma_{\mu\nu}{}^\lambda \dot{q}^\mu \dot{q}^\nu. \tag{10.94}$$

A further time derivative determines

$$\dddot{q}^\lambda = -(\partial_\sigma \Gamma_{\mu\nu}{}^\lambda - 2\Gamma_{\mu\nu}{}^\tau \Gamma_{\{\sigma\tau\}}{}^\lambda)\dot{q}^\mu \dot{q}^\nu \dot{q}^\sigma. \tag{10.95}$$

Inserting these expressions into (10.93) and inverting the expansion, we obtain $\dot{q}^\lambda$ at the final time t_n expanded in powers of Δq. Using (10.91) and (10.92) we arrive at the mapping of the finite coordinate differences:

$$\begin{aligned} \Delta x^i &= e^i{}_\lambda \dot{q}^\lambda \Delta t \qquad (10.96)\\ &= e^i{}_\lambda \left[\Delta q^\lambda - \frac{1}{2!}\Gamma_{\mu\nu}{}^\lambda \Delta q^\mu \Delta q^\nu + \frac{1}{3!}\left(\partial_\sigma \Gamma_{\mu\nu}{}^\lambda + \Gamma_{\mu\nu}{}^\tau \Gamma_{\{\sigma\tau\}}{}^\lambda\right)\Delta q^\mu \Delta q^\nu \Delta q^\sigma + \ldots\right], \end{aligned}$$

where $e^i{}_\lambda$ and $\Gamma_{\mu\nu}{}^\lambda$ are evaluated at the postpoint.

It is useful to introduce

$$\Delta\xi^\mu \equiv e_i{}^\mu \Delta x^i \tag{10.97}$$

as *autoparallel coordinates* or *normal coordinates* to parametrize the neighborhood of a point q. If the space has no torsion, they are also called *Riemann normal coordinates* or *geodesic coordinates*.

The normal coordinates (10.97) are expanded in (10.150) in powers of Δq^μ around the postpoint. There exists also a prepoint version of $\Delta\xi$ in which all signs of Δq are simply the opposite. The prepoint version, for instance, has the expansion:

$$\Delta\xi^\lambda = \Delta q^\lambda + \frac{1}{2!}\Gamma_{\mu\nu}{}^\lambda \Delta q^\mu \Delta q^\nu + \frac{1}{3!}\left(\partial_\sigma \Gamma_{\mu\nu}{}^\lambda + \Gamma_{\mu\nu}{}^\tau \Gamma_{\{\sigma\tau\}}{}^\lambda\right)\Delta q^\mu \Delta q^\nu \Delta q^\sigma + \ldots. \tag{10.98}$$

In contrast to the finite differences Δq^μ, the normal coordinates $\Delta\xi^\mu$ in the neighborhood of a point are vectors and thus allow for a *covariant Taylor expansion* of a function $f(q^\mu + \Delta q^\mu)$. Its form is found by performing an ordinary Taylor expansion of a function $F(x)$ in Cartesian coordinates

$$F(x + \Delta x) = F(x) + \partial_i F(x)\Delta x^i + \frac{1}{2!}\partial_i \partial_j F(x)\Delta x^i \Delta x^j + \ldots\,, \tag{10.99}$$

and transforming this to coordinates q^μ. The function $F(x)$ becomes $f(q) = F(x(q))$, and the derivatives $\partial_{i_1}\partial_{i_2}\cdots\partial_{i_n} f(x)$ go over into covariant derivatives:

$e_{i_1}{}^{\mu} e_{i_2}{}^{\mu} \cdots e_{i_n}{}^{\mu} D_{\mu_1} D_{\mu_2} \cdots D_{\mu_n} f(q)$. For instance, $\partial_i F(x) = e_i{}^{\mu} \partial_\mu f(q) = e_i{}^{\mu} D_\mu f(q)$, and

$$\begin{aligned} \partial_i \partial_j f(q) &= e_i{}^{\mu} \partial_\mu e_j{}^{\nu} \partial_\nu f(q) = [e_i{}^{\mu} e_j{}^{\nu} \partial_\mu \partial_\nu + e_i{}^{\mu} (\partial_\mu e_j{}^{\nu}) \partial_\nu] \, f(q) \\ &= e_i{}^{\mu} e_j{}^{\nu} \Big[\partial_\mu \partial_\nu - \Gamma_{\mu\nu}{}^{\lambda} \partial_\lambda \Big] f(q) = e_i{}^{\mu} e_j{}^{\nu} D_\mu \partial_\nu f(q) = e_i{}^{\mu} e_j{}^{\nu} D_\mu D_\nu f(q), \end{aligned} \tag{10.100}$$

where we have used (10.17) to express $\partial_\mu e_j{}^{\nu} = -\Gamma_{\mu\sigma}{}^{\nu} e_j{}^{\sigma}$, and changed dummy indices. The differences Δx^i in (10.99) are replaced by $e^i{}_\mu \Delta \xi^\mu$ with the prepoint expansion (10.98). In this way we arrive at the covariant Taylor expansion

$$f(q+\Delta q) = F(x) + D_\mu f(q) \Delta \xi^i + \frac{1}{2!} D_\mu D_\nu f(q) \Delta \xi^\mu \Delta \xi^\nu + \dots \, . \tag{10.101}$$

Indeed, re-expanding the right-hand side in powers of Δq^μ via (10.98) we may verify that the affine connections cancel against those in the covariant derivatives of $f(q)$, so that (10.101) reduces to the ordinary Taylor expansion of $f(q+\Delta q)$ in powers of Δq.

Note that the expansion (10.96) differs only slightly from a naive Taylor expansion of the difference around the postpoint:

$$\Delta x^i = x^i(q) - x^i(q-\Delta q) = e^i{}_\lambda \Delta q^\lambda - \frac{1}{2} e^i{}_{\nu,\mu} \Delta q^\mu \Delta q^\nu + \frac{1}{3!} e^i{}_{\nu,\mu\sigma} \Delta q^\mu \Delta q^\nu \Delta q^\sigma + \dots \, , \tag{10.102}$$

where a subscript λ separated by a comma denotes the partial derivative $\partial_\lambda = \partial/\partial q^\lambda$, i.e., $f_{,\lambda} \equiv \partial_\lambda f$. The right-hand side can be rewritten with the help of the completeness relation (10.13) as

$$\Delta x^i = e^i{}_\lambda \left[\Delta q^\lambda - \frac{1}{2} e_j{}^{\lambda} e^j{}_{\nu,\mu} \Delta q^\mu \Delta q^\nu + \frac{1}{3!} e_j{}^{\lambda} e^j{}_{\nu,\mu\sigma} \Delta q^\mu \Delta q^\nu \Delta q^\sigma + \dots \right]. \tag{10.103}$$

The expansion coefficients can be expressed in terms of the affine connection (10.16), using the derived relation

$$e_i{}^{\lambda} e^i{}_{\nu,\mu\sigma} \;=\; \partial_\sigma (e_i{}^{\lambda} e^i{}_{\nu,\mu}) - e^{i\tau} e^i{}_{\nu,\mu} e^j{}_\tau e^{j\lambda}{}_{,\sigma} = \partial_\sigma \Gamma_{\mu\nu}{}^{\lambda} + \Gamma_{\mu\nu}{}^{\tau} \Gamma_{\sigma\tau}{}^{\lambda}. \tag{10.104}$$

Thus we obtain

$$\Delta x^i = e^i{}_\lambda \Big[\Delta q^\lambda - \frac{1}{2!} \Gamma_{\mu\nu}{}^{\lambda} \Delta q^\mu \Delta q^\nu + \frac{1}{3!} \Big(\partial_\sigma \Gamma_{\mu\nu}{}^{\lambda} + \Gamma_{\mu\nu}{}^{\tau} \Gamma_{\sigma\tau}{}^{\lambda} \Big) \Delta q^\mu \Delta q^\nu \Delta q^\sigma + \dots \Big]. \tag{10.105}$$

This differs from the true expansion (10.150) only by the absence of the symmetrization of the indices in the last affine connection.

Inserting (10.96) into the short-time amplitude (10.90), we obtain

$$K_0^\epsilon(\Delta \mathbf{x}) = \langle \mathbf{x} | \left(-\frac{i}{\hbar} \epsilon \hat{H} \right) | \mathbf{x} - \Delta \mathbf{x} \rangle = \frac{1}{\sqrt{2\pi i \epsilon \hbar / M}^D} e^{i \mathcal{A}^\epsilon_{>}(q, q-\Delta q)/\hbar}, \tag{10.106}$$

with the short-time postpoint action

$$
\begin{aligned}
\mathcal{A}^{\epsilon}_{>}(q, q-\Delta q) &= (\Delta x^i)^2 = \epsilon \frac{M}{2} g_{\mu\nu} \dot{q}^{\mu} \dot{q}^{\nu} \\
&= \frac{M}{2\epsilon} \Big\{ g_{\mu\nu} \Delta q^{\mu} \Delta q^{\nu} - \Gamma_{\mu\nu\lambda} \Delta q^{\mu} \Delta q^{\nu} \Delta q^{\lambda} \\
&\quad + \left[\frac{1}{3} g_{\mu\tau} \left(\partial_{\kappa} \Gamma_{\lambda\nu}{}^{\tau} + \Gamma_{\lambda\nu}{}^{\delta} \Gamma_{\{\kappa\delta\}}{}^{\tau} \right) + \frac{1}{4} \Gamma_{\lambda\kappa}{}^{\sigma} \Gamma_{\mu\nu\sigma} \right] \Delta q^{\mu} \Delta q^{\nu} \Delta q^{\lambda} \Delta q^{\kappa} + \ldots \Big\}.
\end{aligned}
\tag{10.107}
$$

Separating the affine connection into Christoffel symbol and torsion, this can also be written as

$$
\begin{aligned}
\mathcal{A}^{\epsilon}_{>}(q, q-\Delta q) &= \frac{M}{2\epsilon} \Big\{ g_{\mu\nu} \Delta q^{\mu} \Delta q^{\nu} - \bar{\Gamma}_{\mu\nu\lambda} \Delta q^{\mu} \Delta q^{\nu} \Delta q^{\lambda} \\
&+ \left[\frac{1}{3} g_{\mu\tau} \left(\partial_{\kappa} \bar{\Gamma}_{\lambda\nu}{}^{\tau} + \bar{\Gamma}_{\lambda\nu}{}^{\delta} \bar{\Gamma}_{\delta\kappa}{}^{\tau} \right) + \frac{1}{4} \bar{\Gamma}_{\lambda\kappa}{}^{\sigma} \bar{\Gamma}_{\mu\nu\sigma} + \frac{1}{3} S^{\sigma}{}_{\lambda\kappa} S_{\sigma\mu\nu} \right] \Delta q^{\mu} \Delta q^{\nu} \Delta q^{\lambda} \Delta q^{\kappa} + \ldots \Big\}.
\end{aligned}
\tag{10.108}
$$

Note that in contrast to the formulas for the short-time action derived in Chapter 8, the right-hand side contains only *intrinsic* quantities of q-space. For the systems treated there (which all lived in a Euclidean space parametrized with curvilinear coordinates), the present intrinsic result reduces to the previous one.

Take, for example, a two-dimensional Euclidean space parametrized by radial coordinates treated in Section 8.1. The postpoint expansion (10.96) reads for the components r, ϕ of $\dot{q}^{\lambda}$

$$
\dot{r} = \frac{\Delta r}{\epsilon} + \frac{r(\Delta\phi)^2}{2\epsilon} - \frac{\Delta r (\Delta\phi)^2}{\epsilon} + \ldots\,, \tag{10.109}
$$

$$
\dot{\phi} = \frac{\Delta\phi}{\epsilon} - \frac{\Delta r \Delta\phi}{\epsilon r} - \frac{(\Delta\phi)^3}{6\epsilon} + \ldots\,. \tag{10.110}
$$

Inserting these into the short-time action which is here simply

$$
\mathcal{A}^{\epsilon} = \frac{M}{2} \epsilon (\dot{r}^2 + r^2 \dot{\phi}^2), \tag{10.111}
$$

we find the time-sliced action

$$
\mathcal{A}^{\epsilon} = \frac{M}{2\epsilon} \left[\Delta r^2 + r^2 (\Delta\phi)^2 - r \Delta r (\Delta\phi)^2 - \frac{1}{12} r^2 (\Delta\phi)^4 + \ldots \right]. \tag{10.112}
$$

A symmetrization of the postpoint expressions using the fact that r^2 stands for

$$
r_n^2 = r_n (r_{n-1} + \Delta r_n), \tag{10.113}
$$

leads to the short-time action displaying the subscripts n

$$
\mathcal{A}^{\epsilon} = \frac{M}{2\epsilon} \left[\Delta r_n^2 + r_n r_{n-1} (\Delta\phi_n)^2 - \frac{1}{12} r_n r_{n-1} (\Delta\phi_n)^4 + \ldots \right]. \tag{10.114}
$$

This agrees with the previous expansion of the time-sliced action in Eq. (8.53). While the previous result was obtained from a transformation of the time-sliced

Euclidean action to radial coordinates, the short-time action here is found from a purely intrinsic formulation. The intrinsic method has the obvious advantage of not being restricted to a Euclidean initial space and therefore has the chance of being true in an arbitrary metric-affine space.

At this point we observe that the final short-time action (10.107) could also have been introduced without any reference to the flat reference coordinates x^i. Indeed, the same action is obtained by evaluating the continuous action (10.3) for the small time interval $\Delta t = \epsilon$ along the classical orbit between the points q_{n-1} and q_n. Due to the equations of motion (10.18), the Lagrangian

$$L(q, \dot{q}) = \frac{M}{2} g_{\mu\nu}(q(t))\, \dot{q}^\mu(t) \dot{q}^\nu(t) \tag{10.115}$$

is independent of time (this is true for autoparallels as well as geodesics). The short-time action

$$\mathcal{A}^\epsilon(q, q') = \frac{M}{2} \int_{t-\epsilon}^{t} dt\, g_{\mu\nu}(q(t)) \dot{q}^\mu(t) \dot{q}^\nu(t) \tag{10.116}$$

can therefore be written in either of the three forms

$$\mathcal{A}^\epsilon = \frac{M}{2} \epsilon g_{\mu\nu}(q) \dot{q}^\mu \dot{q}^\nu = \frac{M}{2} \epsilon g_{\mu\nu}(q') \dot{q}'^\mu \dot{q}'^\nu = \frac{M}{2} \epsilon g_{\mu\nu}(\bar{q}) \dot{\bar{q}}^\mu \dot{\bar{q}}^\nu, \tag{10.117}$$

where $q^\mu, q'^\mu, \bar{q}^\mu$ are the coordinates at the final time t_n, the initial time t_{n-1}, and the average time $(t_n + t_{n-1})/2$, respectively. The first expression obviously coincides with (10.107). The others can be used as a starting point for deriving equivalent prepoint or midpoint actions. The prepoint action $\mathcal{A}^\epsilon_<$ arises from the postpoint one $\mathcal{A}^\epsilon_>$ by exchanging Δq by $-\Delta q$ and the postpoint coefficients by the prepoint ones. The midpoint action has the most simple-looking appearance:

$$\begin{aligned} \bar{\mathcal{A}}^\epsilon(\bar{q} + \frac{\Delta q}{2}, \bar{q} - \frac{\Delta q}{2}) = & \\ \frac{M}{2\epsilon} \left[g_{\mu\nu}(\bar{q}) \Delta q^\mu \Delta q^\nu + \frac{1}{12} g_{\kappa\tau} \left(\partial_\lambda \Gamma_{\mu\nu}{}^\tau + \Gamma_{\mu\nu}{}^\delta \Gamma_{\{\lambda\delta\}}{}^\tau \right) \Delta q^\mu \Delta q^\nu \Delta q^\lambda \Delta q^\kappa + \ldots \right], & \end{aligned} \tag{10.118}$$

where the affine connection can be evaluated at any point in the interval (t_{n-1}, t_n). The precise position is irrelevant to the amplitude, producing changes only in higher than the relevant orders of ϵ.

We have found the postpoint action most useful since it gives ready access to the time evolution of amplitudes, as will be seen below. The prepoint action is completely equivalent to it and useful if one wants to describe the time evolution backwards. Some authors favor the midpoint action because of its symmetry and the absence of cubic terms in Δq^μ in the expression (10.118).

The different completely equivalent "anypoint" formulations of the same short-time action, which is *universally defined* by the nonholonomic mapping procedure, must be distinguished from various so-called time-slicing "prescriptions" found in the literature when setting up a lattice approximation to the Lagrangian (10.115). There, a *midpoint prescription* is often favored, in which one approximates L by

$$L(q,\dot{q}) \to L^{\epsilon}(q,\Delta q/\epsilon) = \frac{M}{2\epsilon^2} g_{\mu\nu}(\bar{q})\, \Delta q^{\mu}(t)\Delta q^{\nu}(t), \tag{10.119}$$

and uses the associated short-time action

$$\bar{\mathcal{A}}^{\epsilon}_{\rm mpp} = \epsilon L^{\epsilon}(q,\Delta q/\epsilon) \tag{10.120}$$

in the exponent of the path integrand. The motivation for this prescription lies in the popularity of H. Weyl's ordering prescription for products of position and momenta in operator quantum mechanics. From the discussion in Section 1.13 we know, however, that the Weyl prescription for the operator order in the kinetic energy $g^{\mu\nu}(\hat{q})\hat{p}_{\mu}\hat{p}_{\nu}/2M$ does not lead to the correct Laplace-Beltrami operator in general coordinates. The discussion in this section, on the other hand, will show that the Weyl-ordered action (10.120) differs from the midpoint form (10.118) of the correct short-time action by an additional forth-order term in Δq^{μ}, implying that the short-time action $\bar{\mathcal{A}}^{\epsilon}_{\rm mpp}$ does not lead to the correct physics. Worse shortcomings are found when slicing the short-time action following a pre- or postpoint prescription. There is, in fact, no freedom of choice of different slicing prescriptions, in contrast to ubiquitous statements in the literature. The short-time action is completely fixed as being the unique nonholonomic image of the Euclidean time-sliced action. This also solves uniquely the operator-ordering problem which has plagued theorists for many decades.

In the following, the action $\mathcal{A}^{\epsilon}$ without subscript will always denote the preferred postpoint expression (10.107):

$$\mathcal{A}^{\epsilon} \equiv \mathcal{A}^{\epsilon}_{>}(q, q-\Delta q). \tag{10.121}$$

10.3.2 Measure of Path Integration

We now turn to the integration measure in the Cartesian path integral (10.89)

$$\frac{1}{\sqrt{2\pi i\epsilon\hbar/M}^{D}} \prod_{n=1}^{N} d^D x_n.$$

This has to be transformed to the general metric-affine space. We imagine evaluating the path integral starting out from the latest time and performing successively the integrations over $x_N, x_{N-1}, \dots$, i.e., in each short-time amplitude we integrate over the earlier position coordinate, the prepoint coordinate. For the purpose of this discussion, we relabel the product $\prod_{n=1}^{N} d^D x^i_n$ by $\prod_{n=2}^{N+1} dx^i_{n-1}$, so that the integration in each time slice (t_n, t_{n-1}) with $n = N+1, N, \dots$ runs over dx^i_{n-1}.

In a flat space parametrized with curvilinear coordinates, the transformation of the integrals over $d^D x^i_{n-1}$ into those over $d^D q^{\mu}_{n-1}$ is obvious:

$$\prod_{n=2}^{N+1} \int d^D x^i_{n-1} = \prod_{n=2}^{N+1} \left\{ \int d^D q^{\mu}_{n-1} \det\left[e^i{}_{\mu}(q_{n-1})\right] \right\}. \tag{10.122}$$

The determinant of $e^i{}_\mu$ is the square root of the determinant of the metric $g_{\mu\nu}$:

$$\det\,(e^i{}_\mu) = \sqrt{\det g_{\mu\nu}(q)} \equiv \sqrt{g(q)}, \tag{10.123}$$

and the measure may be rewritten as

$$\prod_{n=2}^{N+1} \int d^D x^i_{n-1} = \prod_{n=2}^{N+1} \left[\int d^D q^\mu_{n-1}\, \sqrt{g(q_{n-1})}\right]. \tag{10.124}$$

This expression is not directly applicable. When trying to do the $d^D q^\mu_{n-1}$-integrations successively, starting from the final integration over dq^μ_N, the integration variable q_{n-1} appears for each n in the argument of det $\left[e^i_\mu(q_{n-1})\right]$ or $g_{\mu\nu}(q_{n-1})$. To make this q_{n-1}-dependence explicit, we expand in the measure (10.122) $e^i_\mu(q_{n-1}) = e^i{}_\mu(q_n - \Delta q_n)$ around the postpoint q_n into powers of Δq_n. This gives

$$dx^i = e^i_\mu(q-\Delta q)dq^\mu = e^i_\mu dq^\mu - e^i{}_{\mu,\nu} dq^\mu \Delta q^\nu + \frac{1}{2} e^i{}_{\mu,\nu\lambda} dq^\mu \Delta q^\nu \Delta q^\lambda + \ldots \tag{10.125}$$

omitting, as before, the subscripts of q_n and Δq_n. Thus the Jacobian of the coordinate transformation from dx^i to dq^μ is

$$J_0 = \det\,(e^i{}_\kappa)\ \det\left[\delta^\kappa{}_\mu - e_i{}^\kappa e^i{}_{\mu,\nu}\Delta q^\nu + \frac{1}{2} e_i{}^\kappa e^i{}_{\mu,\nu\lambda}\Delta q^\nu \Delta q^\lambda\right], \tag{10.126}$$

giving the relation between the infinitesimal integration volumes $d^D x^i$ and $d^D q^\mu$:

$$\prod_{n=2}^{N+1} \int d^D x^i_{n-1} = \prod_{n=2}^{N+1} \left\{\int d^D q^\mu_{n-1}\, J_{0n}\right\}. \tag{10.127}$$

The well-known expansion formula

$$\det\,(1+B) = \exp \operatorname{tr} \log(1+B) = \exp \operatorname{tr}(B - B^2/2 + B^3/3 - \ldots) \tag{10.128}$$

allows us now to rewrite J_0 as

$$J_0 = \det\,(e^i{}_\kappa) \exp\left(\frac{i}{\hbar}\mathcal{A}^\epsilon_{J_0}\right), \tag{10.129}$$

with the determinant $\det\,(e^i_\mu) = \sqrt{g(q)}$ evaluated at the postpoint. This equation defines an effective action associated with the Jacobian, for which we obtain the expansion

$$\frac{i}{\hbar}\mathcal{A}^\epsilon_{J_0} = -e_i{}^\kappa e^i{}_{\kappa,\mu}\Delta q^\mu + \frac{1}{2}\left[e_i{}^\mu e^i_{\mu,\nu\lambda} - e_i{}^\mu e^i{}_{\kappa,\nu} e_j{}^\kappa e^j{}_{\mu,\lambda}\right]\Delta q^\nu \Delta q^\lambda + \ldots . \tag{10.130}$$

The expansion coefficients are expressed in terms of the affine connection (10.16) using the relations:

$$e_{i\nu,\mu} e^i{}_{\kappa,\lambda} = e_i{}^\sigma e^i{}_{\nu,\mu} e_{j\sigma} e^j{}_{\kappa,\lambda} = \Gamma_{\mu\nu}{}^\sigma \Gamma_{\lambda\kappa\sigma} \tag{10.131}$$

$$\begin{aligned} e_{i\mu} e^i{}_{\nu,\lambda\kappa} &= g_{\mu\tau}[\partial_\kappa(e_i{}^\tau e^i{}_{\nu,\lambda}) - e^{i\sigma} e^i{}_{\nu,\lambda} e^j{}_\sigma e^{j\tau}{}_{,\kappa}] \\ &= g_{\mu\tau}(\partial_\kappa \Gamma_{\lambda\nu}{}^\tau + \Gamma_{\lambda\nu}{}^\sigma \Gamma_{\kappa\sigma}{}^\tau). \end{aligned} \tag{10.132}$$

The Jacobian action becomes therefore:

$$\frac{i}{\hbar}\mathcal{A}^{\epsilon}_{J_0} \;=\; -\Gamma_{\mu\nu}{}^{\nu}\Delta q^{\mu}+\frac{1}{2}\partial_{\mu}\Gamma_{\nu\kappa}{}^{\kappa}\Delta q^{\nu}\Delta q^{\mu}+\dots\;. \tag{10.133}$$

The same result would, incidentally, be obtained by writing the Jacobian in accordance with (10.124) as

$$J_0=\sqrt{g(q-\Delta q)}, \tag{10.134}$$

which leads to the alternative formula for the Jacobian action

$$\exp\left(\frac{i}{\hbar}\mathcal{A}^{\epsilon}_{J_0}\right)=\frac{\sqrt{g(q-\Delta q)}}{\sqrt{g(q)}}. \tag{10.135}$$

An expansion in powers of Δq gives

$$\exp\left(\frac{i}{\hbar}\mathcal{A}^{\epsilon}_{\bar{J}_0}\right)=1-\frac{1}{\sqrt{g(q)}}\sqrt{g(q)}_{,\mu}\Delta q^{\mu}+\frac{1}{2\sqrt{g(q)}}\sqrt{g(q)}_{,\mu\nu}\Delta q^{\mu}\Delta q^{\nu}+\dots\;. \tag{10.136}$$

Using the formula

$$\frac{1}{\sqrt{g}}\partial_{\mu}\sqrt{g}=\frac{1}{2}g^{\sigma\tau}\partial_{\mu}g_{\sigma\tau}=\bar{\Gamma}_{\mu\nu}{}^{\nu}, \tag{10.137}$$

this becomes

$$\exp\left(\frac{i}{\hbar}\mathcal{A}^{\epsilon}_{\bar{J}_0}\right)=1-\bar{\Gamma}_{\mu\nu}{}^{\nu}\Delta q^{\mu}+\frac{1}{2}(\partial_{\mu}\bar{\Gamma}_{\nu\lambda}{}^{\lambda}+\bar{\Gamma}_{\mu\sigma}{}^{\sigma}\bar{\Gamma}_{\nu\lambda}{}^{\lambda})\Delta q^{\mu}\Delta q^{\nu}+\dots, \tag{10.138}$$

so that

$$\frac{i}{\hbar}\mathcal{A}^{\epsilon}_{\bar{J}_0}=-\bar{\Gamma}_{\mu\nu}{}^{\nu}\Delta q^{\mu}+\frac{1}{2}\partial_{\mu}\bar{\Gamma}_{\nu\lambda}{}^{\lambda}\Delta q^{\mu}\Delta q^{\nu}+\dots\;. \tag{10.139}$$

In a space without torsion where $\bar{\Gamma}^{\lambda}_{\mu\nu}\equiv\Gamma_{\mu\nu}{}^{\lambda}$, the Jacobian actions (10.133) and (10.139) are trivially equal to each other. But the equality holds also in the presence of torsion. Indeed, when inserting the decomposition (10.27), $\Gamma_{\mu\nu}{}^{\lambda}=\bar{\Gamma}_{\mu\nu}{}^{\lambda}+K_{\mu\nu}{}^{\lambda}$, into (10.133), the contortion tensor drops out since it is antisymmetric in the last two indices and these are contracted in both expressions.

In terms of $\mathcal{A}^{\epsilon}_{J_{0n}}$, we can rewrite the transformed measure (10.122) in the more useful form

$$\prod_{n=2}^{N+1}\int d^D x^i_{n-1}=\prod_{n=2}^{N+1}\left\{\int d^D q^{\mu}_{n-1}\,\det\left[e^i_{\mu}(q_n)\right]\exp\left(\frac{i}{\hbar}\mathcal{A}^{\epsilon}_{J_{0n}}\right)\right\}. \tag{10.140}$$

In a flat space parametrized in terms of curvilinear coordinates, the right-hand sides of (10.122) and (10.140) are related by an ordinary coordinate transformation, and both give the correct measure for a time-sliced path integral. In a general metric-affine space, however, this is no longer true. Since the mapping $dx^i \to dq^\mu$ is nonholonomic, there are in principle infinitely many ways of transforming the path integral measure from Cartesian coordinates to a non-Euclidean space. Among these, there exists a preferred mapping which leads to the correct quantum-mechanical amplitude in all known physical systems. This will serve to solve the path integral of the Coulomb system in Chapter 13.

The clue for finding the correct mapping is offered by an unaesthetic feature of Eq. (10.125): The expansion contains both differentials dq^μ and differences Δq^μ. This is somehow inconsistent. When time-slicing the path integral, the differentials dq^μ in the action are increased to finite differences Δq^μ. Consequently, the differentials in the measure should also become differences. A relation such as (10.125) containing simultaneously differences and differentials should not occur.

It is easy to achieve this goal by changing the starting point of the nonholonomic mapping and rewriting the initial flat space path integral (10.89) as

$$(\mathbf{x}\,t|\mathbf{x}'t') = \frac{1}{\sqrt{2\pi i\epsilon\hbar/M}^D} \prod_{n=1}^{N} \left[\int_{-\infty}^{\infty} d\Delta x_n\right] \prod_{n=1}^{N+1} K_0^\epsilon(\Delta \mathbf{x}_n). \tag{10.141}$$

Since x_n are Cartesian coordinates, the measures of integration in the time-sliced expressions (10.89) and (10.141) are certainly identical:

$$\prod_{n=1}^{N} \int d^D x_n \equiv \prod_{n=2}^{N+1} \int d^D \Delta x_n. \tag{10.142}$$

Their images under a nonholonomic mapping, however, are different so that the initial form of the time-sliced path integral is a matter of choice. The initial form (10.141) has the obvious advantage that the integration variables are precisely the quantities Δx_n^i which occur in the short-time amplitude $K_0^\epsilon(\Delta x_n)$.

Under a nonholonomic transformation, the right-hand side of Eq. (10.142) leads to the integral measure in a general metric-affine space

$$\prod_{n=2}^{N+1} \int d^D \Delta x_n \to \prod_{n=2}^{N+1} \left[\int d^D \Delta q_n\, J_n\right], \tag{10.143}$$

with the Jacobian following from (10.96) (omitting n)

$$J = \frac{\partial(\Delta x)}{\partial(\Delta q)} = \det\left(e^i{}_\kappa\right) \det\left[\delta_\mu{}^\lambda - \Gamma_{\{\mu\nu\}}{}^\lambda \Delta q^\nu + \frac{1}{2}\left(\partial_{\{\sigma}\Gamma_{\mu\nu\}}{}^\lambda + \Gamma_{\{\mu\nu}{}^\tau \Gamma_{\{\tau|\sigma\}\}}{}^\lambda\right)\Delta q^\nu \Delta q^\sigma + \ldots\right]. \tag{10.144}$$

In a space with curvature and torsion, the measure on the right-hand side of (10.143) replaces the flat-space measure on the right-hand side of (10.124). The curly double brackets around the indices ν, κ, σ, μ indicate a symmetrization in τ and σ followed by a symmetrization in μ, ν, and σ. With the help of formula (10.128) we now calculate the Jacobian action

$$\frac{i}{\hbar}\mathcal{A}_J^\epsilon = -\Gamma_{\{\mu\nu\}}{}^\mu \Delta q^\nu + \frac{1}{2}\left[\partial_{\{\mu}\Gamma_{\nu\kappa\}}{}^\kappa + \Gamma_{\{\nu\kappa}{}^\sigma\Gamma_{\{\sigma|\mu\}\}}{}^\kappa - \Gamma_{\{\nu\kappa\}}{}^\sigma\Gamma_{\{\sigma\mu\}}{}^\kappa\right]\Delta q^\nu \Delta q^\mu + \dots . \tag{10.145}$$

This expression differs from the earlier Jacobian action (10.133) by the symmetrization symbols. Dropping them, the two expressions coincide. This is allowed if q^μ are curvilinear coordinates in a flat space. Since then the transformation functions $x^i(q)$ and their first derivatives $\partial_\mu x^i(q)$ are integrable and possess commuting derivatives, the two Jacobian actions (10.133) and (10.145) are identical.

There is a further good reason for choosing (10.142) as a starting point for the nonholonomic transformation of the measure. According to Huygens' principle of wave optics, each point of a wave front is a center of a new spherical wave propagating from that point. Therefore, in a time-sliced path integral, the differences Δx_n^i play a more fundamental role than the coordinates themselves. Intimately related to this is the observation that in the canonical form, a short-time piece of the action reads

$$\int \frac{dp_n}{2\pi\hbar} \exp\left[\frac{i}{\hbar}p_n(x_n - x_{n-1}) - \frac{ip_n^2}{2M\hbar}t\right].$$

Each momentum is associated with a coordinate difference $\Delta x_n \equiv x_n - x_{n-1}$. Thus, we should expect the spatial integrations conjugate to p_n to run over the coordinate differences $\Delta x_n = x_n - x_{n-1}$ rather than the coordinates x_n themselves, which makes the important difference in the subsequent nonholonomic coordinate transformation.

We are thus led to postulate the following time-sliced path integral in q-space:

$$\langle q|e^{-i(t-t')\hat{H}/\hbar}|q'\rangle = \frac{1}{\sqrt{2\pi i\hbar\epsilon/M}^D}\prod_{n=2}^{N+1}\left[\int d^D\Delta q_n \frac{\sqrt{g(q_n)}}{\sqrt{2\pi i\epsilon\hbar/M}^D}\right] e^{i\sum_{n=1}^{N+1}(\mathcal{A}^\epsilon + \mathcal{A}_J^\epsilon)/\hbar}, \tag{10.146}$$

where the integrals over Δq_n may be performed successively from $n = N$ down to $n = 1$.

Let us emphasize that this expression has not been *derived* from the flat space path integral. It is the result of a specific new *quantum equivalence principle* which rules how a flat space path integral behaves under nonholonomic coordinate transformations.

It is useful to re-express our result in a different form which clarifies best the relation with the naively expected measure of path integration (10.124), the product of integrals

$$\prod_{n=1}^{N}\int d^D x_n = \prod_{n=1}^{N}\left[\int d^D q_n \sqrt{g(q_n)}\right]. \tag{10.147}$$

The measure in (10.146) can be expressed in terms of (10.147) as

$$\prod_{n=2}^{N+1}\left[\int d^D\Delta q_n\sqrt{g(q_n)}\right]=\prod_{n=1}^{N}\left[\int d^D q_n\,\sqrt{g(q_n)}e^{-i\mathcal{A}^\epsilon_{J_0}/\hbar}\right]. \tag{10.148}$$

The corresponding expression for the entire time-sliced path integral (10.146) in the metric-affine space reads

$$\langle q|e^{-i(t-t')\hat{H}/\hbar}|q'\rangle=\frac{1}{\sqrt{2\pi i\hbar\epsilon/M}^D}\prod_{n=1}^{N}\left[\int d^Dq_n\frac{\sqrt{g(q_n)}}{\sqrt{2\pi i\hbar\epsilon/M}^D}\right]e^{i\sum_{n=1}^{N+1}(\mathcal{A}^\epsilon+\Delta\mathcal{A}^\epsilon_J)/\hbar}, \tag{10.149}$$

where $\Delta\mathcal{A}^\epsilon_J$ is the difference between the correct and the wrong Jacobian actions in Eqs. (10.133) and (10.145):

$$\Delta\mathcal{A}^\epsilon_J\equiv\mathcal{A}^\epsilon_J-\mathcal{A}^\epsilon_{J_0}. \tag{10.150}$$

In the absence of torsion where $\Gamma_{\{\mu\nu\}}{}^\lambda=\bar{\Gamma}_{\mu\nu}{}^\lambda$, this simplifies to

$$\frac{i}{\hbar}\Delta\mathcal{A}^\epsilon_J=\frac{1}{6}\bar{R}_{\mu\nu}\Delta q^\mu\Delta q^\nu, \tag{10.151}$$

where $\bar{R}_{\mu\nu}$ is the Ricci tensor associated with the Riemann curvature tensor, i.e., the contraction (10.41) of the Riemann curvature tensor associated with the Christoffel symbol $\bar{\Gamma}_{\mu\nu}{}^\lambda$.

Being quadratic in Δq, the effect of the additional action can easily be evaluated perturbatively using the methods explained in Chapter 8, according to which $\Delta q^\mu\Delta q^\nu$ may be replaced by its lowest order expectation

$$\langle\Delta q^\mu\Delta q^\nu\rangle_0=i\epsilon\hbar g^{\mu\nu}(q)/M.$$

Then $\Delta\mathcal{A}^\epsilon_J$ yields the additional effective potential

$$V_{\rm eff}(q)=-\frac{\hbar^2}{6M}\bar{R}(q), \tag{10.152}$$

where $\bar{R}$ is the Riemann curvature scalar. By including this potential in the action, the path integral in a curved space can be written down in the naive form (10.147) as follows:

$$\langle q|e^{-i(t-t')\hat{H}/\hbar}|q'\rangle=\frac{1}{\sqrt{2\pi i\hbar\epsilon/M}^D}\prod_{n=1}^{N}\left[\int d^Dq_n\frac{\sqrt{g(q_n)}}{\sqrt{2\pi i\epsilon\hbar/M}^D}e^{i\hbar R(q_n)/6M}\right]e^{i\sum_{n=1}^{N+1}\mathcal{A}^\epsilon[q_n]/\hbar}. \tag{10.153}$$

This time-sliced expression will from now on be the definition of a path integral in curved space written in the continuum notation as

$$\langle q|e^{-i(t-t')\hat{H}/\hbar}|q'\rangle \;=\; \int\mathcal{D}^Dq\sqrt{g(q)}\,e^{i\int_{t_a}^{t_b}dt\,\mathcal{A}[q]/\hbar}. \tag{10.154}$$

The integrals over q_n in (10.153) are conveniently performed successively downwards over $\Delta q_{n+1} = q_{n+1} - q_n$ at fixed q_{n+1}. The weights $\sqrt{g(q_n)} = \sqrt{g(q_{n+1} - \Delta q_{n+1})}$ require a postpoint expansion leading to the naive Jacobian J_0 of (10.126) and the Jacobian action $\mathcal{A}^\epsilon_{J_0}$ of Eq. (10.133).

It is important to observe that the above time-sliced definition is automatically invariant under coordinate transformations. This is an immediate consequence of the definition via the nonholonomic mapping from a flat-space path integral.

It goes without saying that the path integral (10.153) also has a phase space version. It is obtained by omitting all $(M/2\epsilon)(\Delta q_n)^2$ terms in the short-time actions $\mathcal{A}^\epsilon$ and extending the multiple integral by the product of momentum integrals

$$\prod_{n=1}^{N+1}\left[\frac{dp_n}{2\pi\hbar\sqrt{g(q_n)}}\right] e^{(i/\hbar)\sum_{n=1}^{N+1}\left[p_{n\mu}\Delta q^\mu - \epsilon\frac{1}{2M}g^{\mu\nu}(q_n)p_{n\mu}p_{n\nu}\right]}. \tag{10.155}$$

When using this expression, all problems which were encountered in the literature with canonical transformations of path integrals disappear.

An important property of the definition of the path integral in spaces with curvature and torsion as a nonholonomic image of a Euclidean path integral is that this image is automatically invariant under ordinary holonomic coordinate transformations.

10.4 Completing the Solution of Path Integral on Surface of Sphere in D Dimensions

The measure of path integration in Eq. (10.146) allows us to finally complete the calculation, initiated in Sections 8.7–8.9, of the path integrals of a point particle on the surface of a sphere on group spaces in any number of dimensions. Indeed, using the result (10.152) we are now able to solve the problems discussed in Section 8.7 in conjunction with the energy formula (8.225). Thus we are finally in a position to find the correct energies and amplitudes of these systems.

A sphere of radius r embedded in D dimensions has an intrinsic dimension $D' \equiv D-1$ and a curvature scalar

$$\bar{R} = \frac{(D'-1)D'}{r^2}. \tag{10.156}$$

This is most easily derived as follows. Consider a line element in D dimensions

$$(d\mathbf{x})^2 = (dx^1)^2 + (dx^2)^2 + \ldots + (dx^D)^2 \tag{10.157}$$

and restrict the motion to a spherical surface

$$(x^1)^2 + (x^2)^2 + \ldots + (x^D)^2 = r^2, \tag{10.158}$$

by eliminating x^D. This brings (10.157) to the form

$$(d\mathbf{x})^2 = (dx^1)^2 + (dx^2)^2 + \ldots + (dx^{D'})^2 + \frac{(x^1dx^1 + dx^2 + \ldots + x^{D'}dx^{D'})^2}{r^2 - r'^2}, \tag{10.159}$$

where $r'^2 \equiv (x^1)^2 + (x^2)^2 + \ldots + (x^{D'})^2$. The metric on the D'-dimensional surface is therefore

$$g_{\mu\nu}(x) = \delta_{\mu\nu} + \frac{x^\mu x^\nu}{r^2 - r'^2}. \tag{10.160}$$

Since $\bar{R}$ will be constant on the spherical surface, we may evaluate it for small x^μ ($\mu = 1, \ldots, D'$) where $g_{\mu\nu}(x) \approx \delta_{\mu\nu} + x^\mu x^\nu / r^2$ and the Christoffel symbols (10.7) are $\Gamma_{\mu\nu}{}^\lambda \approx \Gamma_{\mu\nu\lambda} \approx \delta_{\mu\nu} x_\lambda / r^2$. Inserting this into (10.35) we obtain the curvature tensor for small x^μ:

$$\bar{R}_{\mu\nu\lambda\kappa} \approx \frac{1}{r^2} \left(\delta_{\mu\kappa}\delta_{\nu\lambda} - \delta_{\mu\lambda}\delta_{\nu\kappa} \right). \tag{10.161}$$

This can be extended covariantly to the full surface of the sphere by replacing $\delta_{\mu\lambda}$ by the metric $g_{\mu\lambda}(x)$:

$$\bar{R}_{\mu\nu\lambda\kappa}(x) = \frac{1}{r^2} \left[g_{\mu\kappa}(x) g_{\nu\lambda}(x) - g_{\mu\lambda}(x) g_{\nu\kappa}(x) \right], \tag{10.162}$$

so that Ricci tensor is [recall (10.41)]

$$\bar{R}_{\nu\kappa}(x) = \bar{R}_{\mu\nu\kappa}{}^\mu(x) = \frac{D'-1}{r^2} g_{\nu\kappa}(x). \tag{10.163}$$

Contracting this with $g^{\nu\kappa}$ [recall (10.42)] yields indeed the curvature scalar (10.156).

The effective potential (10.152) is therefore

$$V_{\text{eff}} = -\frac{\hbar^2}{6Mr^2}(D-2)(D-1). \tag{10.164}$$

It supplies precisely the missing energy which changes the energy (8.225) *near* the sphere, corrected by the expectation of the quartic term ϑ_n^4 in the action, to the proper value

$$E_l = \frac{\hbar^2}{2Mr^2} l(l+D-2). \tag{10.165}$$

Astonishingly, this elementary result of Schrödinger quantum mechanics was found only a decade ago by path integration [5]. Other time-slicing procedures yield extra terms proportional to the scalar curvature $\bar{R}$, which are absent in our theory. Here the scalar curvature is a trivial constant, so it would remain undetectable in atomic experiments which measure only energy differences. The same result will be derived in general in Eqs. (11.25) and in Section 11.3.

An important property of this spectrum is that the ground state energy vanished for all dimensions D. This property would not have been found in the naive measure of path integration on the right-hand side of Eq. (10.147) which is used in most works on this subject. The correction term (10.151) coming from the nonholonomic mapping of the flat-space measure is essential for the correct result.

More evidence for the correctness of the measure in (10.146) will be supplied in Chapter 13 where we solve the path integrals of the most important atomic system, the hydrogen atom.

We remark that for $t \to t'$, the amplitude (10.153) shows the states $|q\rangle$ to obey the covariant orthonormality relation

$$\langle q|q'\rangle = \sqrt{g(q)}^{-1} \delta^{(D)}(q-q'). \tag{10.166}$$

The completeness relation reads

$$\int d^D q \sqrt{g(q)} |q\rangle\langle q| = 1. \tag{10.167}$$

10.5 External Potentials and Vector Potentials

An important generalization of the above path integral formulas (10.146), (10.149), (10.153) of a point particle in a space with curvature and torsion includes the presence of an external potential and a vector potential. These allow us to describe, for instance, a particle in external electric and magnetic fields. The classical action is then

$$\mathcal{A}_{\rm em} = \int_{t_a}^{t_b} dt \left[\frac{e}{c} A_\mu(q(t))\dot{q}^\mu - V(q(t)) \right]. \tag{10.168}$$

To find the time-sliced action we proceed as follows. First we set up the correct time-sliced expression in Euclidean space and Cartesian coordinates. For a single slice it reads, in the postpoint form,

$$\mathcal{A}^\epsilon = \frac{M}{2\epsilon}(\Delta x^i)^2 + \frac{e}{c} A_i(\mathbf{x})\Delta x^i - \frac{e}{2c} A_{i,j}(\mathbf{x})\Delta x^i \Delta x^j - \epsilon V(\mathbf{x}) + \dots\ . \tag{10.169}$$

As usual, we have neglected terms which do not contribute in the continuum limit. The derivation of this time-sliced expression proceeds by calculating, as in (10.116), the action

$$\mathcal{A}^\epsilon = \int_{t-\epsilon}^{t} dt L(t) \tag{10.170}$$

along the classical trajectory in Euclidean space, where

$$L(t) = \frac{M}{2}\dot{\mathbf{x}}^2(t) + \frac{e}{c}\mathbf{A}(\mathbf{x}(t))\dot{\mathbf{x}}(t) - V(\mathbf{x}(t)) \tag{10.171}$$

is the classical Lagrangian. In contrast to (10.116), however, the Lagrangian has now a nonzero time derivative (omitting the time arguments):

$$\frac{d}{dt}L = M\dot{\mathbf{x}}\ddot{\mathbf{x}} + \frac{e}{c}\mathbf{A}(\mathbf{x})\ddot{\mathbf{x}} + \frac{e}{c}A_{i,j}(\mathbf{x})\dot{x}^i\dot{x}^j - V_i(\mathbf{x})\dot{x}^i. \tag{10.172}$$

For this reason we cannot simply write down an expression such as (10.117) but we have to expand the Lagrangian around the postpoint leading to the series

$$\mathcal{A}^\epsilon = \int_{t-\epsilon}^{t} dt L(t) = \epsilon L(t) - \frac{1}{2}\epsilon^2 \frac{d}{dt} L(t) + \dots \,. \tag{10.173}$$

The evaluation makes use of the equation of motion

$$M\ddot{x}^i = -\frac{e}{c}(A_{i,j}(\mathbf{x}) - A_{j,i}(\mathbf{x}))\dot{x}^j - V_i(\mathbf{x}), \tag{10.174}$$

from which we derive the analog of Eq. (10.96): First we have the postpoint expansion

$$\begin{aligned} \Delta x^i &= -\epsilon\dot{x}^i + \frac{1}{2}\epsilon^2\ddot{x}^i + \dots \\ &= -\epsilon\dot{x}^i - \frac{e}{2Mc}\epsilon^2\left[(A_{i,j} - A_{j,i})\dot{x}^j + V_i(\mathbf{x})\right] + \dots \,. \end{aligned} \tag{10.175}$$

Inverting this gives

$$\dot{x}^i = -\frac{\Delta x^i}{\epsilon} - \frac{e}{2Mc}(A_{i,j} - A_{j,i})\Delta x^j + \dots \,. \tag{10.176}$$

When inserted into (10.173), this yields indeed the time-sliced short-time action (10.169).

The quadratic term $\Delta x^i \Delta x^j$ in the action (10.169) can be replaced by the perturbative expectation value

$$\Delta x^i \Delta x^j \to \langle \Delta x^i \Delta x^j \rangle = \delta_{ij}\, i\frac{\hbar\epsilon}{M}, \tag{10.177}$$

so that $\mathcal{A}^\epsilon$ becomes

$$\mathcal{A}^\epsilon = \frac{M}{2\epsilon}(\Delta x^i)^2 + \frac{e}{c}A_i(\mathbf{x})\Delta x^i - i\epsilon\frac{\hbar e}{2Mc}A_{i,i}(\mathbf{x}) - \epsilon V(\mathbf{x}) + \dots \,. \tag{10.178}$$

Incidentally, the action (10.169) could also have been written as

$$\mathcal{A}^\epsilon = \frac{M}{2\epsilon}(\Delta x^i)^2 + \frac{e}{c}A_i(\bar{\mathbf{x}})\Delta x^i - \epsilon V(\mathbf{x}) + \dots \,, \tag{10.179}$$

where $\bar{\mathbf{x}}$ is the midpoint value of the slice coordinates

$$\bar{\mathbf{x}} = \mathbf{x} - \frac{1}{2}\Delta\mathbf{x}, \tag{10.180}$$

i.e., more explicitly,

$$\bar{\mathbf{x}}(t_n) \equiv \frac{1}{2}[\mathbf{x}(t_n) + \mathbf{x}(t_{n-1})]. \tag{10.181}$$

Thus, with an external vector potential in Cartesian coordinates, a midpoint "prescription" for $\mathcal{A}^\epsilon$ happens to yield the correct expression (10.179).

Having found the time-sliced action in Cartesian coordinates, it is easy to go over to spaces with curvature and torsion. We simply insert the nonholonomic transformation (10.96) for the differentials Δx^i. This gives again the short-time action (10.107), extended by the interaction due to the potentials

$$\mathcal{A}^\epsilon_{\rm em} = \frac{e}{c} A_\mu \Delta q^\mu - \frac{e}{2c} \partial_\nu A_\mu \Delta q^\mu \Delta q^\nu - \epsilon V(q) + \dots \ . \tag{10.182}$$

The second term can be evaluated perturbatively leading to

$$\mathcal{A}^\epsilon_{\rm em} = \frac{e}{c} A_\mu \Delta q^\mu - i\epsilon \frac{\hbar e}{2Mc} \partial_\mu A^\mu - \epsilon V(q) + \dots \ . \tag{10.183}$$

The sum over all slices,

$$\mathcal{A}^N_{\rm em} = \sum_{n=1}^{N+1} \mathcal{A}^\epsilon_{\rm em} \, , \tag{10.184}$$

has to be added to the action in each time-sliced expression (10.146), (10.149), and (10.153).

10.6 Perturbative Calculation of Path Integrals in Curved Space

In Sections 2.15 and 3.21 we have given a perturbative definition of path integrals which does not require the rather cumbersome time slicing but deals directly with a continuous time. We shall now extend this definition to curved space in such a way that it leads to the same result as the time-sliced definition given in Section 10.3. In particular, we want to ensure that this definition preserves the fundamental property of coordinate independence achieved in the time-sliced definition via the nonholonomic mapping principle, as observed at the end of Subsection 10.3.2. In a perturbative calculation, this property will turn out to be highly nontrivial. In addition, we want to be sure that the ground state energy of a particle on a sphere is zero in any number of dimensions, just as in the time-sliced calculation leading to Eq. (10.165). This implies that also in the perturbative definition of path integral, the operator-ordering problem will be completely solved.

10.6.1 Free and Interacting Parts of Action

The partition function of a point particle in a curved space with an intrinsic dimension D is given by the path integral over all periodic paths on the imaginary-time axis τ:

$$Z = \int \mathcal{D}^D q \sqrt{g} \, e^{-\mathcal{A}[q]}, \tag{10.185}$$

where $\mathcal{A}[q]$ is the Euclidean action

$$\mathcal{A}[q] = \int_0^\beta d\tau \left[\frac{1}{2}g_{\mu\nu}(q(\tau))\dot{q}^\mu(\tau)\dot{q}^\nu(\tau) + V(q(\tau))\right]. \tag{10.186}$$

We have set $\hbar$ and the particle mass M equal to unity. For a space with constant curvature, this is a generalization of the action for a particle on a sphere (8.147), also called a *nonlinear σ-model* (see p. 746). The perturbative definition of Sections 2.15 and 3.21 amounts to the following calculation rules. Expand the metric $g_{\mu\nu}(q)$ and the potential $V(q)$ around some point q_a^μ in powers of $\delta q^\mu \equiv q^\mu - q_a^\mu$. After this, separate the action $\mathcal{A}[q]$ into a harmonically fluctuating part

$$\mathcal{A}^{(0)}[q_a;\delta q] \equiv \frac{1}{2}\int_0^\beta d\tau\, g_{\mu\nu}(q_a)\left[\delta\dot{q}^\mu(\tau)\delta\dot{q}^\nu(\tau) + \omega^2\delta q^\mu(\tau)\delta q^\nu(\tau)\right], \tag{10.187}$$

and an interacting part

$$\mathcal{A}^{\rm int}[q_a;\delta q] \equiv \mathcal{A}[q] - \mathcal{A}^{(0)}[q_a;\delta q]. \tag{10.188}$$

The second term in (10.187) is called *frequency term* or *mass term*. It is not invariant under coordinate transformations. The implications of this will be seen later. When studying the partition function in the limit of large β, the frequency ω cannot be set equal to zero since this would lead to infinities in the perturbation expansion, as we shall see below.

A delicate problem is posed by the square root of the determinant of the metric in the functional integration measure in (10.185). In a purely formal continuous definition of the measure, we would write it as

$$\int \mathcal{D}^D q\sqrt{g} \equiv \prod_\tau \int d^D q(\tau)\sqrt{g(\tau)} = \left[\prod_\tau \int d^D q(\tau)\sqrt{g(q_a)}\right] \exp\left[\frac{1}{2}\sum_\tau \log\frac{g(q(\tau))}{g(q_a)}\right]. \tag{10.189}$$

The formal sum over all continuous times τ in the exponent corresponds to an integral $\int d\tau$ divided by the spacing of the points, which on a sliced time axis would be the slicing parameter ϵ. Here it is $d\tau$. The ratio $1/d\tau$ may formally be identified with $\delta(0)$, in accordance with the defining integral $\int d\tau\, \delta(\tau) = 1$. The infinity of $\delta(0)$ may be regularized in some way, for instance by a cutoff in the Fourier representation $\delta(0) \equiv \int d\omega/(2\pi)$ at large frequencies ω, a so-called *UV-cutoff*. Leaving the regularization unspecified, we rewrite the measure (10.189) formally as

$$\int \mathcal{D}^D q\sqrt{g} \equiv \left[\prod_\tau \int d^D q(\tau)\sqrt{g(q_a)}\right] \exp\left[\frac{1}{2}\delta(0)\int_0^\beta d\tau \log\frac{g(q(\tau))}{g(q_a)}\right], \tag{10.190}$$

and further as

$$\int \mathcal{D}^D q\,\sqrt{g(q_a)}\,e^{-\mathcal{A}^g[q]}, \tag{10.191}$$

where we have introduced an effective action associated with the measure:

$$\mathcal{A}_g[q] = -\frac{1}{2}\delta(0)\int_0^\beta d\tau\, \log\frac{g(q(\tau))}{g(q_a)}. \tag{10.192}$$

For a perturbative treatment, this action is expanded in powers of $\delta q(\tau)$ and is a functional of this variable:

$$\mathcal{A}_g[q_a, \delta q] = -\frac{1}{2}\delta(0)\int_0^\beta d\tau\, [\log g(q_a + \delta q(\tau)) - \log g(q_a)]. \tag{10.193}$$

This is added to (10.188) to yield the total interaction

$$\mathcal{A}^{\rm int}_{\rm tot}[q_a, \delta q] = \mathcal{A}^{\rm int}[q_a, \delta q] + \mathcal{A}_g[q_a, \delta q]. \tag{10.194}$$

The path integral for the partition function is now written as

$$Z \;=\; \int \mathcal{D}^D q\, \sqrt{g(q_a)}\, e^{-\mathcal{A}^{(0)}[q]}\, e^{-\mathcal{A}^{\rm int}_{\rm tot}[q]}. \tag{10.195}$$

According to the rules of perturbation theory, we expand the factor $e^{-\mathcal{A}^{\rm int}_{\rm tot}}$ in powers of the total interaction, and obtain the perturbation series

$$\begin{aligned} Z \;&=\; \int \mathcal{D}^D q\, \sqrt{g(q_a)} \left(1 - \mathcal{A}^{\rm int}_{\rm tot} + \frac{1}{2}\mathcal{A}^{\rm int\,2}_{\rm tot} - \ldots\right) e^{-\mathcal{A}^{(0)}[q]} \\ &=\; Z_\omega \left[1 - \left\langle \mathcal{A}^{\rm int}_{\rm tot}\right\rangle + \frac{1}{2!}\left\langle \mathcal{A}^{\rm int\,2}_{\rm tot}\right\rangle - \ldots\right], \end{aligned} \tag{10.196}$$

where

$$Z_\omega \equiv e^{-\beta F_\omega} = \int \mathcal{D}^D q \sqrt{g(q_a)}\, e^{-\mathcal{A}^{(0)}[q]} \tag{10.197}$$

is the path integral over the free part, and the symbol $\langle \ldots \rangle$ denotes the expectation values in this path integral

$$\langle \ldots \rangle = Z_\omega^{-1} \int \mathcal{D}q\, \sqrt{g(q_a)}\, (\ldots)\, e^{-\mathcal{A}^{(0)}[q]}. \tag{10.198}$$

With the usual definition of the cumulants $\left\langle \mathcal{A}^{\rm int}_{\rm tot}\right\rangle_c = \left\langle \mathcal{A}^{\rm int}_{\rm tot}\right\rangle$, $\left\langle \mathcal{A}^{\rm int\,2}_{\rm tot}\right\rangle_c = \left\langle \mathcal{A}^{\rm int\,2}_{\rm tot}\right\rangle - \left\langle \mathcal{A}^{\rm int}_{\rm tot}\right\rangle^2, \ldots$ [recall (3.485), (3.486)], this can be written as

$$Z \equiv e^{-\beta F} = \exp\left[-\beta F_\omega - \left\langle \mathcal{A}^{\rm int}_{\rm tot}\right\rangle_c + \frac{1}{2!}\left\langle \mathcal{A}^{\rm int\,2}_{\rm tot}\right\rangle_c - \ldots\right], \tag{10.199}$$

where $F_\omega \equiv -\beta^{-1}\log Z$ the free energy associated with Z_ω.

The cumulants are now calculated according to Wick's rule order by order in $\hbar$, treating the δ-function at the origin $\delta(0)$ as if it were finite. The perturbation series will contain factors of $\delta(0)$ and its higher powers. Fortunately, these unpleasant terms will turn out to cancel each other at each order in a suitably defined expansion parameter. On account of these cancellations, we may ultimately discard all terms containing $\delta(0)$, or set $\delta(0)$ equal to zero, in accordance with Veltman's rule (2.508).

The harmonic path integral (10.197) is performed using formulas (2.489) and (2.505). Assuming for a moment what we shall prove below that we may choose coordinates in which $g_{\mu\nu}(q_a) = \delta_{\mu\nu}$, we obtain directly in D dimensions

$$Z_\omega = \int \mathcal{D}q\, e^{-\mathcal{A}_\omega[q]} = \exp\left[-\frac{D}{2}\mathrm{Tr}\log(-\partial^2 + \omega^2)\right] \equiv e^{-\beta F_\omega}. \tag{10.200}$$

The expression in brackets specifies the free energy F_ω of the harmonic oscillator at the inverse temperature β.

10.6.2 Zero Temperature

For simplicity, we fist consider the limit of zero temperature or $\beta \to \infty$. Then F_ω becomes equal to the sum of D ground state energies $\omega/2$ of the oscillator, one for each dimension:

$$F_\omega = \frac{1}{\beta}\frac{D}{2}\mathrm{Tr}\log(-\partial^2 + \omega^2) \underset{\beta\to\infty}{\to} \frac{D}{2}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\log(k^2 + \omega^2) = \frac{D}{2}\omega. \tag{10.201}$$

The Wick contractions in the cumulants $\left\langle \mathcal{A}_{\mathrm{tot}}^{\mathrm{int}\,2} \right\rangle_c$ of the expansion (10.198) contain only connected diagrams. They contain temporal integrals which, after suitable partial integrations, become products of the following basic correlation functions

$$G^{(2)}_{\mu\nu}(\tau,\tau') \equiv \langle q_\mu(\tau) q_\nu(\tau')\rangle = \ , \tag{10.202}$$
$$\partial_\tau G^{(2)}_{\mu\nu}(\tau,\tau') \equiv \langle \dot q_\mu(\tau) q_\nu(\tau')\rangle = \ , \tag{10.203}$$
$$\partial_{\tau'} G^{(2)}_{\mu\nu}(\tau,\tau') \equiv \langle q_\mu(\tau) \dot q_\nu(\tau')\rangle = \ , \tag{10.204}$$
$$\partial_\tau\partial_{\tau'} G^{(2)}_{\mu\nu}(\tau,\tau') \equiv \langle \dot q_\mu(\tau) \dot q_\nu(\tau')\rangle = \ . \tag{10.205}$$

The right-hand sides define line symbols to be used for drawing Feynman diagrams for the interaction terms.

Under the assumption $g_{\mu\nu}(q_a) = \delta_{\mu\nu}$, the correlation function $G^{(2)}_{\mu\nu}(\tau,\tau')$ factorizes as

$$G^{(2)}_{\mu\nu}(\tau,\tau') = \delta_{\mu\nu}\,\Delta(\tau - \tau'), \tag{10.206}$$

with $\Delta(\tau - \tau')$ abbreviating the correlation the zero-temperature Green function $G^{\mathrm{p}}_{\omega^2,\mathrm{e}}(\tau)$ of Eq. (3.249) (remember the present units with $M = \hbar = 1$):

$$\Delta(\tau - \tau') = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{e^{ik(\tau-\tau')}}{k^2 + \omega^2} = \frac{1}{2\omega}e^{-\omega|\tau-\tau'|}. \tag{10.207}$$

As a consequence, the second correlation function (10.203) has a discontinuity

$$\partial_\tau G^{(2)}_{\mu\nu}(\tau,\tau') = \delta_{\mu\nu}\, i\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{e^{ik(\tau-\tau')}k}{k^2 + \omega^2} = \delta_{\mu\nu}\dot\Delta(\tau - \tau') \equiv -\frac{1}{2}\delta_{\mu\nu}\epsilon(\tau - \tau')e^{-\omega|\tau-\tau'|}, \tag{10.208}$$

where $\epsilon(\tau-\tau')$ is the distribution defined in Eq. (1.315) which has a jump at $\tau=\tau'$ from -1 to 1. It can be written as an integral over a δ-function:

$$\epsilon(\tau-\tau') \equiv -1+2\int_{-\infty}^{\tau} d\tau''\,\delta(\tau''-\tau'). \tag{10.209}$$

The third correlation function (10.204) is simply the negative of (10.203):

$$\partial_{\tau'} G^{(2)}_{\mu\nu}(\tau,\tau') = -\partial_\tau G^{(2)}_{\mu\nu}(\tau,\tau') = -\delta_{\mu\nu}\dot{\Delta}(\tau-\tau'). \tag{10.210}$$

At the point $\tau=\tau'$, the momentum integral (10.208) vanishes by antisymmetry:

$$\partial_\tau G^{(2)}_{\mu\nu}(\tau,\tau')|_{\tau=\tau'} = -\partial_{\tau'} G^{(2)}_{\mu\nu}(\tau,\tau')|_{\tau=\tau'} = \delta_{\mu\nu}\, i\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{k}{k^2+\omega^2} = \delta_{\mu\nu}\dot{\Delta}(0)=0. \tag{10.211}$$

The fourth correlation function (10.205) contains a δ-function:

$$\begin{aligned}\partial_\tau\partial_{\tau'} G^{(2)}_{\mu\nu}(\tau,\tau') &= -\partial_\tau^2 G^{(2)}_{\mu\nu}(\tau,\tau') = \delta_{\mu\nu}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{e^{ik(\tau-\tau')}k^2}{k^2+\omega^2} = -\delta_{\mu\nu}\ddot{\Delta}(\tau-\tau') \qquad (10.212)\\ &= \delta_{\mu\nu}\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ik(\tau-\tau')}\left(1-\frac{\omega^2}{k^2+\omega^2}\right) = \delta_{\mu\nu}\delta(\tau-\tau') - \omega^2 G^{(2)}_{\mu\nu}(\tau,\tau').\end{aligned}$$

The Green functions for $\mu=\nu$ are plotted in Fig. 10.4.

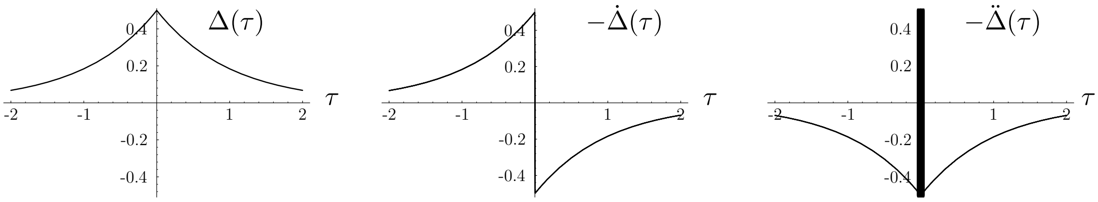


Figure 10.4 Green functions for perturbation expansions in curvilinear coordinates in natural units with $\omega=1$. The third contains a δ-function at the origin.

The last equation is actually the defining equation for the Green function, which is always the solution of the inhomogeneous equation of motion associated with the harmonic action (10.187), which under the assumption $g_{\mu\nu}(q_a)=\delta_{\mu\nu}$ reads for each component:

$$-\ddot{q}(\tau)+\omega^2 q(\tau) = \delta(\tau-\tau'). \tag{10.213}$$

The Green function $\Delta(\tau-\tau')$ solves this equation, satisfying

$$\ddot{\Delta}(\tau) = \omega^2\Delta(\tau)-\delta(\tau). \tag{10.214}$$

When trying to evaluate the different terms produced by the Wick contractions, we run into a serious problem. The different terms containing products of time derivatives of Green functions contain effectively products of δ-functions and

Heaviside functions. In the mathematical theory of distributions, such integrals are undefined. We shall offer two ways to resolve this problem. One is based on extending the integrals over the time axis to integrals over a d-dimensional *time space*, and continuing the results at the end back to $d = 1$. The extension makes the path integral a *functional integral* of the type used in quantum field theories. It will turn out that this procedure leads to well-defined finite results, also for the initially divergent terms coming from the effective action of the measure (10.193). In addition, and very importantly, it guarantees that the perturbatively defined path integral is invariant under coordinate transformations.

For the time-sliced definition in Section 10.3, coordinate independence was an automatic consequence of the nonholonomic mapping from a flat-space path integral. In the perturbative definition, the coordinate independence has been an outstanding problem for many years, and was only solved recently in Refs. [23]–[25].

In d-dimensional quantum field theory, path integrals between two and four spacetime dimensions have been defined by perturbation expansions for a long time. Initial difficulties in guaranteeing coordinate independence were solved by 't Hooft and Veltman [29] using dimensional regularization with minimal subtractions. For a detailed description of this method see the textbook [30]. Coordinate independence emerges after calculating all Feynman integrals in an arbitrary number of dimensions d, and continuing the results to the desired physical integer value. Infinities occuring in the limit are absorbed into parameters of the action. In contrast, and surprisingly, numerous attempts [31]–[36] to define the simpler *quantum-mechanical* path integrals in curved space by perturbation expansions encountered problems. Although all final results are finite and unique, the Feynman integrals in the expansions are highly singular and mathematically undefined. When evaluated in momentum space, they yield different results depending on the order of integration. Various definitions chosen by the earlier authors were not coordinate-independent, and this could only be cured by adding coordinate-dependent "correction terms" to the classical action — a highly unsatisfactory procedure violating the basic Feynman postulate that physical amplitudes should consist of a sum over all paths with phase factors $e^{i\mathcal{A}/\hbar}$ containing only the classical actions along the paths.

The calculations in d spacetime dimensions and the continuation to $d = 1$ will turn out to be somewhat tedious. We shall therefore find in Subsection 10.11.4 a method of doing the calculations directly for $d = 1$.

10.7 Model Study of Coordinate Invariance

Let us consider first a simple model which exhibits typical singular Feynman integrals encountered in curvilinear coordinates and see how these can be turned into a finite perturbation expansion which is invariant under coordinate transformations. For simplicity, we consider an ordinary harmonic oscillator in one dimension, with the action

$$\mathcal{A}_\omega = \frac{1}{2}\int_0^\beta d\tau\left[\dot{x}^2(\tau) + \omega^2 x^2(\tau)\right]. \tag{10.215}$$

The partition function of this system is exactly given by (10.201):

$$Z_\omega = \int \mathcal{D}x\, e^{-\mathcal{A}_\omega[x]} = \exp\left[-\frac{D}{2}\mathrm{Tr}\log(-\partial^2 + \omega^2)\right] \equiv e^{-\beta F_\omega}. \tag{10.216}$$

A nonlinear transformation of $x(\tau)$ to some other coordinate $q(\tau)$ turns (10.216) into a path integral of the type (10.185) which has a singular perturbation expansion. For simplicity we assume a specific simple coordinate transformation preserving the reflection symmetry $x \longleftrightarrow -x$ of the initial oscillator, whose power series expansion starts out like

$$x(\tau) = f_\eta(\eta q(\tau)) \equiv \frac{1}{\eta} f(\eta q(\tau)) = q - \frac{\eta}{3}q^3 + a\,\frac{\eta^2}{5}q^5 - \cdots\ , \tag{10.217}$$

where η is an expansion parameter which will play the role of a coupling constant counting the orders of the perturbation expansion. An extra parameter a is introduced for the sake of generality. We shall see that it does not influence the conclusions.

The transformation changes the partition function (10.216) into

$$Z = \int \mathcal{D}q(\tau)\, e^{-\mathcal{A}_J[q]} e^{-\mathcal{A}[q]}, \tag{10.218}$$

where is $\mathcal{A}[q]$ is the transformed action, whereas

$$\mathcal{A}_J[q] = -\delta(0)\int d\tau\ \log\ \frac{\partial f(q(\tau))}{\partial q(\tau)} \tag{10.219}$$

is an effective action coming from the Jacobian of the coordinate transformation

$$J = \prod_\tau \sqrt{\frac{\partial f(q(\tau))}{\partial q(\tau)}}. \tag{10.220}$$

The Jacobian plays the role of the square root of the determinant of the metric in (10.185), and $\mathcal{A}_J[q]$ corresponds to the effective action $\mathcal{A}_g[\delta q]$ in Eq. (10.193).

The transformed action is decomposed into a free part

$$\mathcal{A}_\omega[q] = \frac{1}{2}\int_0^\beta d\tau\, [\dot{q}^2(\tau) + \omega^2 q^2(\tau)], \tag{10.221}$$

and an interacting part corresponding to (10.188), which reads to second order in η:

$$\begin{aligned}\mathcal{A}^{\rm int}[q] = \int_0^\beta d\tau \Bigg\{ & -\eta\left[q^2(\tau)\dot{q}^2(\tau) + \frac{\omega^2}{3}q^4(\tau)\right] \\ & + \eta^2\left[\left(\frac{1}{2}+a\right)q^4(\tau)\dot{q}^2(\tau) + \omega^2\left(\frac{1}{18}+\frac{2a}{5}\right)q^6(\tau)\right]\Bigg\}. \end{aligned} \tag{10.222}$$

This is found from (10.188) by inserting the one-dimensional metric

$$g_{00}(q) = g(q) = [f'(\eta q)]^2 = 1 - 2\eta q^2 + (1+2a)\eta^2 q^4 + \dots\ . \tag{10.223}$$

To the same order in η, the Jacobian action (10.219) is

$$\mathcal{A}_J[q] = -\delta(0)\int_0^\beta d\tau\left[-\eta q^2(\tau)+\eta^2\left(a-\frac{1}{2}\right)q^4(\tau)\right], \tag{10.224}$$

and the perturbation expansion (10.199) is to be performed with the total interaction

$$\mathcal{A}^{\rm int}_{\rm tot}[q] = \mathcal{A}^{\rm int}[q] + \mathcal{A}_J[q]. \tag{10.225}$$

For $\eta = 0$, the transformed partition function (10.218) coincides trivially with (10.216). When expanding Z of Eq. (10.218) in powers of η, we obtain sums of Feynman diagrams contributing to each order η^n. This sum must vanish to ensure coordinate independence of the path integral. From the connected diagrams in the cumulants in (10.199) we obtain the free energy

$$\beta F = \beta F_\omega + \beta\sum_{n=1}\eta^n\,F_n = \beta F_\omega + \left\langle \mathcal{A}^{\rm int}_{\rm tot}\right\rangle_c - \frac{1}{2!}\left\langle \mathcal{A}^{\rm int\,2}_{\rm tot}\right\rangle_c + \dots\ . \tag{10.226}$$

The perturbative treatment is coordinate-independent if F does not depend on the parameters η and a of the coordinate transformation (10.217). Hence all expansion terms F_n must vanish. This will indeed happen, albeit in a quite nontrivial way.

10.7.1 Diagrammatic Expansion

The graphical expansion for the ground state energy will be carried here only up to three loops. At any order η^n, there exist different types of Feynman diagrams with $L = n+1, n$, and $n-1$ number of loops coming from the interaction terms (10.222) and (10.224), respectively. The diagrams are composed of the three types of lines in (10.202)–(10.205), and new interaction vertices for each power of η. The diagrams coming from the Jacobian action (10.224) are easily recognized by an accompanying power of $\delta(0)$.

First we calculate the contribution to the free energy of the first cumulant $\left\langle \mathcal{A}^{\rm int}_{\rm tot}\right\rangle_c$ in the expansion (10.226). The associated diagrams contain only lines whose end points have equal times. Such diagrams will be called *local*.

To lowest order in η, the cumulant contains the terms

$$\beta F_1 = \eta\int_0^\beta d\tau\left\langle -\left[q^2(\tau)\dot q^2(\tau)+\frac{\omega^2}{3}q^4(\tau)\right]+\delta(0)q^2(\tau)\right\rangle_c .$$

There are two diagrams originating from the interaction, one from the Jacobian action:

$\beta F_1 = -\,\eta$  $-\,\eta\,\omega^2$  $+\,\eta\,\delta(0)$  . (10.227)

The first cumulant contains also terms of order η^2:

$$\eta^2 \int_0^\beta d\tau \left\langle \left[\left(\frac{1}{2}+a\right) q^4(\tau)\dot{q}^2(\tau) + \omega^2 \left(\frac{1}{18}+\frac{2a}{5}\right) q^6(\tau) \right] - \delta(0) \left(a-\frac{1}{2}\right) q^4(\tau) \right\rangle_c .$$

The interaction gives rise to two three-loop diagrams, the Jacobian action to a single two-loop diagram:

$$\beta F_2^{(1)} = \eta^2 \left[3\left(\frac{1}{2}+a\right) \; + 15\,\omega^2 \left(\frac{1}{18}+\frac{a}{5}\right) \; - 3\left(a-\frac{1}{2}\right)\delta(0) \; \right]. \quad (10.228)$$

We now come to the contribution of the second cumulant $\left\langle \mathcal{A}_{\rm tot}^{\rm int\,2}\right\rangle_c$ in the expansion (10.226). Keeping only terms contributing to order η^2 we have to calculate the expectation value

$$-\frac{1}{2!}\eta^2 \int_0^\beta d\tau \int_0^\beta d\tau' \left\langle \left[-q^2(\tau)\dot{q}^2(\tau) - \frac{\omega^2}{3} q^4(\tau) + \delta(0) q^2(\tau) \right] \right.$$
$$\left. \times \left[-q^2(\tau')\dot{q}^2(\tau') - \frac{\omega^2}{3} q^4(\tau') + \delta(0) q^2(\tau') \right] \right\rangle_c . \quad (10.229)$$

Only the connected diagrams contribute to the cumulant, and these are necessarily nonlocal. The simplest diagrams are those containing factors of $\delta(0)$:

$$\beta F_2^{(2)} = -\frac{\eta^2}{2!} \left\{ 2\delta^2(0) \; - 4\delta(0) \left[\; + \; + 2\,\omega^2 \; \right] \right\}. \quad (10.230)$$

The remaining diagrams have either the form of three bubble in a chain, or of a watermelon, each with all possible combinations of the three line types (10.202)–(10.205). The sum of the three-bubbles diagrams is

$$\beta F_2^{(3)} = -\frac{\eta^2}{2!} \left[4 \; + 2 \; + 2 \; + 8\,\omega^2 \; + 8\omega^2 \; + 8\omega^4 \; \right], \quad (10.231)$$

while the watermelon-like diagrams contribute

$$\beta F_2^{(4)} = -\frac{\eta^2}{2!}\, 4 \left[\frac{2}{3}\omega^4 \; + \; + 4 \; + \; + 4\omega^2 \; \right]. \quad (10.232)$$

Since the equal-time expectation value $\langle \dot{q}(\tau)\, q(\tau) \rangle$ vanishes according to Eq. (10.211), diagrams with a local contraction of a mixed line (10.203) vanish trivially, and have been omitted.

We now show that if we calculate all Feynman integrals in $d = 1 - \varepsilon$ time dimensions and take the limit $\varepsilon \to 0$ at the end, we obtain unique finite results. These have the desired property that the sum of all Feynman diagrams contributing to each order η^n vanishes, thus ensuring invariance of the perturbative expressions (10.196) and (10.199) under coordinate transformations.

10.7.2 Diagrammatic Expansion in d Time Dimensions

As a first step, we extend the dimension of the τ-axis to d, assuming τ to be a vector $\tau \equiv (\tau^0, \dots, \tau^d)$, in which the zeroth component is the physical imaginary time, the others are auxiliary coordinates to be eliminated at the end. Then we replace the harmonic action (10.215) by

$$\mathcal{A}_\omega = \frac{1}{2} \int d^d\tau \left[\partial_\alpha x(\tau)\partial_\alpha x(\tau) + \omega^2 x^2(\tau)\right], \tag{10.233}$$

and the terms $\dot{q}^2$ in the transformed action (10.222) accordingly by $\partial_a q(\tau)\partial_a q(\tau)$.

The correlation functions (10.206), (10.208), and (10.212) are replaced by two-point functions

$$G^{(2)}(\tau,\tau') \;=\; \langle q(\tau)q(\tau')\rangle \qquad = \Delta(\tau-\tau') \quad = \int \frac{d^dk}{(2\pi)^d}\frac{e^{ik(\tau-\tau')}}{k^2+\omega^2}, \tag{10.234}$$

and its derivatives

$$G^{(2)}_\alpha(\tau,\tau') \;=\; \langle \partial_\alpha q(\tau)q(\tau')\rangle \qquad = \Delta_\alpha(\tau-\tau') \quad = \int \frac{d^dk}{(2\pi)^d}\frac{ik_\alpha}{k^2+\omega^2}e^{ik(\tau-\tau')}, \tag{10.235}$$

$$G^{(2)}_{\alpha\beta}(\tau,\tau') \;=\; \langle \partial_\alpha q(\tau)\partial_\beta q(\tau')\rangle \;\; = \Delta_{\alpha\beta}(\tau-\tau') = \int \frac{d^dk}{(2\pi)^d}\frac{k_\alpha k_\beta}{k^2+\omega^2}e^{ik(\tau-\tau')}. \tag{10.236}$$

The configuration space is still one-dimensional so that the indices μ,ν and the corresponding tensors in Eqs. (10.206), (10.208), and (10.212) are absent.

The analytic continuation to $d = 1-\varepsilon$ time dimensions is most easily performed if the Feynman diagrams are calculated in momentum space. The three types of lines represent the analytic expressions

$= \dfrac{1}{p^2+\omega^2}$, $= i\dfrac{p_\alpha}{p^2+\omega^2}$, $= \dfrac{p_\alpha p_\beta}{p^2+\omega^2}$. (10.237)

Most diagrams in the last section converge in one-dimensional momentum space, thus requiring no regularization to make them finite, as we would expect for a quantum-mechanical system. Trouble arises, however, in some multiple momentum integrals, which yield different results depending on the order of their evaluation.

As a typical example, take the Feynman integral

 $= -\displaystyle\int d^d\tau_1 \Delta(\tau_1-\tau_2)\Delta_\alpha(\tau_1-\tau_2)\Delta_\beta(\tau_1-\tau_2)\Delta_{\alpha\beta}(\tau_1-\tau_2)$. (10.238)

For the ordinary one-dimensional Euclidean time, a Fourier transformation yields the triple momentum space integral

$$X = \int \frac{dk}{2\pi}\frac{dp_1}{2\pi}\frac{dp_2}{2\pi}\frac{k^2\,(p_1p_2)}{(k^2+\omega^2)(p_1^2+\omega^2)(p_2^2+\omega^2)[(k+p_1+p_2)^2+\omega^2]}. \tag{10.239}$$

Integrating this first over k, then over p_1 and p_2 yields $1/32\omega$. In the order first p_1, then p_2 and k, we find $-3/32\omega$, whereas the order first p_1, then k and p_2, gives again $1/32\omega$. As we shall see below in Eq. (10.284), the correct result is $1/32\omega$.

The unique correct evaluation will be possible by extending the momentum space to d dimensions and taking the limit $d \to 1$ at the end. The way in which the ambiguity will be resolved may be illustrated by a typical Feynman integral

$$Y_d = \int \frac{d^d k}{(2\pi)^d} \frac{d^d p_1}{(2\pi)^d} \frac{d^d p_2}{(2\pi)^d} \frac{k^2\,(p_1 p_2) - (kp_1)(kp_2)}{(k^2+\omega^2)(p_1^2+\omega^2)(p_2^2+\omega^2)[(k+p_1+p_2)^2+\omega^2]}, \tag{10.240}$$

whose numerator vanished trivially in $d = 1$ dimensions. Due to the different contractions in d dimensions, however, Y_0 will be seen to have the nonzero value $Y_0 = 1/32\omega - (-1/32\omega)$ in the limit $d \to 1$, the result being split according to the two terms in the numerator [to appear in the Feynman integrals (10.282) and (10.284); see also Eq. (10.355)].

The diagrams which need a careful treatment are easily recognized in configuration space, where the one-dimensional correlation function (10.234) is the continuous function (10.207). Its first derivative (10.208) which has a jump at equal arguments is a rather unproblematic distribution, as long as the remaining integrand does not contain δ-functions or their derivatives. These appear with second derivatives of $\Delta(\tau, \tau')$, where the d-dimensional evaluation must be invoked to obtain a unique result.

10.8 Calculating Loop Diagrams

The loop integrals encountered in d dimensions are based on the basic one-loop integral

$$I \equiv \int \frac{\bar{d}^d k}{k^2+\omega^2} = \frac{\omega^{d-2}}{(4\pi)^{d/2}} \Gamma\left(1 - d/2\right) \underset{d=1}{=} \frac{1}{2\omega}, \tag{10.241}$$

where we have abbreviated $\bar{d}^d k \equiv d^d k/(2\pi)^d$ by analogy with $\hbar \equiv h/2\pi$. The integral exists only for $\omega \neq 0$ since it is otherwise divergent at small k. Such a divergence is called infrared divergence (IR-divergence) and ω plays the role of an infrared (IR) cutoff.

By differentiation with respect to ω^2 we can easily generalize (10.241) to

$$I_\alpha^\beta \equiv \int \frac{\bar{d}^d k\,(k^2)^\beta}{(k^2+\omega^2)^\alpha} = \frac{\omega^{d+2\beta-2\alpha}}{(4\pi)^{d/2}} \frac{\Gamma\left(d/2+\beta\right)\Gamma\left(\alpha-\beta-d/2\right)}{\Gamma\left(d/2\right)\Gamma(\alpha)}. \tag{10.242}$$

Note that for consistency of dimensional regularization, all integrals over a pure power of the momentum must vanish:

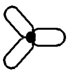
$$I_0^\beta = \int \bar{d}^d k\,(k^2)^\beta = 0. \tag{10.243}$$

We recognize *Veltman's rule* of Eq. (2.508).

With the help of Eqs. (10.241) and (10.242) we calculate immediately the local expectation values (10.234) and (10.236) and thus the local diagrams in (10.227) and (10.228):


$$= \left\langle q^2 \right\rangle = \int \frac{d\!\!\bar{}^d k}{k^2+\omega^2} \underset{d=1}{=} \frac{1}{2\omega}, \tag{10.244}$$


$$= \left\langle q^2 \right\rangle^2 = \left(\int \frac{d\!\!\bar{}^d k}{k^2+\omega^2}\right)^2 \underset{d=1}{=} \frac{1}{4\omega^2}, \tag{10.245}$$


$$= \left\langle q^2 \right\rangle^3 = \left(\int \frac{d\!\!\bar{}^d k}{k^2+\omega^2}\right)^3 \underset{d=1}{=} \frac{1}{8\omega^3}, \tag{10.246}$$

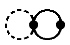

$$= \left\langle q^2 \right\rangle \langle \partial q\, \partial q \rangle = \int \frac{d\!\!\bar{}^d k}{k^2+\omega^2} \int \frac{d\!\!\bar{}^d p\, p^2}{p^2+\omega^2} \underset{d=1}{=} -\frac{1}{4}, \tag{10.247}$$


$$= \langle q q \rangle^2 \langle \partial q \partial q \rangle = \left(\int \frac{d\!\!\bar{}^d k}{k^2+\omega^2}\right)^2 \int \frac{d\!\!\bar{}^d p\, p^2}{p^2+\omega^2} \underset{d=1}{=} -\frac{1}{8\omega}. \tag{10.248}$$

The two-bubble diagrams in Eq. (10.230) can also be easily computed


$$= \int d^d\tau_1\, \Delta^2(\tau_1-\tau_2) \int \frac{d\!\!\bar{}^d p}{(p^2+\omega^2)^2} \underset{d=1}{=} \frac{1}{4\omega^3}, \tag{10.249}$$


$$= \int d^d\tau_1\, \Delta(\tau_1-\tau_1)\Delta_\alpha^2(\tau_1-\tau_2) \int \frac{d\!\!\bar{}^d k}{k^2+\omega^2} \int \frac{d\!\!\bar{}^d p\, p^2}{(p^2+\omega^2)^2} \underset{d=1}{=} \frac{1}{8\omega^2}, \tag{10.250}$$


$$= \int d^d\tau_1\, \Delta_{\alpha\alpha}(\tau_1-\tau_1)\Delta^2(\tau_1-\tau_2) \int \frac{d\!\!\bar{}^d k\, k^2}{k^2+\omega^2} \int \frac{d\!\!\bar{}^d p}{(p^2+\omega^2)^2} \underset{d=1}{=} -\frac{1}{8\omega^2}, \tag{10.251}$$


$$= \int d^d\tau_1\, \Delta(\tau_1-\tau_1)\Delta^2(\tau_1-\tau_2) \int \frac{d\!\!\bar{}^d k}{k^2+\omega^2} \int \frac{d\!\!\bar{}^d p}{(p^2+\omega^2)^2} \underset{d=1}{=} \frac{1}{8\omega^4}. \tag{10.252}$$

For the three-bubble diagrams in Eq. (10.231) we find

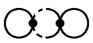

$$\begin{aligned} &= \int d^d\tau_1\, \Delta(\tau_1-\tau_1)\Delta_{\alpha\alpha}^2(\tau_1-\tau_2)\Delta(\tau_2-\tau_2) \\ &= \left(\int \frac{d\!\!\bar{}^d q}{q^2+\omega^2}\right)^2 \int \frac{d\!\!\bar{}^d p (p^2)^2}{(p^2+\omega^2)^2} \underset{d=1}{=} -\frac{3}{16\omega}, \end{aligned} \tag{10.253}$$


$$\begin{aligned} &= \int d^d\tau_1\, \Delta_{\alpha\alpha}(\tau_1-\tau_1)\Delta^2(\tau_1-\tau_2)\Delta_{\beta\beta}(\tau_2-\tau_2) \\ &= \left[\int \frac{d\!\!\bar{}^d q\, (q^2)^2}{q^2+\omega^2}\right]^2 \int \frac{d\!\!\bar{}^d k}{(k^2+\omega^2)^2} \underset{d=1}{=} \frac{1}{16\omega}, \end{aligned} \tag{10.254}$$

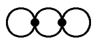

$$\begin{aligned} &= \int d^d\tau_1\, \Delta(\tau_1-\tau_1)\Delta_\alpha^2(\tau_1-\tau_2)\Delta_{\beta\beta}(\tau_2-\tau_2) \\ &= \int \frac{d\!\!\bar{}^d k}{k^2+\omega^2} \int \frac{d\!\!\bar{}^d p\, p^2}{(p^2+\omega^2)^2} \int \frac{d\!\!\bar{}^d q\, q^2}{q^2+\omega^2} \underset{d=1}{=} -\frac{1}{16\omega}, \end{aligned} \tag{10.255}$$


$$= \int d^d\tau_1\, \Delta_{\alpha\alpha}(\tau_1-\tau_1)\Delta^2(\tau_1-\tau_2)\Delta(\tau_2-\tau_2)$$

$$
= \int \frac{\bar{d}^d k\, k^2}{k^2+\omega^2} \int \frac{\bar{d}^d p}{(p^2+\omega^2)^2} \int \frac{\bar{d}^d q}{q^2+\omega^2} \underset{d=1}{=} -\frac{1}{16\omega^3}, \tag{10.256}
$$



$$
\begin{aligned}
&= \int d^d\tau_1\, \Delta(\tau_1-\tau_1)\Delta^2_{\alpha}(\tau_1-\tau_2)\Delta(\tau_2-\tau_2)\\
&= \int \frac{\bar{d}^d k}{k^2+\omega^2} \int \frac{\bar{d}^d p\, p^2}{(p^2+\omega^2)^2} \int \frac{\bar{d}^d q}{q^2+\omega^2} \underset{d=1}{=} \frac{1}{16\omega^3},
\end{aligned} \tag{10.257}
$$



$$
\begin{aligned}
&= \int d^d\tau_1\, \Delta(\tau_1-\tau_1)\Delta^2(\tau_1-\tau_2)\Delta(\tau_2-\tau_2)\\
&= \int \frac{\bar{d}^d k}{k^2+\omega^2} \int \frac{\bar{d}^d p}{(p^2+\omega^2)^2} \int \frac{\bar{d}^d q}{q^2+\omega^2} \underset{d=1}{=} \frac{1}{16\omega^5}.
\end{aligned} \tag{10.258}
$$

In these diagrams, it does not make any difference if we replace $\Delta^2_{\alpha\alpha}$ by $\Delta^2_{\alpha\beta}$.

We now turn to the watermelon-like diagrams in Eq. (10.232) which are more tedious to calculate. They require a further basic integral [26]:

$$
\begin{aligned}
J(p^2) &= \int \frac{\bar{d}^d k}{(k^2+\omega^2)[(k+p)^2+\omega^2]} = \int_0^1 dx \int \frac{\bar{d}^d k}{[k^2+p^2x(1-x)+\omega^2]^2}\\
&= \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}} \left(\frac{p^2+4\omega^2}{4}\right)^{d/2-2} F\left(2-\frac{d}{2},\frac{1}{2};\frac{3}{2};\frac{p^2}{p^2+4\omega^2}\right),
\end{aligned} \tag{10.259}
$$

where $F(a,b;c;z)$ is the hypergeometric function (1.453). For $d=1$, the result is simply

$$
J(p^2) = \frac{1}{\omega(p^2+4\omega^2)}. \tag{10.260}
$$

We also define the more general integrals

$$
J_{\alpha_1\ldots\alpha_n}(p) = \int \frac{\bar{d}^d k\;\; k_{\alpha_1}\cdots k_{\alpha_n}}{(k^2+\omega^2)[(k+p)^2+\omega^2]}, \tag{10.261}
$$

and further

$$
J_{\alpha_1\ldots\alpha_n,\beta_1\ldots\beta_m}(p) = \int \frac{\bar{d}^d k\;\; k_{\alpha_1}\cdots k_{\alpha_n}\,(k+p)_{\beta_1}\cdots(k+p)_{\beta_m}}{(k^2+\omega^2)[(k+p)^2+\omega^2]}. \tag{10.262}
$$

The latter are linear combinations of momenta and the former, for instance

$$
J_{\alpha,\beta}(p) = J_\alpha(p)\, p_\beta + J_{\alpha\beta}(p). \tag{10.263}
$$

Using Veltman's rule (10.243), all integrals (10.262) can be reduced to combinations of $p, I, J(p^2)$. Relevant examples for our discussion are

$$
J_\alpha(p) = \int \frac{\bar{d}^d k\;\; k_\alpha}{(k^2+\omega^2)[(k+p)^2+\omega^2]} = -\frac{1}{2}\, p_\alpha\, J(p^2), \tag{10.264}
$$

and

$$J_{\alpha\beta}(p) \;=\; \int \frac{\bar{d}^d k\ k_\alpha k_\beta}{(k^2+\omega^2)[(k+p)^2+\omega^2]} = \left[\delta_{\alpha\beta} + (d-2)\frac{p_\alpha p_\beta}{p^2}\right]\frac{I}{2(d-1)}$$
$$+\left[-\delta_{\alpha\beta}(p^2+4\omega^2) + \frac{p_\alpha p_\beta}{p^2}\left(d\,p^2+4\omega^2\right)\right]\frac{J(p^2)}{4(d-1)}, \qquad (10.265)$$

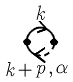
whose trace is

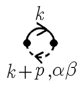
$$J_{\alpha\alpha}(p) = \int \frac{\bar{d}^d k\ k^2}{(k^2+\omega^2)[(k+p)^2+\omega^2]} = I - \omega^2\, J(p^2). \qquad (10.266)$$

Similarly we expand

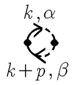
$$J_{\alpha\alpha\beta}(p) \;=\; \int \frac{\bar{d}^d k\ k^2 k_\beta}{(k^2+\omega^2)[(k+p)^2+\omega^2]} = \frac{1}{2}p_\beta[-I+\omega^2 J(p^2)]. \qquad (10.267)$$

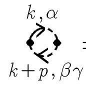
The integrals appear in the following subdiagrams

$= J(p^2),$ 

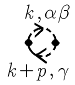
$= J_{,\alpha}(p),$ 

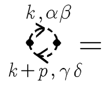
$= J_{,\alpha\beta}(p),$

$= J_\alpha(p),$ 

$= J_{\alpha,\beta}(p),$ 

$= J_{\alpha,\beta\gamma}(p),$ (10.268)

$= J_{\alpha\beta}(p),$ 

$= J_{\alpha\beta,\gamma}(p),$ 

$= J_{\alpha\beta,\gamma\delta}(p).$

All two- and three-loop integrals needed for the calculation can be brought to the generic form

$$K(a,b) = \int \bar{d}^d p\ (p^2)^a J^b(p^2), \quad a\geq 0, \quad b \geq 1, \quad a \leq b, \qquad (10.269)$$

and evaluated recursively as follows [27]. From the Feynman parametrization of the first line of Eq. (10.259) we observe that the two basic integrals (10.241) and (10.259) satisfy the differential equation

$$J(p^2) = -\frac{\partial I}{\partial\omega^2} + \frac{1}{2}p^2\frac{\partial J(p^2)}{\partial\omega^2} - 2p^2\frac{\partial J(p^2)}{\partial p^2}. \qquad (10.270)$$

Differentiating $K(a+1,b)$ from Eq. (10.269) with respect to ω^2, and using Eq. (10.270), we find the recursion relation

$$K(a,b) = \frac{2b(d/2-1)\,I\,K(a-1,b-1) - 2\omega^2(2a-2-b+d)K(a-1,b)}{(b+1)d/2 - 2b + a}, \qquad (10.271)$$

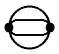
which may be solved for increasing a starting with

$$K(0,0) = 0, \qquad K(0,1) = \int \bar{d}^d p\, J(p^2) = I^2,$$
$$K(0,2) = \int \bar{d}^d p\, J^2(p^2) = A, \;\; \dots, \qquad (10.272)$$

where A is the integral

$$A \equiv \int \frac{\bar{d}^d p_1 \bar{d}^d p_2 \bar{d}^d k}{(p_1^2+\omega^2)(p_2^2+\omega^2)(k^2+\omega^2)[(p_1+p_2+k)^2+\omega^2]}. \tag{10.273}$$

This integral will be needed only in $d = 1$ dimensions where it can be calculated directly from the configuration space version of this integral. For this we observe that the first watermelon-like diagram in (10.232) corresponds to an integral over the product of two diagrams $J(p^2)$ in (10.268):



$$= \int d^d\tau_1 \Delta(\tau_1-\tau_2)\Delta(\tau_1-\tau_2)\Delta(\tau_1-\tau_2)\Delta(\tau_1-\tau_2) = \int \bar{d}^d k\, J^2(k) = A. \tag{10.274}$$

Thus we find A in $d = 1$ dimensions from the simple τ-integral

$$A = \int_{-\infty}^{\infty} d\tau\, \Delta^4(\tau,0) = \int_{-\infty}^{\infty} dx \left(\frac{1}{2\omega} e^{-\omega|x|}\right)^4 = \frac{1}{32\omega^5}. \tag{10.275}$$

Since this configuration space integral contains no δ-functions, the calculation in $d = 1$ dimension is without subtlety.

With the help of Eqs. (10.271), (10.272), and Veltman's rule (10.243), according to which

$$K(a,0) \equiv 0, \tag{10.276}$$

we find further the integrals

$$\begin{aligned}
\int \bar{d}^d p\, p^2 J(p^2) &= K(1,1) = -2\omega^2 I^2, && (10.277)\\
\int \bar{d}^d p\; p^2\, J^2(p^2) &= K(1,2) = \frac{4}{3}(I^3 - \omega^2 A), && (10.278)\\
\int \bar{d}^d p\, (p^2)^2 J^2(p^2) &= K(2,2) = -8\omega^2 \frac{(6-5d)I^3 + 2d\omega^2 A}{3(4-3d)}. && (10.279)
\end{aligned}$$

We are thus prepared to calculate all remaining three-loop contributions from the watermelon-like diagrams in Eq. (10.232). The second is an integral over the product of subdiagrams $J_{\alpha\beta}$ in (10.268) and yields

$$\begin{aligned}
&= \int d^d\tau_1\, \Delta^2(\tau_1-\tau_2)\Delta^2_{\alpha\beta}(\tau_1-\tau_2)\\
&= \int \bar{d}^d p\, \bar{d}^d k\, \bar{d}^d q \frac{(pk)^2}{(p^2+\omega^2)(k^2+\omega^2)(q^2+\omega^2)[(p+k+q)^2+\omega^2]}\\
&\underset{q\to q-p}{=} \int \bar{d}^d q\, J_{\alpha\beta}(q) J_{\alpha\beta}(q) = \int \bar{d}^d k\, \frac{1}{16}(k^2)^2 J^2(k)\\
&\qquad + \int \bar{d}^d k \frac{1}{4(d-1)} \left\{ dI^2 + \left[(d-2)k^2 - 4\omega^2\right] I\, J(k) + \frac{1}{4}(k^2+4\omega^2)^2 J^2(k)\right\}\\
&= -\frac{\omega^2}{2}\frac{(6-5d)I^3 + 2d\omega^2 A}{3(4-3d)} - \frac{\omega^2}{6(4-3d)}\left[(6-5d)I^3 + 2d\omega^2 A\right] \qquad (10.280)\\
&= -\frac{\omega^2}{3(4-3d)}\left[(8-7d)I^3 + (d+4)\omega^2 A\right] \underset{d=1}{=} -\frac{\omega^2}{3}(I^3 + 5\omega^2 A) = -\frac{3}{32\omega}.
\end{aligned}$$

The third diagram contains two mixed lines. It is an integral over a product of the diagrams $J_\alpha(p)$ and $J_{\beta,\alpha\beta}$ in (10.268) and gives

$$
\begin{aligned}
&= -\int d^d\tau_1 \Delta(\tau_1-\tau_2)\Delta_\alpha(\tau_1-\tau_2)\Delta_\beta(\tau_1-\tau_2)\Delta_{\alpha\beta}(\tau_1-\tau_2)\\
&= \int \bar{d}^dk\bar{d}^dp_1\bar{d}^dp_2 \frac{(kp_1)(kp_2)}{(k^2+\omega^2)(p_1^2+\omega^2)(p_2^2+\omega^2)\,[(k+p_1+p_2)^2+\omega^2]}\\
&\underset{p_2\to p_2-k}{=} \int \bar{d}^dp\;[p_\beta J_\alpha(p)J_{\alpha\beta}(p)+J_\alpha(p)J_{\beta\alpha\beta}(p)]\\
&= -\frac{1}{8}\int \bar{d}^dp\; p^2 J(p^2)\left[(p^2+2\omega^2)J(p^2)-2I\right] \qquad (10.281)\\
&= -\frac{\omega^2}{6(4-3d)}\left[(8-5d)\,I^3-2(4-d)\omega^2A\right] \underset{d=1}{=} -\frac{\omega^2}{2}(I^3-2\omega^2A) = -\frac{1}{32\omega}.
\end{aligned}
$$

The fourth diagram contains four mixed lines and is evaluated as follows:

$$
= -\int d^d\tau_1 \Delta_\alpha(\tau_1-\tau_2)\Delta_\alpha(\tau_1-\tau_2)\Delta_\beta(\tau_1-\tau_2)\Delta_\beta(\tau_1-\tau_2). \qquad (10.282)
$$

Since the integrand is regular and vanishes at infinity, we can perform a partial integration and rewrite the configuration space integral as

$$
\begin{aligned}
&= \int d^d\tau_1 \Delta(\tau_1-\tau_2)\Delta_{\alpha\alpha}(\tau_1-\tau_2)\Delta_\beta(\tau_1-\tau_2)\Delta_\beta(\tau_1-\tau_2)\\
&+\; 2\int d^d\tau_1 \Delta(\tau_1-\tau_2)\Delta_\alpha(\tau_1-\tau_2)\Delta_\beta(\tau_1-\tau_2)\Delta_{\alpha\beta}(\tau_1-\tau_2). \qquad (10.283)
\end{aligned}
$$

The second integral has just been evaluated in (10.282). The first is precisely the integral Eq. (10.239) discussed above. It is calculated as follows:

$$
\begin{aligned}
&\int d^d\tau_1 \Delta(\tau_1-\tau_2)\Delta_{\alpha\alpha}(\tau_1-\tau_2)\Delta_\beta(\tau_1-\tau_2)\Delta_\beta(\tau_1-\tau_2)\\
&= \int \bar{d}^dk\,\bar{d}^dp_1\bar{d}^dp_2 \frac{k^2\;(p_1p_2)}{(k^2+\omega^2)(p_1^2+\omega^2)(p_2^2+\omega^2)[(k+p_1+p_2)^2+\omega^2]}\\
&= \int \bar{d}^dp\,[p_\alpha J_\alpha(p)J_{\beta\beta}+J_\alpha(p)J_{\beta\alpha\beta}(p)] = \frac{\omega^2}{4}\int \bar{d}^dp\,p^2J^2(p^2)\\
&= -\frac{\omega^2}{3}(I^3-\omega^2A) \underset{d=1}{=} \frac{1}{32\omega}. \qquad (10.284)
\end{aligned}
$$

Hence we obtain

$$
= \frac{1}{32\omega}. \qquad (10.285)
$$

The fifth diagram in (10.232) is an integral of the product of two subdiagrams $J_\alpha(p)$ in (10.268) and yields

$$
= \int d^d\tau_1 \Delta(\tau_1-\tau_2)\Delta_\alpha^2(\tau_1-\tau_2)\Delta(\tau_1-\tau_2)
$$

$$
\begin{aligned}
&= -\int d\!\!{}^{-}{}^d k\, d\!\!{}^{-}{}^d p_1 d\!\!{}^{-}{}^d p_2 \frac{p_1 p_2}{(k^2+\omega^2)(p_1^2+\omega^2)(p_2^2+\omega^2)\,[(k+p_1+p_2)^2+\omega^2]}\\
&\underset{k\to k-p_2}{=} -\int d\!\!{}^{-}{}^d k\, d\!\!{}^{-}{}^d p_1 d\!\!{}^{-}{}^d p_2 \frac{p_1 p_2}{[(k-p_2)^2+\omega^2]\,(p_1^2+\omega^2)(p_2^2+\omega^2)\,[(k+p_1)^2+\omega^2]}\\
&\underset{p_2\to -p_2}{=} \int d\!\!{}^{-}{}^d k\, d\!\!{}^{-}{}^d p_1 \frac{p_{1\alpha}}{(p_1^2+\omega^2)\,[(p_1+k)^2+\omega^2]} \int d\!\!{}^{-}{}^d p_2 \frac{p_{2\alpha}}{(p_2^2+\omega^2)\,[(p_2+k)^2+\omega^2]}\\
&= \int d\!\!{}^{-}{}^d k\, J_\alpha^2(k) = \frac{1}{4}\int d\!\!{}^{-}{}^d k\, k^2 J^2(k^2)\\
&= \frac{1}{4}\frac{4}{3}(I^3-\omega^2 A) \underset{d=1}{=} \frac{1}{32\omega^3}.
\end{aligned}
\tag{10.286}
$$

We can now sum up all contributions to the free energy in Eqs. (10.227)–(10.232). An immediate simplification arises from the Veltman's rule (10.243). This implies that all δ-functions at the origin are zero in dimensional regularization:

$$
\delta^{(d)}(0) = \int \frac{d^d k}{(2\pi)^d} = 0. \tag{10.287}
$$

The first-order contribution (10.227) to the free energy is obviously zero by Eqs. (10.245) and (10.247).

The first second-order contribution $\beta F_2^{(1)}$ becomes, from (10.246) and (10.248):

$$
F_2^{(1)} = \eta^2\left[3\left(\frac{1}{2}+a\right)\left(-\frac{1}{8\omega}\right)+15\omega^2\left(\frac{1}{18}+\frac{a}{5}\right)\right] = -\frac{\eta^2}{12\omega}. \tag{10.288}
$$

The parameter a has disappeared from this equation.

The second second-order contribution $\beta F_2^{(2)}$ vanishes trivially, by Veltman's rule (10.287).

The third second-order contribution $\beta F_2^{(3)}$ in (10.231) vanishes nontrivially using (10.253)–(10.258):

$$
\begin{aligned}
F_2^{(3)} = -\frac{\eta^2}{2!}\Bigg[\; 4 \left(-\frac{1}{16\omega}\right) + 2\left(-\frac{3}{16\omega}\right)+2\left(\frac{1}{16\omega}\right)\\
+8\omega^2\left(-\frac{1}{16\omega^3}\right)+8\omega^2\left(\frac{1}{16\omega^3}\right)+8\omega^4\left(\frac{1}{16\omega^5}\right)\Bigg] = 0.
\end{aligned}
\tag{10.289}
$$

The fourth second-order contribution, finally, associated with the watermelon-like diagrams in (10.232) yield via (10.281), (10.282), (10.285), (10.286), and (10.274):

$$
\beta F_2^{(4)} = -\frac{\eta^2}{2!}\left[-\frac{3}{32\omega}+4\left(-\frac{1}{32\omega}\right)+\frac{1}{32\omega}+4\omega^2\left(\frac{1}{32\omega^3}\right)+\frac{2}{3}\omega^4\left(\frac{1}{32\omega^5}\right)\;\right] = \frac{\eta^2}{12\omega}, \tag{10.290}
$$

canceling (10.288), and thus the entire free energy. This proves the invariance of the perturbatively defined path integral under coordinate transformations.

10.8.1 Reformulation in Configuration Space

The Feynman integrals in momentum space in the last section corresponds in τ-space to integrals over products of distributions. For many applications it is desirable to do the calculations directly in τ-space. This will lead to an extension of distribution theory which allows us to do precisely that.

In dimensional regularization, an important simplification came from Veltman's rule (10.287), according to which the delta function at the origin vanishes. In the more general calculations to come, we shall encounter generalized δ-functions, which are multiple derivatives of the ordinary δ-function:

$$\delta^{(d)}_{\alpha_1\ldots\alpha_n}(\tau) \quad\equiv\quad \partial_{\alpha_1\ldots\alpha_n}\delta^{(d)}(\tau) = \int đ^d k (ik)_{\alpha_1}\ldots(ik)_{\alpha_n} e^{ikx}, \tag{10.291}$$

with $\partial_{\alpha_1\ldots\alpha_n} \equiv \partial_{\alpha_1}\ldots\partial_{\alpha_n}$ and $đ^d k \equiv d^d k/(2\pi)^d$. By Veltman's rule (10.243), all these vanish at the origin:

$$\delta^{(d)}_{\alpha_1\ldots\alpha_n}(0) = \int đ^d k (ik)_{\alpha_1}\ldots(ik)_{\alpha_n} = 0. \tag{10.292}$$

In the extended coordinate space, the correlation function $\Delta(\tau,\tau')$ in (10.234), which we shall also write as $\Delta(\tau-\tau')$, is at equal times given by the integral [compare (10.241)]

$$\Delta(0) = \int \frac{đ^d k}{k^2+\omega^2} = \frac{\omega^{d-2}}{(4\pi)^{d/2}}\Gamma\left(1-\frac{d}{2}\right) = I \underset{d=1}{=} \frac{1}{2\omega}. \tag{10.293}$$

The extension (10.235) of the time derivative (10.208),

$$\Delta_\alpha(\tau) = \int đ^d k \frac{ik_\alpha}{k^2+\omega^2} e^{ik\tau} \tag{10.294}$$

vanishes at equal times, just like (10.211):

$$\Delta_\alpha(0) = 0. \tag{10.295}$$

This follows directly from a Taylor series expansion of $1/(k^2+\omega^2)$ in powers of k^2, after imposing (10.292).

The second derivative of $\Delta(\tau)$ has the Fourier representation (10.236). Contracting the indices yields

$$\Delta_{\alpha\alpha}(\tau) = -\int đ^d k\, \frac{k^2}{k^2+\omega^2} e^{ikx} = -\delta^{(d)}(\tau) + \omega^2\Delta(\tau)\,. \tag{10.296}$$

This equation is a direct consequence of the definition of the correlation function as a solution to the inhomogeneous field equation

$$(-\partial_\alpha^2 + \omega^2) q(\tau) = \delta^{(d)}(\tau). \tag{10.297}$$

Inserting Veltman's rule (10.287) into (10.296), we obtain

$$\Delta_{\alpha\alpha}(0) = \omega^2\,\Delta(0) \underset{d=1}{=} \frac{\omega}{2}. \tag{10.298}$$

This ensures the vanishing of the first-order contribution (10.227) to the free energy

$$F_1 = -g\,\eta\left[-\Delta_{\alpha\alpha}(0) + \omega^2\Delta(0)\right]\,\Delta(0) = 0. \tag{10.299}$$

The same equation (10.296) allows us to calculate immediately the second-order contribution (10.228) from the local diagrams

$$
\begin{aligned}
F_2^{(1)} &= -\eta^2\,3g^2\left[\left(\frac{1}{2}+a\right)\Delta_{\alpha\alpha}(0)-5\left(\frac{1}{18}+\frac{a}{5}\right)\omega^2\Delta(0)\right]\Delta^2(0)\\
&= -\eta^2\frac{2}{3}\,\omega^2\Delta^3(0)\underset{d=1}{=}-\frac{\eta^2}{12\omega}. \qquad (10.300)
\end{aligned}
$$

The other contributions to the free energy in the expansion (10.226) require rules for calculating products of two and four distributions, which we are now going to develop.

10.8.2 Integrals over Products of Two Distributions

The simplest integrals are

$$
\begin{aligned}
\int d^d\tau\,\Delta^2(\tau) &= \int d\!\!{}^-{}^d p\, d\!\!{}^-{}^d k\,\frac{\delta^{(d)}(k+p)}{(p^2+\omega^2)(k^2+\omega^2)}\\
&= \int\frac{d\!\!{}^-{}^d k}{(k^2+\omega^2)^2}=\frac{\omega^{d-4}}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)=\frac{(2-d)}{2\omega^2}\,\Delta(0), \qquad (10.301)
\end{aligned}
$$

and

$$
\begin{aligned}
\int d^d\tau\,\Delta_\alpha^2(\tau) &= -\int d^d\tau\,\Delta(\tau)\left[-\delta^{(d)}(\tau)+\omega^2\Delta(\tau)\right]=\Delta(0)-\omega^2\int d^d\tau\,\Delta^2(\tau)\\
&= \frac{d}{2}\,\Delta(0). \qquad (10.302)
\end{aligned}
$$

To obtain the second result we have performed a partial integration and used (10.296).

In contrast to (10.301) and (10.302), the integral

$$
\begin{aligned}
\int d^d\tau\,\Delta_{\alpha\beta}^2(\tau) &= \int d\!\!{}^-{}^d p\, d\!\!{}^-{}^d k\,\frac{(kp)^2\,\delta^{(d)}(k+p)}{(k^2+\omega^2)(p^2+\omega^2)}\\
&= \int d\!\!{}^-{}^d k\,\frac{(k^2)^2}{(k^2+\omega^2)^2}=\int d^d\tau\,\Delta_{\alpha\alpha}^2(\tau) \qquad (10.303)
\end{aligned}
$$

diverges formally in $d=1$ dimension. In dimensional regularization, however, we may decompose $(k^2)^2=(k^2+\omega^2)^2-2\omega^2(k^2+\omega^2)+\omega^4$, and use (10.292) to evaluate

$$
\begin{aligned}
\int d^d\tau\,\Delta_{\alpha\alpha}^2(\tau) &= \int d\!\!{}^-{}^d k\frac{(k^2)^2}{(k^2+\omega^2)^2}=-2\omega^2\int\frac{d\!\!{}^-{}^d k}{(k^2+\omega^2)}+\omega^4\int\frac{d\!\!{}^-{}^d k}{(k^2+\omega^2)^2}\\
&= -2\,\omega^2\Delta(0)+\omega^4\int d^d\tau\Delta^2(\tau). \qquad (10.304)
\end{aligned}
$$

Together with (10.301), we obtain the relation between integrals of products of two distributions

$$
\begin{aligned}
\int d^d\tau\,\Delta_{\alpha\beta}^2(\tau) &= \int d^d\tau\,\Delta_{\alpha\alpha}^2(\tau)=-2\omega^2\Delta(0)+\omega^4\int d^d\tau\,\Delta^2(\tau)\\
&= -\,(1+d/2)\,\omega^2\Delta(0)\,. \qquad (10.305)
\end{aligned}
$$

An alternative way of deriving the equality (10.303) is to use partial integrations and the identity

$$\partial_\alpha \Delta_{\alpha\beta}(\tau) = \partial_\beta \Delta_{\alpha\alpha}(\tau), \tag{10.306}$$

which follows directly from the Fourier representation (10.294).

Finally, from Eqs. (10.301), (10.302), and (10.305), we observe the useful identity

$$\int d^d\tau \, \left[\Delta^2_{\alpha\beta}(\tau) + 2\omega^2 \, \Delta^2_\alpha(\tau) + \omega^4 \, \Delta^2(\tau)\right] \, = \, 0 \, , \tag{10.307}$$

which together with the inhomogeneous field equation (10.296) reduces the calculation of the second-order contribution of all three-bubble diagrams (10.231) to zero:

$$F_2^{(3)} = -g^2 \Delta^2(0) \int d^d\tau \, \left[\Delta^2_{\alpha\beta}(\tau) + 2\omega^2 \, \Delta^2_\alpha(\tau) + \omega^4 \, \Delta^2(\tau)\right] \, = \, 0 \, . \tag{10.308}$$

10.8.3 Integrals over Products of Four Distributions

Consider now the more delicate integrals arising from watermelon-like diagrams in (10.232) which contain products of four distributions, a nontrivial tensorial structure, and overlapping divergences. We start from the second to fourth diagrams:

 $$= \int d^d\tau \, \Delta^2(\tau) \Delta^2_{\alpha\beta}(\tau), \tag{10.309}$$

4  $$= 4 \int d^d\tau \Delta(\tau) \Delta_\alpha(\tau) \Delta_\beta(\tau) \Delta_{\alpha\beta}(\tau), \tag{10.310}$$

 $$= \int d^d\tau \Delta_\alpha(\tau) \Delta_\alpha(\tau) \Delta_\beta(\tau) \Delta_\beta(\tau). \tag{10.311}$$

To isolate the subtleties with the tensorial structure exhibited in Eq. (10.240), we introduce the integral

$$Y_d = \int d^d\tau \, \Delta^2(\tau) \left[\Delta^2_{\alpha\beta}(\tau) - \Delta^2_{\alpha\alpha}(\tau)\right] . \tag{10.312}$$

In $d = 1$ dimension, the bracket vanishes formally, but the limit $d \to 1$ of the integral is nevertheless finite. We now decompose the Feynman diagram (10.309), into the sum

$$\int d^d\tau \, \Delta^2(\tau) \Delta^2_{\alpha\beta}(\tau) = \int d^d\tau \, \Delta^2(\tau) \Delta^2_{\alpha\alpha}(\tau) + Y_d \, . \tag{10.313}$$

To obtain an analogous decomposition for the other two diagrams (10.310) and (10.311), we derive a few useful relations using the inhomogeneous field equation (10.296), partial integrations, and Veltman's rules (10.287) or (10.292). From the inhomogeneous field equation, there is the relation

$$-\int d^d\tau \, \Delta_{\alpha\alpha}(\tau) \Delta^3(\tau) = \Delta^3(0) - \omega^2 \int d^d\tau \, \Delta^4(\tau). \tag{10.314}$$

By a partial integration, the left-hand side becomes

$$\int d^d\tau\, \Delta_{\alpha\alpha}(\tau)\Delta^3(\tau) = -3\int d^d\tau\, \Delta_\alpha^2(\tau)\Delta^2(\tau), \tag{10.315}$$

leading to

$$\int d^d\tau\, \Delta_\alpha^2(\tau)\Delta^2(\tau) = \frac{1}{3}\Delta^3(0) - \frac{1}{3}\omega^2\int d^d\tau\, \Delta^4(\tau). \tag{10.316}$$

Invoking once more the inhomogeneous field equation (10.296) and Veltman's rule (10.287), we obtain the integrals

$$\int d^d\tau\, \Delta_{\alpha\alpha}^2(\tau)\Delta^2(\tau) - \omega^4\int d^d\tau\, \Delta^4(\tau) + 2\omega^2\Delta^3(0) = 0, \tag{10.317}$$

and

$$\int d^d\tau\, \Delta_{\alpha\alpha}(\tau)\Delta_\beta^2(\tau)\Delta(\tau) = \omega^2\int d^d\tau\, \Delta_\beta^2(\tau)\Delta^2(\tau). \tag{10.318}$$

Using (10.316), the integral (10.318) takes the form

$$\int d^d\tau\, \Delta_{\alpha\alpha}(\tau)\Delta_\beta^2(\tau)\Delta(\tau) \;=\; \frac{1}{3}\omega^2\Delta^3(0) - \frac{1}{3}\omega^4\int d^d\tau\, \Delta^4(\tau). \tag{10.319}$$

Partial integration, together with Eqs. (10.317) and (10.319), leads to

$$\begin{aligned}\int d^d\tau\, \partial_\beta\Delta_{\alpha\alpha}(\tau)\Delta_\beta(\tau)\Delta^2(\tau) &= -\int d^d\tau\, \Delta_{\alpha\alpha}^2(\tau)\Delta^2(\tau) - 2\int d^d\tau\, \Delta_{\alpha\alpha}(\tau)\Delta_\beta^2(\tau)\Delta(\tau)\\ &= \frac{4}{3}\omega^2\Delta^3(0) - \frac{1}{3}\omega^4\int d^d\tau\, \Delta^4(\tau).\end{aligned} \tag{10.320}$$

A further partial integration, and use of Eqs. (10.306), (10.318), and (10.320) produces the decompositions of the second and third Feynman diagrams (10.310) and (10.311):

$$4\int d^d\tau\, \Delta(\tau)\Delta_\alpha(\tau)\Delta_\beta(\tau)\Delta_{\alpha\beta}(\tau) = 4\omega^2\int d^d\tau\, \Delta^2(\tau)\Delta_\alpha^2(\tau) - 2\,Y_d, \tag{10.321}$$

and

$$\int d^d\tau\, \Delta_\alpha^2(\tau)\Delta_\beta^2(\tau) = -3\omega^2\int d^d\tau\, \Delta^2(\tau)\Delta_\alpha^2(\tau) + Y_d. \tag{10.322}$$

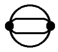
We now make the important observation that the subtle integral Y_d of Eq. (10.312) appears in Eqs. (10.313), (10.321), and (10.322) in such a way that it drops out from the sum of the watermelon-like diagrams in (10.232):

 $+4$  $+$ 

$$= \int d^d\tau\, \Delta^2(\tau)\Delta_{\alpha\alpha}^2(\tau) + \omega^2\int d^d\tau\, \Delta^2(\tau)\Delta_\alpha^2(\tau). \tag{10.323}$$

Using now the relations (10.316) and (10.317), the right-hand side becomes a sum of completely regular integrals involving only products of propagators $\Delta(\tau)$.

We now add to this sum the first and last watermelon-like diagrams in Eq. (10.232)

$$\frac{2}{3}\omega^4$$  $$= \frac{2}{3}\omega^4 \int d^d\tau\, \Delta^4(\tau), \tag{10.324}$$

and

$$4\omega^2$$  $$= 4\omega^2 \int d^d\tau\, \Delta^2(\tau)\Delta^2_\alpha(\tau), \tag{10.325}$$

and obtain for the total contribution of all watermelon-like diagrams in (10.232) the simple expression for $\eta = 1$:

$$\begin{aligned} F_2^{(4)} &= -2\eta^2 g^2 \int d^d\tau\, \Delta^2(\tau) \left[\frac{2}{3}\omega^4\, \Delta^2(\tau) + \Delta^2_{\alpha\alpha}(\tau) + 5\omega^2\, \Delta^2_\alpha(\tau)\right] \\ &= \eta^2 \frac{2}{3}\,\omega^2 \Delta^3(0) \underset{d=1}{=} \frac{\eta^2}{12\omega}. \end{aligned} \tag{10.326}$$

This cancels the finite contribution (10.300), thus making also the second-order free energy in (10.222) vanish, and confirming the invariance of the perturbatively defined path integral under coordinate transformations up to this order.

Thus we have been able to relate all diagrams involving singular time derivatives of correlation functions to integrals over products of the regular correlation function (10.234), where they can be replaced directly by their $d = 1$ -version (10.207). The disappearance of the ambiguous integral Y_d in the combination of watermelon-like diagrams (10.323) has the pleasant consequence that ultimately all calculations can be done in $d = 1$ dimensions after all. This leads us to expect that the dimensional regularization may be made superfluous by a more comfortable calculation procedure. This is indeed so and the rules will be developed in Section 10.11. Before we come to this it is useful, however, to point out a pure x-space way of finding the previous results.

10.9 Distributions as Limits of Bessel Function

In dimensional regularization it is, of course, possible to perform the above configuration space integrals over products of distributions without any reference to momentum space integrals. For this we express all distributions explicitly in terms of modified Bessel functions $K_\alpha(y)$.

10.9.1 Correlation Function and Derivatives

The basic correlation function in d-dimension is obtained from the integral in Eq. (10.234), as

$$\Delta(\tau) = c_d\, y^{1-d/2}\, K_{1-d/2}(y), \tag{10.327}$$

where $y \equiv m\,|\tau|$ is reduced length of τ_α, with the usual Euclidean norm $|x| = \sqrt{\tau_1^2 + \ldots + \tau_d^2}$, and $K_{1-d/2}(y)$ is the modified Bessel function. The constant factor in front is

$$c_d = \frac{\omega^{d-2}}{(2\pi)^{d/2}}. \tag{10.328}$$

In one dimension, the correlation function (10.327) reduces to (10.202). The short-distance properties of the correlation functions is governed by the small-y behavior of Bessel function at origin[4]

$$K_\beta(y) \underset{y\approx 0}{\approx} \frac{1}{2}\Gamma(\beta)(y/2)^{\mp\beta}, \quad \text{Re}\,\beta \gtrless 0. \tag{10.329}$$

In the application to path integrals, we set the dimension equal to $d = 1 - \varepsilon$ with a small positive ε, whose limit $\varepsilon \to 0$ will yield the desired results in $d = 1$ dimension. In this regime, Eq. (10.329) shows that the correlation function (10.327) is regular at the origin, yielding once more (10.293). For $d = 1$, the result is $\Delta(0) = 1/2\omega$, as stated in Eq. (10.298).

The first derivative of the correlation function (10.327), which is the d-dimensional extension of time derivative (10.203), reads

$$\Delta_\alpha(\tau) = -c_d y^{1-d/2}\, K_{d/2}(y)\,\partial_\alpha y, \tag{10.330}$$

where $\partial_\alpha y = m\,\tau_\alpha/|x|$. By Eq. (10.329), this is regular at the origin for $\varepsilon > 0$, such that the antisymmetry $\Delta_\alpha(-x) = -\Delta_\alpha(\tau)$ makes $\Delta_\alpha(0) = 0$, as observed after Eq. (10.294).

Explicitly, the small-τ behavior of the correlation function and its derivative is

$$\Delta(\tau) \propto \text{const.}, \qquad \Delta_\alpha(\tau) \propto |\tau|^\varepsilon\, \partial_\alpha|\tau|. \tag{10.331}$$

In contrast to these two correlation functions, the second derivative

$$\Delta_{\alpha\beta}(\tau) = \Delta(\tau)\,(\partial_\alpha y)(\partial_\beta y) + \frac{c_d}{(d-2)}\, y^{d/2}\, K_{d/2}(y)\,\partial_{\alpha\beta} y^{2-d}, \tag{10.332}$$

is singular at short distance. The singularity comes from the second term in (10.332):

$$\partial_{\alpha\beta} y^{2-d} = (2-d)\frac{\omega^{2-d}}{|y|^d}\left(\delta_{\alpha\beta} - d\,\frac{y_\alpha y_\beta}{y^2}\right), \tag{10.333}$$

which is a distribution that is ambiguous at origin, and defined up to the addition of a $\delta^{(d)}(\tau)$-function. It is regularized in the same way as the divergence in the Fourier representation (10.292). Contracting the indices α and β in Eq. (10.333), we obtain

$$\partial^2 y^{2-d} = (2-d)\omega^{2-d}\, S_d\, \delta^{(d)}(\tau), \tag{10.334}$$

where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface of a unit sphere in d dimensions [recall Eq. (1.558)]. As a check, we take the trace of $\Delta_{\alpha\beta}(\tau)$ in Eq. (10.332), and reproduce the inhomogeneous field equation (10.296):

$$\begin{aligned}\Delta_{\alpha\alpha}(\tau) &= \omega^2\,\Delta(\tau) - c_d\, m^{2-d}\, S_d\, \frac{1}{2}\,\Gamma\,(d/2)\, 2^{d/2}\,\delta^{(d)}(\tau) \\ &= \omega^2\,\Delta(\tau) - \delta^{(d)}(\tau).\end{aligned} \tag{10.335}$$

Since $\delta^{(d)}(\tau)$ vanishes at the origin by (10.292), we find once more Eq. (10.298).

A further relation between distributions is found from the derivative

$$\partial_\alpha\,\Delta_{\alpha\beta}(\tau) = \partial_\beta\left[-\delta^{(d)}(\tau) + \omega^2\,\Delta(\tau)\right] + \omega S_d\left[\Delta(\tau)|y|^{d-1}(\partial_\beta y)\right]\delta^{(d)}(\tau) = \partial_\beta\Delta_{\lambda\lambda}(\tau). \tag{10.336}$$

[4] M. Abramowitz and I. Stegun, *op. cit.*, Formula 9.6.9.

10.9.2 Integrals over Products of Two Distributions

Consider now the integrals over products of such distributions. If an integrand $f(|x|)$ depends only on $|x|$, we may perform the integrals over the directions of the vectors

$$\int d^d\tau\, f(\tau) = S_d \int_0^\infty dr\, r^{d-1}\, f(r), \qquad r \equiv |x|. \tag{10.337}$$

Using the integral formula[5]

$$\int_0^\infty dy\, y\, K_\beta^2(y) = \frac{1}{2}\frac{\pi\beta}{\sin\pi\beta} = \frac{1}{2}\Gamma(1+\beta)\Gamma(1-\beta), \tag{10.338}$$

we can calculate directly:

$$\begin{aligned}\int d^d\tau\; \Delta^2(\tau) &= \omega^{-d}\, c_d^2\, S_d \int_0^\infty dy\, y\, K_{1-d/2}^2(y)\\ &= \omega^{-d}\, c_d^2\, S_d \frac{1}{2}\,(1-d/2)\,\Gamma(1-d/2)\,\Gamma(d/2) = \frac{2-d}{2\omega^2}\Delta(0),\end{aligned} \tag{10.339}$$

and

$$\begin{aligned}\int d^d\tau\, \Delta_\alpha^2(\tau) &= \omega^{2-d}\, c_d^2\, S_d \int_0^\infty dy\, y K_{d/2}^2(y)\\ &= \omega^{2-d}\, c_d^2\, S_d\, \frac{1}{2}\Gamma\,(1+d/2)\,\Gamma\,(1-d/2) = \frac{d}{2}\Delta(0),\end{aligned} \tag{10.340}$$

in agreement with Eqs. (10.301) and (10.302). Inserting $\Delta(0) = 1/2\omega$ from (10.293), these integrals give us once more the values of the Feynman diagrams (10.249), (10.252), (10.253), (10.256), and (10.258).

Note that due to the relation[6]

$$K_{d/2}(y) = -y^{d/2-1}\,\frac{d}{dy}\left[y^{1-d/2}\,K_{1-d/2}(y)\right], \tag{10.341}$$

the integral over y in Eq. (10.340) can also be performed by parts, yielding

$$\begin{aligned}\int d^d\tau\, \Delta_\alpha^2(\tau) &= -\omega^{2-d}\, c_d^2\, S_d \left(y^{d/2}K_{d/2}\right)\left(y^{1-d/2}K_{1-d/2}\right)\Big|_0^\infty - \omega^2 \int d^d\tau\, \Delta^2(\tau)\\ &= \Delta(0) - \omega^2 \int d^d\tau \Delta^2(\tau).\end{aligned} \tag{10.342}$$

The upper limit on the right-hand side gives zero because of the exponentially fast decrease of the Bessel function at infinity. This was obtained before in Eq. (10.302) from a partial integration and the inhomogeneous field equation (10.296).

Using the explicit representations (10.327) and (10.332), we calculate similarly the integral

$$\begin{aligned}&\int d^d\tau\, \Delta_{\alpha\alpha}^2(\tau) = \int d^d\tau\, \Delta_{\alpha\beta}^2(\tau) = \omega^4 \int d^d\tau\, \Delta^2(\tau) - \omega^{4-d}\, c_d^2\, \Gamma\,(d/2)\,\Gamma\,(1-d/2)\, S_d\\ &= \omega^4 \int d^d\tau\, \Delta^2(\tau) - 2\omega^2\, \Delta(0) = -\,(1+d/2)\,\omega^2 \Delta(0).\end{aligned} \tag{10.343}$$

---

[5]I.S. Gradshteyn and I.M. Ryzhik, *op. cit.*, Formula 6.521.3

[6]ibid., Formulas 8.485 and 8.486.12

The first equality follows from partial integrations. In the last equality we have used (10.339). We have omitted the integral containing the modified Bessel functions

$$(d-1)\left[\int_0^\infty dz K_{d/2}(z)K_{1-d/2}(z)+\frac{d}{2}\int_0^\infty dz\, z^{-1}\, K^2_{d/2}(z)\right], \tag{10.344}$$

since this vanishes in one dimension as follows:

$$-\frac{\pi}{4}\Gamma\left(1-\varepsilon/2\right)\left[\Gamma\left(\varepsilon/2\right)+\Gamma\left(-\varepsilon/2\right)\right]\,\varepsilon^2\,\Gamma(\varepsilon)\underset{\varepsilon\to 0}{=}0.$$

Inserting into (10.343) $\Delta(0)=1/2\omega$ from (10.293), we find once more the value of the right-hand Feynman integral (10.250) and the middle one in (10.253).

By combining the result (10.303) with (10.339) and (10.340), we can derive by proper integrations the fundamental rule in this generalized distribution calculus that the integral over the square of the δ-function vanishes. Indeed, solving the inhomogeneous field equation (10.296) for $\delta^{(d)}(\tau)$, and squaring it, we obtain

$$\int d^d\tau\left[\delta^{(d)}(\tau)\right]^2=\omega^4\int d^d\tau\,\Delta^2(\tau)+2\omega^2\int d^d\tau\,\Delta^2_{\alpha}(\tau)+\int d^d\tau\,\Delta^2_{\beta\beta}(\tau)=0. \tag{10.345}$$

Thus we may formally calculate

$$\int d^d\tau\;\delta^{(d)}(\tau)\delta^{(d)}(\tau)=\delta^{(d)}(0)=0, \tag{10.346}$$

pretending that one of the two δ-functions is an admissible smooth test function $f(\tau)$ of ordinary distribution theory, where

$$\int d^d\tau\;\delta^{(d)}(\tau)f(\tau)=f(0). \tag{10.347}$$

10.9.3 Integrals over Products of Four Distributions

The calculation of the configuration space integrals over products of four distributions in $d=1$ dimension is straightforward as long as they are unique. Only if they are ambiguous, they require a calculation in $d=1-\varepsilon$ dimension, with the limit $\varepsilon\to 0$ taken at the end.

A unique case is

$$\begin{aligned}\int d^d\tau\,\Delta^4(\tau) &= c_d^4\,\omega^{-d}S_d\int_0^\infty dy\,y^{3-d}\,K^4_{1-d/2}(y)\\ &\underset{d=1}{=} c_1^4\,\omega^{-1}\,S_1\frac{\pi^2}{2^4}\Gamma^4\left(\frac{3}{2}-\frac{d}{2}\right)\Gamma(d)=\frac{1}{32\omega^5},\end{aligned} \tag{10.348}$$

where we have set

$$y\equiv\omega\tau. \tag{10.349}$$

Similarly, we derive by partial integration

$$\begin{aligned}\int d^d\tau\,\Delta^2(\tau)\Delta^2_{\alpha}(\tau) &= \omega^{2-d}\,c_d^4\,S_d\int_0^\infty dy y^{3-d}\,K^2_{d/2}(y)K^2_{1-d/2}(y)\\ &= \frac{1}{3}\omega^{2-d}\,c_d^4\,S_d\left[2^{-d-1}\,\Gamma(d/2)\,\Gamma^3(1-d/2)\right.\\ &+ \left.\int_0^\infty dy\left(y^{1-d/2}\,K_{1-d/2}\right)^3\frac{d}{dy}\left(y^{d/2}\,K_{d/2}\right)\right]\\ &= \frac{1}{3}\left[\Delta^3(0)-\omega^2\int d^d\tau\,\Delta^4(\tau)\right]\underset{d=1}{=}\frac{1}{32\omega}.\end{aligned} \tag{10.350}$$

Using (10.327), (10.330), and (10.332), we find for the integral in $d = 1-\varepsilon$ dimensions

$$\int d^d\tau\, \Delta(\tau)\Delta_\alpha(\tau)\Delta_\beta(\tau)\Delta_{\alpha\beta}(\tau) = \omega^2 \int d^d\tau\, \Delta^2(\tau)\Delta_\alpha^2(\tau) - \frac{1}{2}Y_d, \tag{10.351}$$

where Y_d is the integral

$$Y_d \;=\; -2(d-1)\omega^{4-d}\, c_d^4\, S_d \int_0^\infty dy y^{2-d}\, K_{1-d/2}(y)K_{d/2}^3(y). \tag{10.352}$$

In spite of the prefactor $d-1$, this has a nontrivial limit for $d \to 1$, the zero being compensated by a pole from the small-y part of the integral at $y = 0$. In order to see this we use the integral representation of the Bessel function [28]:

$$K_\beta(y) \;=\; \pi^{-1/2}\,(y/2)^{-\beta}\,\Gamma\left(\frac{1}{2}+\beta\right)\int_0^\infty dt(\cosh t)^{-2\beta}\,\cos(y\sinh t). \tag{10.353}$$

In one dimension where $\beta = 1/2$, this becomes simply $K_{1/2}(y) = \sqrt{\pi/2y}e^{-y}$. For $\beta = d/2$ and $\beta = 1-d/2$ written as $\beta = (1\mp\varepsilon)/2$, it is approximately equal to

$$\begin{aligned} K_{(1\mp\varepsilon)/2}(y) \;&=\; \pi^{-1/2}\,(y/2)^{-(1\mp\varepsilon)/2}\,\Gamma\left(1\mp\frac{\varepsilon}{2}\right)\\ &\times\; \left[\frac{\pi}{2}e^{-y} \pm \varepsilon\int_0^\infty dt(\cosh t)^{-1}\,\ln(\cosh t)\,\cos(y\sinh t)\right], \end{aligned} \tag{10.354}$$

where the t-integral is regular at $y = 0$.[7] After substituting (10.354) into (10.352), we obtain the finite value

$$\begin{aligned} Y_d \;&\underset{\varepsilon\approx 0}{\approx}\; 2\left(\omega^{4-d}\,c_d^4\,S_d\right)\varepsilon\,\frac{\pi^2}{4}\Gamma\left(1+\varepsilon/2\right)\Gamma^3\left(1-\varepsilon/2\right)\times 2^{-5\varepsilon}\,\Gamma(2\varepsilon)\\ &\underset{\varepsilon\to 0}{=}\; \left(\frac{1}{2\omega\pi^2}\right)\frac{\pi^2}{4} = \frac{1}{8\omega}. \end{aligned} \tag{10.355}$$

The prefactor $d-1 = -\varepsilon$ in (10.352) has been canceled by the pole in $\Gamma(2\varepsilon)$.

This integral coincides with the integral (10.312) whose subtle nature was discussed in the momentum space formulation (10.240). Indeed, inserting the Bessel expressions (10.327) and (10.332) into (10.312), we find

$$\begin{aligned} &\int d^d\tau\;\Delta^2(\tau)\left[\Delta_{\alpha\beta}^2(\tau)-\Delta_{\alpha\alpha}^2(\tau)\right]\\ &\qquad = -(d-1)\omega^{4-d}c_d^4 S_d\int_0^\infty dy\left[y^{1-d/2}\,K_{1-d/2}(y)\right]^2\frac{d}{dy}K_{d/2}^2(y), \end{aligned} \tag{10.356}$$

and a partial integration

$$\int_0^\infty dy\left[y^{1-d/2}(y)\,K_{1-d/2}(y)\right]^2\frac{d}{dy}K_{d/2}^2(y) = 2\int_0^\infty dy\,y^{2-d}\,K_{1-d/2}(y)\,K_{d/2}^3(y) \tag{10.357}$$

establishes contact with the integral (10.352) for Y_d. Thus Eq. (10.351) is the same as (10.321).

Knowing Y_d, we also determine, after integrations by parts, the integral

$$\int d^d\tau\,\Delta_\alpha^2(\tau)\Delta_\beta^2(\tau) = -3\omega^2\int d^d\tau\,\Delta^2(\tau)\Delta_\alpha^2(\tau) + Y_d, \tag{10.358}$$

[7] I.S. Gradshteyn and I.M. Ryzhik, *op. cit.*, Formulas 3.511.1 and 3.521.2.

which is the same as (10.322). It remains to calculate one more unproblematic integral over four distributions:

$$\int d^d\tau\, \Delta^2(\tau)\Delta^2_{\lambda\lambda}(\tau) = \left[-2\omega^2\,\Delta^3(0) + \omega^4 \int d^d\tau\, \Delta^4(\tau)\right]_{d=1} = -\frac{7}{32\omega}. \tag{10.359}$$

Combining this with (10.355) and (10.358) we find the Feynman diagram (10.281). The combination of (10.351) and (10.358) with (10.355) and (10.350), finally, yields the diagrams (10.325) and (10.324), respectively.

Thus we see that there is no problem in calculating integrals over products of distributions in configuration space which produce the same results as dimensional regularization in momentum space.

10.10 Simple Rules for Calculating Singular Integrals

The above methods of calculating the Feynman integrals in d time dimensions with a subsequent limit $d \to 1$ are obviously quite cumbersome. It is preferable to develop a simple procedure of finding the same results directly working with a one-dimensional time. This is possible if we only keep track of some basic aspects of the d-dimensional formulation [37].

Consider once more the ambiguous integrals coming from the first two watermelon diagrams in Eq. (10.232), which in the one-dimensional formulation represent the integrals

$$I_1 = \int_{-\infty}^{\infty} d\tau\, \ddot\Delta^2(\tau)\Delta^2(\tau)\,, \tag{10.360}$$

$$I_2 = \int_{-\infty}^{\infty} d\tau\, \ddot\Delta(\tau)\dot\Delta^2(\tau)\Delta(\tau), \tag{10.361}$$

evaluated before in the d-dimensional equations (10.284) and (10.282). Consider first the integral (10.360) which contains a square of a δ-function. We separate this out by writing

$$I_1 = \int_{-\infty}^{\infty} d\tau\, \ddot\Delta^2(\tau)\Delta^2(\tau) = I_1^{\rm div} + I_1^R, \tag{10.362}$$

with a divergent and a regular part

$$I_1^{\rm div} = \Delta^2(0)\int_{-\infty}^{\infty} d\tau\, \delta^2(\tau)\,, \qquad I_1^R = \int_{-\infty}^{\infty} d\tau\, \Delta^2(\tau)\left[\ddot\Delta^2(\tau) - \delta^2(\tau)\right]. \tag{10.363}$$

All other watermelon diagrams (10.232) lead to the well-defined integrals

$$\int_{-\infty}^{\infty} d\tau\, \Delta^4(\tau) = \frac{1}{4\omega^2}\,\Delta^3(0), \tag{10.364}$$

$$\int_{-\infty}^{\infty} d\tau\, \dot\Delta^2(\tau)\Delta^2(\tau) = \frac{1}{4}\,\Delta^3(0), \tag{10.365}$$

$$\int_{-\infty}^{\infty} d\tau\, \dot\Delta^4(\tau) = \frac{1}{4}\,\omega^2\,\Delta^3(0), \tag{10.366}$$

whose D-dimensional versions are (10.274), (10.286), and (10.282). Substituting these and (10.361), (10.362) into (10.232) yields the sum of all watermelon diagrams

$$
\begin{aligned}
-\frac{4}{2!}\int_{-\infty}^{\infty} d\tau\Big[\Delta^2(\tau)\ddot{\Delta}^2(\tau)+4\Delta(\tau)\dot{\Delta}^2(\tau)\ddot{\Delta}(\tau)+\dot{\Delta}^4(\tau)+4\omega^2\Delta^2(\tau)\dot{\Delta}^2(\tau)+\frac{2}{3}\omega^4\Delta^4(\tau)\Big] \\
= \; - \; 2\Delta^2(0)\int_{-\infty}^{\infty} d\tau\,\delta^2(\tau)-2\left(I_1^R+4I_2\right)-\frac{17}{6}\,\omega^2\Delta^3(0)\,. \qquad (10.367)
\end{aligned}
$$

Adding these to (10.230), (10.231), we obtain the sum of all second-order connected diagrams

$$
\Sigma\,(\text{all}) = 3\left[\delta(0)-\int_{-\infty}^{\infty} d\tau\,\delta^2(\tau)\right]\Delta^2(0)-2\left(I_1^R+4I_2\right)-\frac{7}{2}\,\omega^2\Delta^3(0)\,, \quad (10.368)
$$

where the integrals I_1^R and I_2 are undefined. The sum has to vanish to guarantee coordinate independence. We therefore equate to zero both the singular and finite contributions in Eq. (10.368). The first yields the rule for the product of two δ-functions: $\delta^2(\tau)=\delta(0)\,\delta(\tau)$. This equality should of course be understood in the distributional sense: it holds after multiplying it with an arbitrary test function and integrating over τ.

$$
\int d\tau\,\delta^2(\tau)f(\tau)\equiv\delta(0)f(0). \qquad (10.369)
$$

The equation leads to a perfect cancellation of all powers of $\delta(0)$ arising from the expansion of the Jacobian action, which is the fundamental reason why the heuristic *Veltman rule* of setting $\delta(0)=0$ is applicable everywhere without problems.

The vanishing of the regular parts of (10.368) requires the integrals (10.361) and (10.362) to satisfy

$$
I_1^R+4I_2=-\frac{7}{4}\,\omega^2\Delta^3(0)=-\frac{7}{32\omega}\,. \qquad (10.370)
$$

At this point we run into two difficulties. First, this single equation (10.370) for the two undefined integrals I_1^R and I_2 is insufficient to specify both integrals, so that the requirement of reparametrization invariance alone is not enough to fix all ambiguous temporal integrals over products of distributions. Second, and more seriously, Eq. (10.370) leads to conflicts with standard integration rules based on the use of partial integration and equation of motion, and the independence of the order in which these operations are performed. Indeed, let us apply these rules to the calculation of the integrals I_1^R and I_2 in different orders. Inserting the equation of motion (10.214) into the finite part of the integral (10.362) and making use of the regular integral (10.364), we find immediately

$$
\begin{aligned}
I_1^R &= \int_{-\infty}^{\infty} d\tau\,\Delta^2(\tau)\left[\ddot{\Delta}^2(\tau)-\delta^2(\tau)\right] \\
&= -2\omega^2\,\Delta^3(0)+\omega^4\int_{-\infty}^{\infty} d\tau\,\Delta^4(\tau)=-\frac{7}{4}\omega^2\Delta^3(0)=-\frac{7}{32\omega}\,. \qquad (10.371)
\end{aligned}
$$

The same substitution of the equation of motion (10.214) into the other ambiguous integral I_2 of (10.361) leads, after performing the regular integral (10.365), to

$$\begin{aligned} I_2 &= -\int_{-\infty}^{\infty} d\tau\, \dot\Delta^2(\tau)\,\Delta(\tau)\,\delta(\tau) + \omega^2 \int_{-\infty}^{\infty} d\tau\, \dot\Delta^2(\tau)\,\Delta^2(\tau) \\ &= -\frac{1}{8\omega}\int_{-\infty}^{\infty} d\tau\, \epsilon^2(\tau)\,\delta(\tau) + \frac{1}{4}\omega^2\Delta^3(0) = \frac{1}{8\omega}\left(-I_{\epsilon^2\delta} + \frac{1}{4}\right), \end{aligned} \tag{10.372}$$

where $I_{\epsilon^2\delta}$ denotes the undefined integral over a product of distributions

$$I_{\epsilon^2\delta} = \int_{-\infty}^{\infty} d\tau\, \epsilon^2(\tau)\delta(\tau)\,. \tag{10.373}$$

The integral I_2 can apparently be fixed by applying partial integration to the integral (10.361) which reduces it to the completely regular form (10.366):

$$I_2 = \frac{1}{3}\int_{-\infty}^{\infty} d\tau\, \Delta(\tau)\frac{d}{d\tau}\left[\dot\Delta^3(\tau)\right] = -\frac{1}{3}\int_{-\infty}^{\infty} d\tau\, \dot\Delta^4(\tau) = -\frac{1}{12}\omega^2\Delta^3(0) = -\frac{1}{96\omega}. \tag{10.374}$$

There are no boundary terms due to the exponential vanishing at infinity of all functions involved. From (10.372) and (10.374) we conclude that $I_{\epsilon^2\delta} = 1/3$. This, however, cannot be correct since the results (10.374) and (10.371) do not obey Eq. (10.370) which is needed for coordinate independence of the path integral. This was the reason why previous authors [32, 35] added a noncovariant correction term $\Delta V = -g^2(q^2/6)$ to the classical action (10.186), which is proportional to $\hbar$ and thus violates Feynman's basic postulate that the phase factors $e^{i\mathcal{A}/\hbar}$ in a path integral should contain only the classical action along the paths.

We shall see below that the correct value of the singular integral I in (10.373) is

$$I_{\epsilon^2\delta} = \int_{-\infty}^{\infty} d\tau\, \epsilon^2(\tau)\delta(\tau) \;= 0. \tag{10.375}$$

From the perspective of the previous sections where all integrals were defined in $d = 1 - \epsilon$ dimensions and continued to $\epsilon \to 0$ at the end, the inconsistency of $I_{\epsilon^2\delta} = 1/3$ is obvious: Arbitrary application of partial integration and equation of motion to one-dimensional integrals is forbidden, and this is the case in the calculation (10.374). Problems arise whenever several dots can correspond to different contractions of partial derivatives ∂_α, $\partial_\beta, \ldots$, from which they arise in the limit $d \to 1$. The different contractions may lead to different integrals.

In the pure one-dimensional calculation of the integrals I_1^R and I_2, all ambiguities can be accounted for by using partial integration and equation of motion (10.214) only according to the following integration rules:

Rule 1. We perform a partial integration which allows us to apply subsequently the equation of motion (10.214).

Rule 2. If the equation of motion (10.214) leads to integrals of the type (10.373), they must be performed using naively the Dirac rule for the δ-function and the property $\epsilon(0)=0$. Examples are (10.375) and the trivially vanishing integrals for all odd powers of $\epsilon(\tau)$:

$$\int d\tau\, \epsilon^{2n+1}(\tau)\,\delta(\tau)=0, \quad n=\text{integer}, \tag{10.376}$$

which follow directly from the antisymmetry of $\epsilon^{2n+1}(\tau)$ and the symmetry of $\delta(\tau)$ contained in the regularized expressions (10.330) and (10.332).

RULE 3. The above procedure leaves in general singular integrals, which must be treated once more with the same rules.

Let us show that calculating the integrals I_1^R and I_2 with these rules is consistent with the coordinate independence condition (10.370). In the integral I_2 of (10.361) we first apply partial integration to find

$$\begin{aligned} I_2 &= \frac{1}{2}\int_{-\infty}^{\infty} d\tau\,\Delta(\tau)\,\dot\Delta(\tau)\frac{d}{d\tau}\left[\dot\Delta^2(\tau)\right] \\ &= -\frac{1}{2}\int_{-\infty}^{\infty} d\tau\,\dot\Delta^4(\tau)-\frac{1}{2}\int_{-\infty}^{\infty} d\tau\,\Delta(\tau)\,\dot\Delta^2(\tau)\,\ddot\Delta(\tau)\,, \end{aligned} \tag{10.377}$$

with no contributions from the boundary terms. Note that the partial integration (10.374) is forbidden since it does not allow for a subsequent application of the equation of motion (10.214). On the right-hand side of (10.377) it can be applied. This leads to a combination of two regular integrals (10.365) and (10.366) and the singular integral I, which we evaluate with the naive Dirac rule to $I=0$, resulting in

$$\begin{aligned} I_2 &= -\frac{1}{2}\int_{-\infty}^{\infty} d\tau\,\dot\Delta^4(\tau)+\frac{1}{2}\int_{-\infty}^{\infty} d\tau\,\dot\Delta^2(\tau)\,\Delta(\tau)\,\delta(\tau)-\frac{1}{2}\omega^2\int_{-\infty}^{\infty} d\tau\,\dot\Delta^2(\tau)\,\Delta^2(\tau) \\ &= \frac{1}{16\omega}\,I-\frac{1}{4}\omega^2\Delta^3(0)=-\frac{1}{32\omega}\,. \end{aligned} \tag{10.378}$$

If we calculate the finite part I_1^R of the integral (10.362) with the new rules we obtain a result different from (10.371). Integrating the first term in brackets by parts and using the equation of motion (10.214), we obtain

$$\begin{aligned} I_1^R &= \int_{-\infty}^{\infty} d\tau\Delta^2(\tau)\left[\ddot\Delta^2(\tau)-\delta^2(\tau)\right] \\ &= \int_{-\infty}^{\infty} d\tau\left[-\dddot\Delta(\tau)\,\dot\Delta(\tau)\,\Delta^2(\tau)-2\ddot\Delta(\tau)\,\dot\Delta^2(\tau)\,\Delta(\tau)-\Delta^2(\tau)\,\delta^2(\tau)\right] \\ &= \int_{-\infty}^{\infty} d\tau\left[\dot\Delta(\tau)\,\Delta^2(\tau)\,\dot\delta(\tau)-\Delta^2(\tau)\,\delta^2(\tau)\right]-2I_2-\omega^2\int_{-\infty}^{\infty} d\tau\dot\Delta^2(\tau)\,\Delta^2(\tau)\,. \end{aligned} \tag{10.379}$$

The last two terms are already known, while the remaining singular integral in brackets must be subjected once more to the same treatment. It is integrated by parts so that the equation of motion (10.214) can be applied to obtain

$$\begin{aligned} \int_{-\infty}^{\infty} d\tau\left[\dot\Delta(\tau)\,\Delta^2(\tau)\,\dot\delta(\tau)-\Delta^2(\tau)\,\delta^2(\tau)\right] &= -\int_{-\infty}^{\infty} d\tau\left[\ddot\Delta(\tau)\,\Delta^2(\tau)+2\dot\Delta^2(\tau)\,\Delta(\tau)\right]\delta(\tau) \\ &\quad -\int_{-\infty}^{\infty} d\tau\Delta^2(\tau)\,\delta^2(\tau)=-\omega^2\,\Delta^3(0)-\frac{1}{4\omega}\,I\,. \end{aligned} \tag{10.380}$$

Inserting this into Eq. (10.379) yields

$$I_1^R = \int_{-\infty}^{\infty} d\tau \Delta^2(\tau) \left[\ddot{\Delta}^2(\tau) - \delta^2(\tau)\right] = -2I_2 - \frac{5}{4}\omega^2 \, \Delta^3(0) - \frac{1}{4\omega} I = -\frac{3}{32\omega} \, , \tag{10.381}$$

the right-hand side following from $I = 0$, which is a consequence of Rule 3.

We see now that the integrals (10.378) and (10.381) calculated with the new rules obey Eq. (10.370) which guarantees coordinate independence of the path integral.

The applicability of Rules 1–3 follows immediately from the previously established dimensional continuation [23, 24]. It avoids completely the cumbersome calculations in $1 - \varepsilon$-dimension with the subsequent limit $\varepsilon \to 0$. Only some intermediate steps of the derivation require keeping track of the d-dimensional origin of the rules. For this, we continue the imaginary time coordinate τ to a d-dimensional spacetime vector $\tau \to \tau^\mu = (\tau^0, \tau^1, \dots, \tau^{d-1})$, and note that the equation of motion (10.214) becomes a scalar field equation of the Klein-Gordon type

$$\left(-\partial_\alpha^2 + \omega^2\right) \Delta(\tau) = \delta^{(d)}(\tau) \, . \tag{10.382}$$

In d dimensions, the relevant second-order diagrams are obtained by decomposing the harmonic expectation value

$$\int d^d\tau \left\langle q_\alpha^2(\tau) \, q^2(\tau) \, q_\beta^2(0) \, q^2(0) \right\rangle \tag{10.383}$$

into a sum of products of four two-point correlation functions according to the Wick rule. The fields $q_\alpha(\tau)$ are the d-dimensional extensions $q_\alpha(\tau) \equiv \partial_\alpha q(\tau)$ of $\dot{q}(\tau)$. Now the d-dimensional integrals, corresponding to the integrals (10.360) and (10.361), are defined uniquely by the contractions

$$\begin{aligned} I_1^d &= \int d^d\tau \left\langle q_\alpha(\tau) q_\alpha(\tau) q(\tau) q(\tau) q_\beta(0) q_\beta(0) q(0) q(0) \right\rangle \\ &= \int d^d\tau \, \Delta^2(\tau) \, \Delta_{\alpha\beta}^2(\tau) \, , \end{aligned} \tag{10.384}$$

$$\begin{aligned} I_2^d &= \int d^d\tau \left\langle q_\alpha(\tau) q_\alpha(\tau) q(\tau) q(\tau) q_\beta(0) q_\beta(0) q(0) q(0) \right\rangle \\ &= \int d^d\tau \, \Delta(\tau) \, \Delta_\alpha(\tau) \, \Delta_\beta(\tau) \, \Delta_{\alpha\beta}(\tau) \, . \end{aligned} \tag{10.385}$$

The different derivatives $\partial_\alpha \partial_\beta$ acting on $\Delta(\tau)$ prevent us from applying the field equation (10.382). This obstacle was hidden in the one-dimensional formulation. It can be overcome by a partial integration. Starting with I_2^d, we obtain

$$I_2^d = -\frac{1}{2} \int d^d\tau \, \Delta_\beta^2(\tau) \left[\Delta_\alpha^2(\tau) + \Delta(\tau) \, \Delta_{\alpha\alpha}(\tau)\right] \, . \tag{10.386}$$

Treating I_1^d likewise we find

$$I_1^d = -2I_2^d + \int d^d\tau \, \Delta^2(\tau) \, \Delta_{\alpha\alpha}^2(\tau) + 2 \int d^d\tau \, \Delta(\tau) \, \Delta_\beta^2(\tau) \, \Delta_{\alpha\alpha}(\tau) \, . \tag{10.387}$$

In the second equation we have used the fact that $\partial_\alpha \Delta_{\alpha\beta} = \partial_\beta \Delta_{\lambda\lambda}$. The right-hand sides of (10.386) and (10.387) contain now the contracted derivatives ∂_α^2 such that we can apply the field equation (10.382). This mechanism works to all orders in the perturbation expansion which is the reason for the applicability of Rules 1 and 2 which led to the results (10.378) and (10.381) ensuring coordinate independence.

The value $I_{\epsilon^2\delta} = 0$ according to the Rule 2 can be deduced from the regularized equation (10.386) in $d = 1 - \varepsilon$ dimensions by using the field equation (10.335) to rewrite I_2^d as

$$\begin{aligned} I_2^d &= -\frac{1}{2}\int d^d\tau \, \Delta_\beta^2(\tau)\left[\Delta_\alpha^2(\tau)+\omega^2\Delta^2(\tau)-\Delta(\tau)\delta^{(d)}(\tau)\right] \\ &\underset{d\approx 1}{\approx} -\frac{1}{32\omega}+\frac{1}{2}\int d^d\tau \, \Delta_\beta^2(\tau)\Delta_\alpha^2(\tau)\Delta(\tau)\delta^{(d)}(\tau) \, . \end{aligned}$$

Comparison with (10.372) yields the regularized expression for $I_{\epsilon^2\delta}$

$$I_{\epsilon^2\delta}^R = \left[\int_{-\infty}^{\infty} d\tau \, \epsilon^2(\tau)\delta(\tau)\right]^R = 8\omega \int d^d\tau \, \Delta_\beta^2(\tau)\, \Delta(\tau)\delta^{(d)}(\tau) = 0, \tag{10.388}$$

the vanishing for all $\varepsilon > 0$ being a consequence of the small-τ behavior $\Delta(\tau)\, \Delta_\alpha^2(\tau) \propto |\tau|^{2\varepsilon}$, which follows directly from (10.331).

Let us briefly discuss an alternative possibility of giving up partial integration completely in ambiguous integrals containing ϵ- and δ-function, or their time derivatives, which makes unnecessary to satisfy Eq. (10.374). This yields a freedom in the definition of integral over product of distribution (10.373) which can be used to fix $I_{\epsilon^2\delta} = 1/4$ from the requirement of coordinate independence [25]. Indeed, this value of I would make the integral (10.372) equal to $I_2 = 0$, such that (10.370) would be satisfied and coordinate independence ensured. In contrast, giving up partial integration, the authors of Refs. [31, 33] have assumed the vanishing $\epsilon^2(\tau)$ at $\tau = 0$ so that the integral $I_{\epsilon^2\delta}$ should vanish as well: $I_{\epsilon^2\delta} = 0$. Then Eq. (10.372) yields $I_2 = 1/32\omega$ which together with (10.371) does not obey the coordinate independence condition (10.370), making yet an another noncovariant quantum correction $\Delta V = g^2(q^2/2)$ necessary in the action, which we reject since it contradicts Feynman's original rules of path integration. We do not consider giving up partial integration as an attractive option since it is an important tool for calculating higher-loop diagrams.

10.11 Perturbative Calculation on Finite Time Intervals

The above calculation rules can be extended with little effort to path integrals of time evolution amplitudes on finite time intervals. We shall use an imaginary time interval with $\tau_a = 0$ and $\tau_b = \beta$ to have the closest connection to statistical mechanics. The ends of the paths will be fixed at τ_a and τ_b to be able to extract quantum-mechanical time evolution amplitudes by a mere replacement $\tau \to -it$. The extension to a finite time interval is nontrivial since the Feynman integrals in frequency space become sums over discrete frequencies whose d-dimensional generalizations can usually not be evaluated with standard formulas. The above ambiguities of the integrals, however, will appear in the sums in precisely the same way as before. The reason is that they stem from ordering ambiguities between q and $\dot{q}$ in the perturbation expansions. These are properties of small time intervals and thus of high frequencies, where the sums can be approximated by integrals. In fact, we have seen in the last section, that all ambiguities can be resolved by a careful treatment of the singularities of the correlation functions at small temporal spacings. For integrals on a time axis it is thus completely irrelevant whether the total time interval is finite or infinite, and the ambiguities can be resolved in the same way as before [38].

This can also be seen technically by calculating the frequency sums in the Feynman integrals of finite-time path integrals with the help of the Euler-Maclaurin formula (2.594) or the equivalent ζ-function methods described in Subsection 2.15.6. The lowest approximation involves the pure frequency integrals whose ambiguities have been resolved in the preceeding sections. The remaining correction terms in powers of the temperature $T = 1/\beta$ are all unique and finite [see Eq. (2.598) or (2.558)].

The calculations of the Feynman integrals will most efficiently proceed in configuration space as described in Subsection 10.8.1. Keeping track of certain minimal features of the unique definition of all singular integrals in d dimensions, we shall develop reduction rules based on the equation of motion and partial integration. These will allow us to bring all singular Feynman integrals to a regular form in which the integrations can be done directly in one dimension. The integration rules will be in complete agreement with much more cumbersome calculations in d dimensions with the limit $d \to 1$ taken at the end.

10.11.1 Diagrammatic Elements

The perturbation expansion for an evolution amplitude over a finite imaginary time proceeds as described in Section 10.6, except that the free energy in Eq. (10.201) becomes [recall (2.526)]

$$\begin{aligned}\beta F_\omega &= \frac{D}{2\beta}\mathrm{Tr}\log(-\partial^2+\omega^2) = \frac{D}{2\beta}\sum_n \log(\omega_n^2+\omega^2)\\ &= \frac{D}{\beta}\log\left[2\sinh\left(\hbar\beta\omega/2\right)\right].\end{aligned}\tag{10.389}$$

As before, the diagrams contain four types of lines representing the correlation functions (10.202)–(10.191). Their explicit forms are, however, different. It will be convenient to let the frequency ω in the free part of the action (10.187) go to zero. Then the free energy (10.389) diverges logarithmically in ω. This divergence is, however, trivial. As explained in Section 2.9, the divergence is removed by replacing ω by the length of the q-axis according to the rule (2.361). For finite time intervals, the correlation functions are no longer given by (10.207) which would not have a finite limit for $\omega \to 0$. Instead, they satisfy Dirichlet boundary conditions, where we can go to $\omega = 0$ without problem. The finiteness of the time interval removes a possible infrared divergence for $\omega \to 0$. The Dirichlet boundary conditions fix the paths at the ends of the time interval $(0,\beta)$ making the fluctuations vanish, and thus also their correlation functions:

$$G^{(2)}_{\mu\nu}(0,\tau') = G^{(2)}_{\mu\nu}(\beta,\tau') = 0, \qquad G^{(2)}_{\mu\nu}(\tau,0) = G^{(2)}_{\mu\nu}(\tau,\beta) = 0.\tag{10.390}$$

The first correlation function corresponding to (10.206) is now

$$G^{(2)}_{\mu\nu}(\tau,\tau') = \delta_{\mu\nu}\Delta(\tau,\tau') = \quad \text{———} \quad,\tag{10.391}$$

where

$$\Delta(\tau,\tau') = \Delta(\tau',\tau) = \frac{1}{\beta}(\beta-\tau_>)\,\tau_< = \frac{1}{2}\left[-\epsilon(\tau-\tau')(\tau-\tau')+\tau+\tau'\right] - \frac{\tau\tau'}{\beta},\tag{10.392}$$

abbreviates the Euclidean version of $G_0(t,t')$ in Eq. (3.39). Being a Green function of the free equation of motion (10.213) for $\omega = 0$, this satisfies the inhomogeneous differential equations

$$\ddot{}\Delta(\tau,\tau') = \Delta\ddot{}(\tau,\tau') = -\delta(\tau-\tau'),\tag{10.393}$$

by analogy with Eq. (10.214) for $\omega = 0$. In addition, there is now an independent equation in which the two derivatives act on the different time arguments:

$$ {}^{\cdot}\Delta^{\cdot}(\tau,\tau') = \delta(\tau-\tau') - 1/\beta. \tag{10.394}$$

For a finite time interval, the correlation functions (10.203) (10.204) differ by more than just a sign [recall (10.210)]. We therefore must distinguish the derivatives depending on whether the left or the right argument are differentiated. In the following, we shall denote the derivatives with respect to τ or τ' by a dot on the left or right, respectively, writing

$$ {}^{\cdot}\Delta\,(\tau,\tau') \equiv \frac{d}{d\tau}\,\Delta(\tau,\tau'), \qquad \Delta^{\cdot}(\tau,\tau') \equiv \frac{d}{d\tau'}\,\Delta(\tau,\tau'). \tag{10.395}$$

Differentiating (10.392) we obtain explicitly

$$ {}^{\cdot}\Delta\,(\tau,\tau') = -\frac{1}{2}\epsilon(\tau-\tau') + \frac{1}{2} - \frac{\tau'}{\beta}, \quad \Delta^{\cdot}(\tau,\tau') = \frac{1}{2}\epsilon(\tau-\tau') + \frac{1}{2} - \frac{\tau}{\beta} = {}^{\cdot}\Delta\,(\tau',\tau)\,. \tag{10.396}$$

The discontinuity at $\tau = \tau'$ which does not depend on the boundary condition is of course the same as before, The two correlation functions (10.208) and (10.210) and their diagrammatic symbols are now

$$ \partial_\tau G^{(2)}_{\mu\nu}(\tau,\tau') \equiv \langle \dot q_\mu(\tau) q_\nu(\tau')\rangle = \delta_{\mu\nu}\,{}^{\cdot}\Delta\,(\tau,\tau') \;=\; , \tag{10.397}$$

$$ \partial_{\tau'} G^{(2)}_{\mu\nu}(\tau,\tau') \equiv \langle q_\mu(\tau) \dot q_\nu(\tau')\rangle = \delta_{\mu\nu}\,\Delta^{\cdot}(\tau,\tau') \;=\; . \tag{10.398}$$

The fourth correlation function (10.212) is now

$$ \partial_\tau\partial_{\tau'} G^{(2)}_{\mu\nu}(\tau,\tau') = \delta_{\mu\nu}\,{}^{\cdot}\Delta^{\cdot}(\tau,\tau') = \;, \tag{10.399}$$

with ${}^{\cdot}\Delta^{\cdot}(\tau,\tau')$ being given by (10.394). Note the close similarity but also the difference of this with respect to the equation of motion (10.393).

10.11.2 Cumulant Expansion of D-Dimensional Free-Particle Amplitude in Curvilinear Coordinates

We shall now calculate the partition function of a point particle in curved space for a finite time interval. Starting point is the integral over the diagonal amplitude of a free point particle of unit mass $(\mathbf{x}_a\beta|\mathbf{x}_a 0)$ in flat D-dimensional space

$$ Z = \int d^D x_a\, (\mathbf{x}_a\beta|\mathbf{x}_a 0), \tag{10.400}$$

with the path integral representation

$$ (\mathbf{x}_a\beta|\mathbf{x}_a 0)_0 = \int \mathcal{D}^D x\, e^{-\mathcal{A}^{(0)}[\mathbf{x}]}, \tag{10.401}$$

where $\mathcal{A}^{(0)}[\mathbf{x}]$ is the free-particle action

$$ \mathcal{A}^{(0)}[x] = \frac{1}{2}\int_0^\beta d\tau\, \dot{\mathbf{x}}^2(\tau). \tag{10.402}$$

Performing the Gaussian path integral leads to

$$ (\mathbf{x}_a\beta|\mathbf{x}_a 0)_0 = e^{-(D/2)\mathrm{Tr}\log(-\partial^2)} = [2\pi\beta]^{-D/2}\,, \tag{10.403}$$

where the trace of the logarithm is evaluated with Dirichlet boundary conditions. The result is of course the D-dimensional imaginary-time version of the fluctuation factor (2.130) in natural units. A coordinate transformation $x^i(\tau) = x^i(q^\mu(\tau))$ mapping $\mathbf{x}_a$ to q_a^μ brings the action (10.402) to the form (10.186) with $V(q(\tau)) = 0$:

$$\mathcal{A}[q] = \frac{1}{2}\int_0^\beta d\tau\, g_{\mu\nu}(q(\tau))\dot{q}^\mu(\tau)\dot{q}^\nu(\tau), \quad \text{with} \quad g_{\mu\nu}(q) \equiv \frac{\partial x^i(q)}{\partial q^\mu}\frac{\partial x^i(q)}{\partial q^\nu}. \tag{10.404}$$

In the formal notation (10.189), the measure transforms as follows:

$$\int \mathcal{D}^D x(\tau) \equiv \prod_\tau \int d^D x(\tau) = J \prod_\tau \int d^D q(\tau) \equiv J \int \mathcal{D}^D q\, \sqrt{g(q_a)}\,, \tag{10.405}$$

where $g(q) \equiv \det g_{\mu\nu}(q)$ and J is the Jacobian of the coordinate transformation generalizing (10.220) and (10.219)

$$J = \prod_\tau \left[\sqrt{\frac{\partial x^i(q(\tau))}{\delta q^\mu(\tau)}} \Bigg/ \sqrt{\frac{\partial x^i(q_a)}{\delta q_0^\mu}}\right] = \exp\left[\frac{1}{2}\delta(0)\int_0^\beta d\tau \log\frac{g(q(\tau))}{g(q_a)}\right]. \tag{10.406}$$

Thus we may write the transformed path integral (10.401) in the form

$$(\mathbf{x}_a\beta|\mathbf{x}_a 0)_0 \equiv (q_a\,\beta|q_a\,0)_0 = \int \mathcal{D}^D q\, e^{-\mathcal{A}_{\rm tot}[x]}, \tag{10.407}$$

with the total action in the exponent

$$\mathcal{A}_{\rm tot}[q] = \int_0^\beta d\tau \left[\frac{1}{2}g_{\mu\nu}(q(\tau))\dot{q}^\mu(\tau)\dot{q}^\nu(\tau) - \frac{1}{2}\delta(0)\log\frac{g(q(\tau))}{g(q_a)}\right]. \tag{10.408}$$

Following the rules described in Subsection 10.6.1 we expand the action in powers of $\delta q^\mu(\tau) = q^\mu(\tau) - q_a^\mu$. The action can then be decomposed into a free part

$$\mathcal{A}^{(0)}[q_a, \delta q] = \frac{1}{2}\int_0^\beta d\tau\, g_{\mu\nu}(q_a)\delta\dot{q}^\mu(\tau)\delta\dot{q}^\nu(\tau) \tag{10.409}$$

and an interacting part written somewhat more explicitly than in (10.194) with (10.188) and (10.193):

$$\begin{aligned}\mathcal{A}_{\rm tot}^{\rm int}[q_a, \delta q] \;&=\; \int_0^\beta d\tau \frac{1}{2}[g_{\mu\nu}(q) - g_{\mu\nu}(q_a)]\delta\dot{q}^\mu\delta\dot{q}^\nu \\ &-\; \int_0^\beta d\tau \frac{1}{2}\delta(0)\left\{\left[\frac{g(q_a+\delta q)}{g(q_a)} - 1\right] - \frac{1}{2}\left[\frac{g(q_a+\delta q)}{g(q_a)} - 1\right]^2 + \dots\right\}.\end{aligned} \tag{10.410}$$

For simplicity, we assume the coordinates to be orthonormal at q_a^μ, i.e., $g_{\mu\nu}(q_a) = \delta_{\mu\nu}$. The path integral (10.407) is now formally defined by a perturbation expansion similar to (10.199):

$$\begin{aligned}(q_a\,\beta|q_a\,0) \;&=\; \int \mathcal{D}^D q\, e^{\mathcal{A}^{(0)}[q] - \mathcal{A}_{\rm tot}^{\rm int}[q]} = \int \mathcal{D}^D q\, e^{-\mathcal{A}^{(0)}[q]}\left(1 - \mathcal{A}_{\rm tot}^{\rm int} + \frac{1}{2}\mathcal{A}_{\rm tot}^{\rm int\,2} - \dots\right) \\ &=\; (2\pi\beta)^{-D/2}\left[1 - \left\langle\mathcal{A}_{\rm tot}^{\rm int}\right\rangle + \frac{1}{2}\left\langle\mathcal{A}_{\rm tot}^{\rm int\,2}\right\rangle - \dots\right], \\ &=\; (2\pi\beta)^{-D/2}\exp\left\{-\left\langle\mathcal{A}_{\rm tot}^{\rm int}\right\rangle_c + \frac{1}{2}\left\langle\mathcal{A}_{\rm tot}^{\rm int\,2}\right\rangle_c - \dots\right\} \equiv e^{-\beta f(q)},\end{aligned} \tag{10.411}$$

with the harmonic expectation values

$$\langle\dots\rangle = (2\pi\beta)^{D/2}\int \mathcal{D}^D q(\tau)(\dots)e^{-\mathcal{A}^{(0)}[q]} \tag{10.412}$$

and their cumulants $\left\langle \mathcal{A}_{\rm tot}^{\rm int\,2}\right\rangle_c = \left\langle \mathcal{A}_{\rm tot}^{\rm int\,2}\right\rangle - \left\langle \mathcal{A}_{\rm tot}^{\rm int}\right\rangle^2, \dots$ [recall (3.485), (3.486)], containing only connected diagrams. To emphasize the analogy with the cumulant expansion of the free energy in (10.199), we have defined the exponent in (10.411) as $-\beta f(q)$. This q-dependent quantity $f(q)$ is closely related to the alternative effective classical potential discussed in Subsection 3.25.4, apart from a normalization factor:

$$e^{-\beta f(q)} = \frac{1}{\sqrt{2\pi\hbar^2/Mk_BT}} e^{-\beta \tilde V_\omega^{\rm eff\,cl}(q)}. \tag{10.413}$$

If our calculation procedure respects coordinate independence, all expansion terms of $\beta f(q)$ must vanish to yield the trivial exact results (10.401).

10.11.3 Propagator in $1-\varepsilon$ Time Dimensions

In the dimensional regularization of the Feynman integrals on an infinite time interval in Subsection 10.7.2 we have continued all Feynman diagrams in momentum space to $d = 1-\varepsilon$ time dimensions. For the present Dirichlet boundary conditions, this standard continuation of quantum field theory is not directly applicable since the integrals in momentum space become sums over discrete frequencies $\nu_n = \pi n/\beta$ [compare (3.64)]. For such sums one has to set up completely new rules for a continuation, and there are many possibilities for doing this. Fortunately, it will not be necessary to make a choice since we can use the method developed in Subsection 10.10 to avoid continuations altogether. and work in a single physical time dimension. For a better understanding of the final procedure it is, however, useful to see how a dimensional continuation could proceed. We extend the imaginary time coordinate τ to a d-dimensional spacetime vector whose zeroth component is τ: $z^\mu = (\tau, z^1, \dots, z^{d-1})$. In $d = 1-\varepsilon$ dimensions, the extended correlation function reads

$$\Delta(\tau, \mathbf{z};\, \tau', \mathbf{z}') = \int \frac{d^\varepsilon k}{(2\pi)^\varepsilon} e^{i\mathbf{k}(\mathbf{z}-\mathbf{z}')}\Delta_\omega(\tau,\tau'), \qquad \text{where} \quad \omega \equiv |\mathbf{k}|. \tag{10.414}$$

Here the extra ε-dimensional space coordinates $\mathbf{z}$ are assumed to live on infinite axes with translational invariance along all directions. Only the original τ-coordinate lies in a finite interval $0 \le \tau \le \beta$, with Dirichlet boundary conditions. The Fourier component in the integrand $\Delta_\omega(\tau,\tau')$ is the usual one-dimensional correlation function of a harmonic oscillator with the $\mathbf{k}$-dependent frequency $\omega = |\mathbf{k}|$. It is the Green function which satisfies on the finite τ-interval the equation of motion

$$-\ddot{\Delta}_\omega(\tau,\tau') + \omega^2\Delta_\omega(\tau,\tau') = \delta(\tau-\tau'), \tag{10.415}$$

with Dirichlet boundary conditions

$$\Delta_\omega(0,\tau) = \Delta_\omega(\beta,\tau) = 0. \tag{10.416}$$

The explicit form was given in Eq. (3.36) for real times. Its obvious continuation to imaginary-time is

$$\Delta_\omega(\tau,\tau') = \frac{\sinh\omega(\beta-\tau_>)\,\sinh\omega\tau_<}{\omega\sinh\omega\beta}, \tag{10.417}$$

where $\tau_>$ and $\tau_<$ denote the larger and smaller of the imaginary times τ and τ', respectively.

In d time dimensions, the equation of motion (10.393) becomes a scalar field equation of the Klein-Gordon type. Using Eq. (10.415) we obtain

$$\begin{aligned} {}_{\mu\mu}\Delta(\tau,\mathbf{z};\,\tau',\mathbf{z}') &= \Delta_{\mu\mu}(\tau,\mathbf{z};\,\tau',\mathbf{z}') = \ddot{\Delta}(\tau,\mathbf{z};\,\tau',\mathbf{z}') + {}_{\mathbf{zz}}\Delta(\tau,\mathbf{z};\,\tau',\mathbf{z}') \\ &= \int \frac{d^\varepsilon k}{(2\pi)^\varepsilon} e^{i\mathbf{k}(\mathbf{z}-\mathbf{z}')}\left[\ddot{\Delta}_\omega(\tau,\tau') - \omega^2\Delta_\omega(\tau,\tau')\right] = \\ &= -\,\delta(\tau-\tau')\,\delta^{(\varepsilon)}(\mathbf{z}-\mathbf{z}') \equiv -\,\delta^{(d)}(z-z'). \end{aligned} \tag{10.418}$$

The important observation is now that for d spacetime dimensions, perturbation expansion of the path integral yields for the second correlation function ${}^{\cdot}\Delta^{\cdot}(\tau,\tau')$ in Eqs. (10.515) and (10.516) the extension ${}_{\mu}\Delta_{\nu}(z,z')$. This function differs from the contracted function ${}_{\mu}\Delta_{\mu}(z,z')$, and from ${}_{\mu\mu}\Delta(z,z')$ which satisfies the field equation (10.418). In fact, all correlation functions ${}^{\cdot}\Delta^{\cdot}(\tau,\tau')$ encountered in the diagrammatic expansion which have different time arguments always turn out to have the d-dimensional extension ${}_{\mu}\Delta_{\nu}(z,z')$. An important exception is the correlation function at *equal* times ${}^{\cdot}\Delta^{\cdot}(\tau,\tau)$ whose d-dimensional extension is always ${}_{\mu}\Delta_{\mu}(z,z)$, which satisfies the right-hand equation (10.393) in the $\varepsilon\to 0$ -limit. Indeed, it follows from Eq. (10.414) that

$$
{}_{\mu}\Delta_{\mu}(z,z)=\int\frac{d^{\varepsilon}k}{(2\pi)^{\varepsilon}}\left[{}^{\cdot}\Delta^{\cdot}{}_{\omega}(\tau,\tau)+\omega^{2}\,\Delta_{\omega}(\tau,\tau)\right]. \tag{10.419}
$$

With the help of Eq. (10.417), the integrand in Eq. (10.419) can be brought to

$$
{}^{\cdot}\Delta^{\cdot}{}_{\omega}(\tau,\tau)+\omega^{2}\Delta_{\omega}(\tau,\tau)=\delta(0)-\frac{\omega\cosh\omega(2\tau-\beta)}{\sinh\omega\beta}. \tag{10.420}
$$

Substituting this into Eq. (10.419), we obtain

$$
{}_{\mu}\Delta_{\mu}(z,z)=\delta^{(d)}(z,z)-I^{\varepsilon}. \tag{10.421}
$$

The integral I^{ε} is calculated as follows

$$
\begin{aligned}
I^{\varepsilon} &= \int\frac{d^{\varepsilon}k}{(2\pi)^{\varepsilon}}\frac{\omega\cosh\omega(2\tau-\beta)}{\sinh\omega\beta}=\frac{1}{\beta}\frac{S_{\varepsilon}}{(2\pi\beta)^{\varepsilon}}\int_{0}^{\infty}dz z^{\varepsilon}\frac{\cosh z(1-2\tau/\beta)}{\sinh z} \\
&= \frac{1}{\beta}\frac{S_{\varepsilon}}{(2\pi\beta)^{\varepsilon}}\frac{\Gamma(\varepsilon+1)}{2^{\varepsilon+1}}\left[\zeta\left(\varepsilon+1,1-\tau/\beta\right)+\zeta\left(\varepsilon+1,\tau/\beta\right)\right],
\end{aligned} \tag{10.422}
$$

where $S_{\varepsilon}=2\pi^{\varepsilon/2}/\Gamma(\varepsilon/2)$ is the surface of a unit sphere in ε dimension [recall Eq. (1.558)], and $\Gamma(z)$ and $\zeta(z,q)$ are gamma and zeta functions, respectively. For small $\varepsilon\to 0$, they have the limits $\zeta(\varepsilon+1,q)\to 1/\varepsilon-\psi(q)$, and $\Gamma(\varepsilon/2)\to 2/\varepsilon$, so that $I^{\varepsilon}\to 1/\beta$, proving that the d-dimensional equation (10.421) at coinciding arguments reduces indeed to the one-dimensional equation (10.393). The explicit d-dimensional form will never be needed, since we can always treat ${}_{\mu}\Delta_{\mu}(z,z)$ as one-dimensional functions ${}^{\cdot}\Delta^{\cdot}(\tau,\tau)$, which can in turn be replaced everywhere by the right-hand side $\delta(0)-1/\beta$ of (10.394).

10.11.4 Coordinate Independence for Dirichlet Boundary Conditions

Before calculating the path integral (10.411) in curved space with Dirichlet boundary conditions, let us first verify its coordinate independence following the procedure in Section 10.7. Thus we consider the perturbation expansion of the short-time amplitude of a free particle in one general coordinate. The free action is (10.221), and the interactions (10.222) and (10.224), all with $\omega=0$. Taking the parameter $a=1$, the actions are

$$
\mathcal{A}^{(0)}[q]=\frac{1}{2}\int_{0}^{\beta}d\tau\,\dot{q}^{2}(\tau), \tag{10.423}
$$

$$
\tag{10.424}
$$

and

$$
\mathcal{A}^{\rm int}_{\rm tot}=\int_{0}^{\beta}d\tau\left\{\left[-\eta q^{2}(\tau)+\frac{3}{2}\eta^{2}q^{4}(\tau)\right]\dot{q}^{2}(\tau)-\delta(0)\left[-\eta q^{2}(\tau)+\frac{1}{2}\eta^{2}q^{4}(\tau)\right]\right\}. \tag{10.425}
$$

We calculate the cumulants $\left\langle\mathcal{A}^{\rm int}_{\rm tot}\right\rangle_{c}=\left\langle\mathcal{A}^{\rm int}_{\rm tot}\right\rangle$, $\left\langle\mathcal{A}^{\rm int\,2}_{\rm tot}\right\rangle_{c}=\left\langle\mathcal{A}^{\rm int\,2}_{\rm tot}\right\rangle-\left\langle\mathcal{A}^{\rm int}_{\rm tot}\right\rangle^{2},\dots$ [recall (3.485), (3.486)] contributing to th quantity βf in Eq. (10.411) order by order in η. For a better comparison with the previous expansion in Subsection 10.7.1 we shall denote the diagrammatic contributions which are analogous to the different free energy terms $\beta F_n^{(m)}$ of order n by corresponding symbols $\beta f_n^{(m)}$. There are two main differences with respect to Subsection 10.7.1: All diagrams with a prefactor ω are absent, and there are new diagrams involving the correlation functions at equal times $\langle \dot q^\mu(\tau) q^\nu(\tau)\rangle$ and $\langle q^\mu(\tau)\dot q^\nu(\tau)\rangle$ which previously vanished because of (10.211). Here they have the nonzero value ${}^{\cdot}\Delta(\tau,\tau) = \Delta^{\cdot}(\tau,\tau) = 1/2 - \tau/\beta$ by Eq. (10.396). To first order in η, the quantity $f(q)$ in Eq. (10.411) receives a contribution from the first cumulant of the linear terms in η of the interaction (10.425):

$$\beta f_1 = \left\langle \mathcal{A}_{\rm tot}^{\rm int}\right\rangle_c = \eta \int_0^\beta d\tau \left\langle -q^2(\tau)\dot q^2(\tau) + \delta(0) q^2(\tau)\right\rangle + \mathcal{O}(\eta^2). \tag{10.426}$$

There exists only three diagrams, two originating from the kinetic term and one from the Jacobian action:

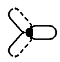
$$\beta f_1 = -\eta$$  $$-2\eta$$  $$+\eta\,\delta(0)$$  . (10.427)

Note the difference with respect to the diagrams (10.227) for infinite time interval with ω^2-term in the action.

The omitted η^2-terms in (10.426) yield the second-order contribution

$$\beta f_2^{(1)} = \eta^2 \int_0^\beta d\tau \left\langle \frac{3}{2}\, q^4(\tau)\dot q^2(\tau) - \delta(0)\frac{1}{2}\, q^4(\tau)\right\rangle_c . \tag{10.428}$$

The associated local diagrams are [compare (10.228)]:

$$\beta f_2^{(1)} = \eta^2 \Bigg[\ \frac{9}{2}$$  $$+\ 18$$  $$-\ \frac{3}{2}\,\delta(0)$$  $$\Bigg].$$ (10.429)

The second cumulant to order η^2 reads

$$-\frac{1}{2!}\eta^2 \int_0^\beta d\tau \int_0^\beta d\tau' \left\langle \left[-q^2(\tau)\dot q^2(\tau) + \delta(0) q^2(\tau)\right]\left[-q^2(\tau')\dot q^2(\tau') + \delta(0) q^2(\tau')\right]\right\rangle_c ,$$

leading to diagrams containing $\delta(0)$:

$$\beta f_2^{(2)} = -\frac{\eta^2}{2!}\Big\{ 2\,\delta^2(0)$$  $$-\ 4\,\delta(0)\big[$$  $$+\ 4$$  $$+$$  $$\big]\Big\}.$$ (10.430)

The remaining diagrams are either of the three-bubble type, or of the watermelon type, each with all possible combinations of the four line types (10.391) and (10.397)–(10.399). The three-bubbles diagrams yield [compare (10.231)]

$$\beta f_2^{(3)} = -\frac{\eta^2}{2!}\Big[4$$  $$+2$$  $$-8$$  $$+4$$  $$+4$$  $$+2$$  $$-8$$  $$\Big].$$ (10.431)

The watermelon-type diagrams contribute the same diagrams as in (10.232) for $\omega = 0$:

$$\beta f_2^{(4)} = -\frac{\eta^2}{2!}\,4\,\Big[$$  $$+\ 4$$  $$+$$  $$\Big].$$ (10.432)

For coordinate independence, the sum of the first-order diagrams (10.427) has to vanish. Analytically, this amounts to the equation

$$\beta f_1 = -\eta \int_0^\beta d\tau \left[\Delta(\tau,\tau)\,{}^{\cdot}\Delta^{\cdot}(\tau,\tau) + 2\,{}^{\cdot}\Delta^{2}(\tau,\tau) - \delta(0)\Delta(\tau,\tau)\right] = 0. \tag{10.433}$$

In the d-dimensional extension, the correlation function ${}^{\cdot}\Delta^{\cdot}(\tau,\tau)$ at equal times is the limit $d\to 1$ of the contracted correlation function ${}_{\mu}\Delta_{\mu}(x,x)$ which satisfies the d-dimensional field equation (10.418). Thus we can use Eq. (10.394) to replace ${}^{\cdot}\Delta^{\cdot}(\tau,\tau)$ by $\delta(0)-1/\beta$. This removes the infinite factor $\delta(0)$ in Eq. (10.433) coming from the measure. The remainder is calculated directly:

$$\int_0^\beta d\tau\left[-\frac{1}{\beta}\Delta(\tau,\tau)+2\,{}^{\cdot}\Delta^{2}(\tau,\tau)\right]=0. \tag{10.434}$$

This result is obtained without subtleties, since by Eqs. (10.392) and (10.396)

$$\Delta(\tau,\tau)=\tau-\frac{\tau^2}{\beta},\qquad {}^{\cdot}\Delta^{2}(\tau,\tau)=\frac{1}{4}-\frac{\Delta(\tau,\tau)}{\beta}, \tag{10.435}$$

whose integrals yield

$$\frac{1}{2\beta}\int_0^\beta d\tau\,\Delta(\tau,\tau)=\int_0^\beta d\tau\,{}^{\cdot}\Delta^{2}(\tau,\tau)=\frac{\beta}{12}. \tag{10.436}$$

Let us evaluate the second-order diagrams in $\beta f_2^{(i)},\ i=1,2,3,4$. The sum of the local diagrams in (10.429) consists of the integrals by

$$\beta f_2^{(1)}=\frac{3}{2}\eta^2\int_0^\beta d\tau\left[3\Delta^2(\tau,\tau){}^{\cdot}\Delta^{\cdot}(\tau,\tau)+12\Delta(\tau,\tau){}^{\cdot}\Delta^{2}(\tau,\tau)-\delta(0)\Delta^2(\tau,\tau)\right]. \tag{10.437}$$

Replacing ${}^{\cdot}\Delta^{\cdot}(\tau,\tau)$ in Eq. (10.437) again by $\delta(0)-1/\beta$, on account of the equation of motion (10.394), and taking into account the right-hand equation (10.435),

$$\beta f_2^{(1)}=\eta^2\left[3\delta(0)\int_0^\beta d\tau\Delta^2(\tau,\tau)\right]=\eta^2\frac{\beta^3}{10}\,\delta(0). \tag{10.438}$$

We now calculate the sum of bubble diagrams (10.430)–(10.432), beginning with (10.430) whose analytic form is

$$\begin{aligned}\beta f_2^{(2)}=-\frac{\eta^2}{2}\int_0^\beta\int_0^\beta d\tau\,d\tau'\,\big\{2\delta^2(0)\Delta^2(\tau,\tau')\\ -4\,\delta(0)\left[\Delta(\tau,\tau){}^{\cdot}\Delta^{2}(\tau,\tau')+4\,{}^{\cdot}\Delta(\tau,\tau)\Delta(\tau,\tau'){}^{\cdot}\Delta(\tau,\tau')+\Delta^2(\tau,\tau'){}^{\cdot}\Delta^{\cdot}(\tau,\tau)\right]\big\}.\end{aligned} \tag{10.439}$$

Inserting Eq. (10.394) into the last equal-time term, we obtain

$$\begin{aligned}\beta f_2^{(2)}=-\frac{\eta^2}{2}\int_0^\beta\int_0^\beta d\tau\,d\tau'\,\big\{-2\delta^2(0)\Delta^2(\tau,\tau')\\ -4\delta(0)\left[\Delta(\tau,\tau){}^{\cdot}\Delta^{2}(\tau,\tau')+4\,{}^{\cdot}\Delta(\tau,\tau)\Delta(\tau,\tau'){}^{\cdot}\Delta(\tau,\tau')-\Delta^2(\tau,\tau')/\beta\right]\big\}.\end{aligned} \tag{10.440}$$

As we shall see below, the explicit evaluation of the integrals in this sum is not necessary. Just for completeness, we give the result:

$$\begin{aligned}\beta f_2^{(2)}&=\frac{\eta^2}{2}\left\{2\delta^2(0)\frac{\beta^4}{90}+4\delta(0)\left[\frac{\beta^3}{45}+4\frac{\beta^3}{180}-\frac{\beta^3}{90}\right]\right\}\\ &=\eta^2\left\{\frac{\beta^4}{90}\delta^2(0)+\frac{\beta^3}{15}\delta(0)\right\}.\end{aligned} \tag{10.441}$$

We now turn to the three-bubbles diagrams (10.432). Only three of these contain the correlation function ${}_{\mu}\Delta_{\nu}(x,x')\to{}^{\cdot}\Delta^{\cdot}(\tau,\tau')$ for which Eq. (10.394) is not applicable: the second, fourth, and sixth diagram. The other three-bubble diagrams in (10.432) containing the generalization ${}_{\mu}\Delta_{\mu}(x,x)$ of the equal-time propagator ${}^{\cdot}\Delta^{\cdot}(\tau,\tau)$ can be calculated using Eq. (10.394).

Consider first a partial sum consisting of the first three three-bubble diagrams in the sum (10.432). This has the analytic form

$$\beta f_2^{(3)}\Big|_{1,2,3} = -\frac{\eta^2}{2}\int_0^\beta\int_0^\beta d\tau\, d\tau' \left\{4\,\Delta(\tau,\tau)\,{}^{\cdot}\Delta^{2}(\tau,\tau')\,{}^{\cdot}\Delta^{\cdot}(\tau',\tau')\right. \\ \left. +\ 2\,{}^{\cdot}\Delta^{\cdot}(\tau,\tau)\Delta^{\cdot 2}(\tau,\tau')\,{}^{\cdot}\Delta^{\cdot}(\tau',\tau') + 16\,{}^{\cdot}\Delta(\tau,\tau)\Delta(\tau,\tau')\,{}^{\cdot}\Delta(\tau,\tau')\,{}^{\cdot}\Delta^{\cdot}(\tau',\tau')\right\}. \tag{10.442}$$

Replacing ${}^{\cdot}\Delta^{\cdot}(\tau,\tau)$ and ${}^{\cdot}\Delta^{\cdot}(\tau',\tau')$ by $\delta(0)-1/\beta$, according to of (10.394), we see that Eq. (10.442) contains, with opposite sign, precisely the previous sum (10.439) of all one-and two-bubble diagrams. Together they give

$$\beta f_2^{(2)} + \beta f_2^{(3)}\Big|_{1,2,3} = -\frac{\eta^2}{2}\int_0^\beta\int_0^\beta d\tau\, d\tau' \left\{-\frac{4}{\beta}\Delta(\tau,\tau)\,{}^{\cdot}\Delta^{2}(\tau,\tau')\right. \\ \left. +\ \frac{2}{\beta^2}\Delta^2(\tau,\tau') - \frac{16}{\beta}\,{}^{\cdot}\Delta(\tau,\tau)\Delta(\tau,\tau')\,{}^{\cdot}\Delta(\tau,\tau')\right\}, \tag{10.443}$$

and can be evaluated directly to

$$\beta f_2^{(2)} + \beta f_2^{(3)}\Big|_{1,2,3} = \frac{\eta^2}{2}\left(\frac{4}{\beta}\frac{\beta^2}{45} - \frac{2}{\beta^2}\frac{\beta^4}{90} + \frac{16}{\beta}\frac{\beta^3}{180}\right) = \frac{\eta^2}{2}\frac{7}{45}\beta^2. \tag{10.444}$$

By the same direct calculation, the Feynman integral in the fifth three-bubble diagram in (10.432) yields

$$: \ I_5 = \int_0^\beta\int_0^\beta d\tau\, d\tau'\, {}^{\cdot}\Delta(\tau,\tau)\,{}^{\cdot}\Delta(\tau,\tau')\,\Delta^{\cdot}(\tau,\tau')\,\Delta^{\cdot}(\tau',\tau') = -\frac{\beta^2}{720}. \tag{10.445}$$

The explicit results (10.444) and (10.445) are again not needed, since the last term in Eq. (10.443) is equal, with opposite sign, to the partial sum of the fourth and fifth three-bubble diagrams in Eq. (10.432). To see this, consider the Feynman integral associated with the sixth three-bubble diagram in Eq. (10.432):

$$: \ I_4 = \int_0^\beta\int_0^\beta d\tau\, d\tau'\, {}^{\cdot}\Delta(\tau,\tau)\Delta(\tau,\tau')\,{}^{\cdot}\Delta^{\cdot}(\tau,\tau')\,\Delta^{\cdot}(\tau',\tau'), \tag{10.446}$$

whose d-dimensional extension is

$$I_4^d = \int_0^\beta\int_0^\beta d^d\tau\, d^d\tau'\, {}_{\alpha}\Delta(\tau,\tau)\Delta(\tau,\tau')\,{}_{\alpha}\Delta_{\beta}(\tau,\tau')\Delta_{\beta}(\tau',\tau'). \tag{10.447}$$

Adding this to the fifth Feynman integral (10.445) and performing a partial integration, we find in one dimension

$$\beta f_2^{(3)}\Big|_{4,5} = -\frac{\eta^2}{2}\,16\,(I_4+I_5) = -\frac{\eta^2}{2}\int_0^\beta\int_0^\beta d\tau\, d\tau'\,\frac{16}{\beta}\,{}^{\cdot}\Delta(\tau,\tau)\,{}^{\cdot}\Delta(\tau,\tau')\Delta(\tau,\tau') \\ = -\eta^2\,\frac{4}{45}\,\beta^2, \tag{10.448}$$

where we have used $d\left[{}^{\cdot}\Delta(\tau,\tau)\right]/d\tau = -1/\beta$ obtained by differentiating (10.435). Comparing (10.448) with (10.443), we find the sum of all bubbles diagrams, except for the sixth and seventh three-bubble diagrams in Eq. (10.432), to be given by

$$\beta f_2^{(2)} + \beta f_2^{(3)}\Big|_{6,7}' = \frac{\eta^2}{2}\frac{2}{15}\beta^2. \tag{10.449}$$

The prime on the sum denotes the exclusion of the diagrams indicated by subscripts. The correlation function ${}^{\cdot}\Delta^{\cdot}(\tau,\tau')$ in the two remaining diagrams of Eq. (10.432), whose d-dimensional extension is ${}_\alpha\Delta_\beta(x,x')$, cannot be replaced via Eq. (10.394), and the expression can only be simplified by applying partial integration to the seventh diagram in Eq. (10.432), yielding

$$
\begin{aligned}
: \ I_7 &= \int_0^\beta\int_0^\beta d\tau\, d\tau'\, \Delta(\tau,\tau)\,{}^{\cdot}\Delta(\tau,\tau')\,{}^{\cdot}\Delta^{\cdot}(\tau,\tau')\,\Delta^{\cdot}(\tau',\tau')\\
&\to \int_0^\beta\int_0^\beta d^d\tau\, d^d\tau'\, \Delta(\tau,\tau)\,{}_\alpha\Delta(\tau,\tau')_\alpha\Delta_\beta(\tau,\tau')\Delta_\beta(\tau',\tau')\\
&= \frac{1}{2}\int_0^\beta\int_0^\beta d^d\tau\, d^d\tau'\, \Delta(\tau,\tau)\Delta_\beta(\tau',\tau')\partial'_\beta\left[{}_\alpha\Delta(\tau,\tau')\right]^2\\
&\to \frac{1}{2}\int_0^\beta\int_0^\beta d\tau\, d\tau'\, \Delta(\tau,\tau)\,{}^{\cdot}\Delta(\tau',\tau')\frac{d}{d\tau'}\left[{}^{\cdot}\Delta^{2}(\tau,\tau')\right]\\
&= \frac{1}{2\beta}\int_0^\beta\int_0^\beta d\tau\, d\tau'\Delta(\tau,\tau)\,{}^{\cdot}\Delta^{2}(\tau,\tau') = \frac{\beta^2}{90}.
\end{aligned}
\tag{10.450}
$$

The sixth diagram in the sum (10.432) diverges linearly. As before, we add and subtract the divergence

$$
\begin{aligned}
: \ I_6 &= \int_0^\beta\int_0^\beta d\tau\, d\tau'\, \Delta(\tau,\tau)\,{}^{\cdot}\Delta^{\cdot 2}(\tau,\tau')\Delta(\tau',\tau')\\
&= \int_0^\beta\int_0^\beta d\tau\, d\tau'\, \Delta(\tau,\tau)\left[{}^{\cdot}\Delta^{\cdot 2}(\tau,\tau') - \delta^2(\tau-\tau')\right]\Delta(\tau',\tau')\\
&+ \int_0^\beta\int_0^\beta d\tau\, d\tau'\, \Delta^2(\tau,\tau)\,\delta^2(\tau-\tau').
\end{aligned}
\tag{10.451}
$$

In the first, finite term we go to d dimensions and replace $\delta(\tau-\tau') \to \delta^{(d)}(\tau-\tau') = -\Delta_{\beta\beta}(\tau,\tau')$ using the field equation (10.418). After this, we apply partial integration and find

$$
\begin{aligned}
I_6^R &\to \int_0^\beta\int_0^\beta d^d\tau\, d^d\tau'\, \Delta(\tau,\tau)\left[{}_\alpha\Delta_\beta^2(\tau,\tau') - \Delta_{\gamma\gamma}^2(\tau,\tau')\right]\Delta(\tau',\tau')\\
&= \int_0^\beta\int_0^\beta d^d\tau\, d^d\tau'\, \left\{-\partial_\alpha\left[\Delta(\tau,\tau)\right]\Delta_\beta(\tau,\tau')\,{}_\alpha\Delta_\beta(\tau,\tau')\Delta(\tau',\tau')\right.\\
&\qquad\qquad \left. + \Delta(\tau,\tau)\Delta_\beta(\tau,\tau')\Delta_{\gamma\gamma}(\tau,\tau')\partial'_\beta\left[\Delta(\tau',\tau')\right]\right\}\\
&\to \int_0^\beta\int_0^\beta d\tau\, d\tau'\, 2\left\{-{}^{\cdot}\Delta(\tau,\tau)\,\Delta^{\cdot}(\tau,\tau')\,{}^{\cdot}\Delta^{\cdot}(\tau,\tau')\Delta(\tau',\tau') + \right.\\
&\qquad\qquad \left. \Delta(\tau,\tau)\,\Delta^{\cdot}(\tau,\tau')\,{}^{\cdot}\Delta(\tau',\tau')\,\Delta^{\cdot\cdot}(\tau,\tau')\right\}.
\end{aligned}
\tag{10.452}
$$

In going to the last line we have used $d[\Delta(\tau,\tau)]/d\tau = 2\,{}^{\cdot}\Delta(\tau,\tau)$ following from (10.435). By interchanging the order of integration $\tau \leftrightarrow \tau'$, the first term in Eq. (10.452) reduces to the integral (10.450). In the last term we replace $\Delta^{\cdot\cdot}(\tau,\tau')$ using the field equation (10.393) and the trivial equation (10.376). Thus we obtain

$$
I_6 = I_6^R + I_6^{\rm div}
\tag{10.453}
$$

with

$$
I_6^R = 2\left(-\frac{\beta^2}{90} - \frac{\beta^2}{120}\right) = \frac{1}{2}\left(-\frac{7\beta^2}{90}\right),
\tag{10.454}
$$

$$
I_6^{\rm div} = \int_0^\beta\int_0^\beta d\tau\, d\tau'\, \Delta^2(\tau,\tau)\,\delta^2(\tau-\tau').
\tag{10.455}
$$

With the help of the identity for distributions (10.369), the divergent part is calculated to be

$$I_6^{\rm div} = \delta(0)\int_0^\beta d\tau\,\Delta^2(\tau,\tau) = \delta(0)\,\frac{\beta^3}{30}. \tag{10.456}$$

Using Eqs. (10.450) and (10.453) yields the sum of the sixth and seventh three-bubble diagrams in Eq. (10.432):

$$\beta f_2^{(3)}\Big|_{6,7} = -\frac{\eta^2}{2}\,(2I_6+16I_7) = -\frac{\eta^2}{2}\left[2\delta(0)\frac{\beta^3}{30}+\frac{\beta^2}{10}\right]. \tag{10.457}$$

Adding this to (10.449), we obtain the sum of all bubble diagrams

$$\beta f_2^{(2)}+\beta f_2^{(3)} = -\frac{\eta^2}{2}\left[2\delta(0)\frac{\beta^3}{30}+\frac{\beta^2}{30}\right]. \tag{10.458}$$

The contributions of the watermelon diagrams (10.432) correspond to the Feynman integrals

$$\begin{aligned}\beta f_2^{(4)} &= -2\eta^2\int_0^\beta\int_0^\beta d\tau d\tau'\,\left[\Delta^2(\tau,\tau')\,{}^{\cdot}\Delta^{\cdot 2}(\tau,\tau')\right.\\ &+\left. 4\,\Delta(\tau,\tau')\,{}^{\cdot}\Delta(\tau,\tau')\,\Delta^{\cdot}(\tau,\tau')\,{}^{\cdot}\Delta^{\cdot}(\tau,\tau')+{}^{\cdot}\Delta^2(\tau,\tau')\,\Delta^{\cdot 2}(\tau,\tau')\right].\end{aligned} \tag{10.459}$$

The third integral is unique and can be calculated directly:

$$: \; I_{10} = \int_0^\beta d\tau\int_0^\beta d\tau'\,{}^{\cdot}\Delta^2(\tau,\tau')\,\Delta^{\cdot 2}(\tau,\tau') = \frac{\beta^2}{90}. \tag{10.460}$$

The second integral reads in d dimensions

$$: \; I_9 = \int\int d^d\tau\,d^d\tau'\,\Delta(\tau,\tau')\,{}_{\alpha}\Delta(\tau,\tau')\Delta_{\beta}(\tau,\tau')\,{}_{\alpha}\Delta_{\beta}(\tau,\tau'). \tag{10.461}$$

This is integrated partially to yield, in one dimension,

$$I_9 = -\frac{1}{2}I_{10}+I_{9'} \equiv -\frac{1}{2}I_{10}-\frac{1}{2}\int\int d\tau\,d\tau'\Delta(\tau,\tau')\,\Delta^{\cdot 2}(\tau,\tau')\,{}^{\cdot\cdot}\Delta(\tau,\tau'). \tag{10.462}$$

The integral on the right-hand side is the one-dimensional version of

$$I_{9'} = -\frac{1}{2}\int\int d^d\tau\,d^d\tau'\,\Delta(\tau,\tau')\Delta_\beta^2(\tau,\tau')\,{}_{\alpha\alpha}\Delta(\tau,\tau'). \tag{10.463}$$

Using the field equation (10.418), going back to one dimension, and inserting $\Delta(\tau,\tau')$, $\Delta^{\cdot}(\tau,\tau')$, and ${}^{\cdot\cdot}\Delta(\tau,\tau')$ from (10.392), (10.396), and (10.393), we perform all unique integrals and obtain

$$I_{9'} = \beta^2\left\{\frac{1}{48}\int d\tau\,\epsilon^2(\tau)\,\delta(\tau)+\frac{1}{240}\right\}. \tag{10.464}$$

According to Eq. (10.375), the integral over the product of distributions vanishes. Inserting the remainder and (10.460) into Eq. (10.462) gives:

$$I_9 = -\frac{\beta^2}{720}. \tag{10.465}$$

We now evaluate the first integral in Eq. (10.459). Adding and subtracting the linear divergence yields

$$\begin{aligned}: \; I_8 &= \int_0^\beta\int_0^\beta d\tau\,d\tau'\,\Delta^2(\tau,\tau')\,{}^{\cdot}\Delta^{\cdot 2}(\tau,\tau') = \int_0^\beta\int_0^\beta d\tau d\tau'\,\Delta^2(\tau,\tau)\delta^2(\tau-\tau')\\ &+\int_0^\beta\int_0^\beta d\tau d\tau'\,\Delta^2(\tau,\tau')\left[{}^{\cdot}\Delta^{\cdot 2}(\tau,\tau')-\delta^2(\tau-\tau')\right].\end{aligned} \tag{10.466}$$

The finite second part of the integral (10.466) has the d-dimensional extension

$$I_8^R = \int\int d^d\tau\, d^d\tau'\, \Delta^2(\tau,\tau')\left[{}_\alpha\Delta_\beta^2(\tau,\tau') - \Delta_{\gamma\gamma}^2(\tau,\tau')\right], \tag{10.467}$$

which after partial integration and going back to one dimension reduces to a combination of integrals Eqs. (10.465) and (10.464):

$$I_8^R = -2I_9 + 2I_{9'} = -\frac{\beta^2}{72}. \tag{10.468}$$

The divergent part of I_8 coincides with $I_6^{\rm div}$ in Eq. (10.455):

$$I_8^{\rm div} = \int_0^\beta \int_0^\beta d\tau d\tau' \Delta^2(\tau,\tau)\delta^2(\tau-\tau') = I_6^{\rm div} = \delta(0)\frac{\beta^3}{30}. \tag{10.469}$$

Inserting this together with (10.460) and (10.465) into Eq. (10.459), we obtain the sum of watermelon diagrams

$$\beta f_2^{(4)} = -2\eta^2(I_8 + 4I_9 + I_{10}) = -\frac{\eta^2}{2}\left\{4\delta(0)\frac{\beta^2}{30} - \frac{\beta^2}{30}\right\}. \tag{10.470}$$

For a flat space in curvilinear coordinates, the sum of the first-order diagrams vanish. To second order, the requirement of coordinate independence implies a vanishing sum of all connected diagrams (10.429)–(10.432). By adding the sum of terms in Eqs. (10.438), (10.458), and (10.470), we find indeed zero, thus confirming coordinate independence. It is not surprising that the integration rules for products of distributions derived in an infinite time interval $\tau \in [0,\infty)$ are applicable for finite time intervals. The singularities in the distributions come in only at a single point of the time axis, so that its total length is irrelevant.

The procedure can easily be continued to higher-loop diagrams to define integrals over higher singular products of ϵ- and δ-functions. At the one-loop level, the cancellation of $\delta(0)$s requires

$$\int d\tau\, \Delta(\tau,\tau)\,\delta(0) = \delta(0)\int d\tau\, \Delta(\tau,\tau). \tag{10.471}$$

The second-order gave, in addition, the rule

$$\int_0^\beta \int_0^\beta d\tau\, d\tau'\, \Delta^2(\tau,\tau)\,\delta^2(\tau-\tau') = \delta(0)\int d\tau\, \Delta^2(\tau,\tau), \tag{10.472}$$

To n-order we can derive the equation

$$\int d\tau_1 \ldots d\tau_n\, \Delta(\tau_1,\tau_2)\,\delta(\tau_1,\tau_2)\cdots\Delta(\tau_n,\tau_1)\,\delta(\tau_n,\tau_1) = \delta(0)\int d\tau\, \Delta^n(\tau,\tau), \tag{10.473}$$

which reduces to

$$\int\int d\tau_1\, d\tau_n\, \Delta^n(\tau_1,\tau_1)\,\delta^2(\tau_1-\tau_n) = \delta(0)\int d\tau\, \Delta^n(\tau,\tau), \tag{10.474}$$

which is satisfied due to the integration rule (10.369). See Appendix 10C for a general derivation of (10.473).

10.11.5 Time Evolution Amplitude in Curved Space

The same Feynman diagrams which we calculated to verify coordinate independence appear also in the perturbation expansion of the time evolution amplitude in curved space if this is performed in normal or geodesic coordinates.

The path integral in curved space is derived by making the mapping from x^i to q_a^μ in Subsection 10.11.2 nonholonomic, so that it can no longer be written as $x^i(\tau) = x^i(q^\mu(\tau))$ but only as $dx^i(\tau) = e^i{}_\mu(q)dq^\mu(\tau)$. Then the q-space may contain curvature and torsion, and the result of the path integral will not longer be trivial one in Eq. (10.403) but depend on $R_{\mu\nu\lambda}{}^\kappa(q_a)$ and $S_{\mu\nu}{}^\lambda(q_a)$. For simplicity, we shall ignore torsion.

Then the action becomes (10.404) with the metric $g_{\mu\nu}(q) = e^i{}_\mu(q)e^i{}_\nu(q)$. It was shown in Subsection 10.3.2 that under nonholonomic coordinate transformations, the measure of a time-sliced path integral transforms from the flat-space form $\prod_n d^Dx_n$ to $\prod_n d^Dq\sqrt{g_n}\exp(\Delta t\bar{R}_n/6)$. This had the consequence, in Section 10.4, that the time evolution amplitude for a particle on the surface of a sphere has an energy (10.165) corresponding to the Hamiltonian (1.418) which governs the Schrödinger equation (1.424). It contains a pure Laplace-Beltrami operator in the kinetic part. There is no extra R-term, which would be allowed if only covariance under ordinary coordinate transformations is required. This issue will be discussed in more detail in Subsection 11.1.1.

Below we shall see that for perturbatively defined path integrals, the nonholonomic transformation must carry the flat-space measure into curved space as follows:

$$\mathcal{D}^Dx \to \mathcal{D}^Dq\,\sqrt{g}\,\exp\left(\int_0^\beta d\tau\,\bar{R}/8\right). \tag{10.475}$$

For a D-dimensional space with a general metric $g_{\mu\nu}(q)$ we can make use of the above proven coordinate invariance to bring the metric to the most convenient normal or geodesic coordinates (10.98) around some point q_a. The advantage of these coordinates is that the derivatives and thus the affine connection vanish at this point. Its derivatives can directly be expressed in terms of the curvature tensor:

$$\partial_\kappa\bar{\Gamma}_{\tau\kappa}{}^\mu(q_a) = -\frac{1}{3}\left[\bar{R}_{\tau\kappa\sigma}{}^\mu(q_a) + \bar{R}_{\sigma\kappa\tau}{}^\mu(q_a)\right], \qquad \text{for normal coordinates.} \tag{10.476}$$

Assuming q_a to lie at the origin, we expand the metric and its determinant in powers of normal coordinates $\Delta\xi^\mu$ around the origin and find, dropping the smallness symbols Δ in front of q and ξ in the transformation (10.98):

$$g_{\mu\nu}(\xi) = \delta_{\mu\nu} + \eta\,\frac{1}{3}\,\bar{R}_{\mu\lambda\nu\kappa}\,\xi^\lambda\xi^\kappa + \eta^2\,\frac{2}{45}\bar{R}_{\lambda\nu\kappa}{}^\delta\,\bar{R}_{\sigma\mu\tau\delta}\,\xi^\lambda\xi^\kappa\xi^\sigma\xi^\tau + \dots, \tag{10.477}$$

$$g(\xi) = 1 - \eta\,\frac{1}{3}\,\bar{R}_{\mu\nu}\,\xi^\mu\xi^\nu + \eta^2\,\frac{1}{18}\left(\bar{R}_{\mu\nu}\,\bar{R}_{\lambda\kappa} + \frac{1}{5}\,\bar{R}_{\mu\sigma\nu}{}^\tau\,\bar{R}_{\lambda\tau\kappa}{}^\sigma\right)\xi^\mu\xi^\nu\xi^\lambda\xi^\kappa + \dots. \tag{10.478}$$

These expansions have obviously the same power content in ξ^μ as the previous one-dimensional expansions (10.223) had in q. The interaction (10.410) becomes in normal coordinates, up to order η^2:

$$\begin{aligned}\mathcal{A}_{\rm tot}^{\rm int}[\xi] \;=\; \int_0^\beta d\tau\Bigg\{&\left[\eta\,\frac{1}{6}\bar{R}_{\mu\lambda\nu\kappa}\,\xi^\lambda\xi^\kappa + \eta^2\frac{1}{45}\bar{R}_{\lambda\mu\kappa}{}^\delta\bar{R}_{\sigma\nu\tau\delta}\,\xi^\lambda\xi^\kappa\xi^\sigma\xi^\tau\right]\dot{\xi}^\mu\dot{\xi}^\nu \\ &+ \eta\,\frac{1}{6}\delta(0)\bar{R}_{\mu\nu}\,\xi^\mu\xi^\nu + \eta^2\frac{1}{180}\delta(0)\bar{R}_{\mu\delta\nu}{}^\sigma\,\bar{R}_{\lambda\sigma\kappa}{}^\delta\,\xi^\mu\xi^\nu\xi^\lambda\xi^\kappa\Bigg\}.\end{aligned} \tag{10.479}$$

This has again the same powers in ξ^μ as the one-dimensional interaction (10.425), leading to the same Feynman diagrams, differing only by the factors associated with the vertices. In one dimension, with the trivial vertices of the interaction (10.425), the sum of all diagrams vanishes. In curved space with the more complicated vertices proportional to $\bar{R}_{\mu\nu\kappa\lambda}$ and $\bar{R}_{\mu\nu}$, the result is nonzero but depends on contractions of the curvature tensor $\bar{R}_{\mu\nu\kappa\lambda}$. The dependence is easily identified for each diagram. All bubble diagrams in (10.430)–(10.432) yield results proportional to $\bar{R}_{\mu\nu}^2$, while the watermelon-like diagrams (10.432) carry a factor $\bar{R}_{\mu\nu\kappa\lambda}^2$.

When calculating the contributions of the first expectation value $\left\langle \mathcal{A}^{\rm int}_{\rm tot}[\xi]\right\rangle$ to the time evolution amplitude it is useful to reduce the D-dimensional expectation values of (10.479) to one-dimensional ones of (10.425) as follows using the contraction rules (8.63) and (8.64):

$$\langle \xi^\mu \xi^\nu\rangle = \delta^{\mu\nu}\,\langle \xi\,\xi\rangle\,, \tag{10.480}$$

$$\left\langle \xi^\lambda\xi^\kappa\xi^\mu \xi^\nu\right\rangle = \left(\delta^{\lambda\kappa}\delta^{\mu\nu}+\delta^{\lambda\mu}\delta^{\nu\kappa}+\delta^{\lambda\nu}\delta^{\kappa\mu}\right)\langle \xi\,\xi\,\xi\,\xi\rangle\,, \tag{10.481}$$

$$\left\langle \xi^\lambda\xi^\kappa\dot\xi^\mu \dot\xi^\nu\right\rangle = \delta^{\lambda\kappa}\delta^{\mu\nu}\,\langle \xi\,\xi\rangle\left\langle \dot\xi\,\dot\xi\right\rangle+\left(\delta^{\lambda\mu}\delta^{\kappa\nu}+\delta^{\lambda\nu}\delta^{\kappa\mu}\right)\left\langle \xi\,\dot\xi\right\rangle\left\langle \xi\,\dot\xi\right\rangle, \tag{10.482}$$

$$\begin{aligned}\left\langle \xi^\lambda\xi^\kappa\xi^\sigma\xi^\tau\dot\xi^\mu \dot\xi^\nu\right\rangle &= \left(\delta^{\lambda\kappa}\delta^{\sigma\tau}+\delta^{\lambda\sigma}\delta^{\kappa\tau}+\delta^{\lambda\tau}\delta^{\kappa\sigma}\right)\delta^{\mu\nu}\,\langle\xi\,\xi\,\rangle\,\langle\xi\,\xi\,\rangle\left\langle \dot\xi\,\dot\xi\right\rangle\\
&+\;\left[\delta^{\lambda\mu}\left(\delta^{\kappa\nu}\delta^{\sigma\tau}+\delta^{\sigma\nu}\delta^{\tau\kappa}+\delta^{\tau\nu}\delta^{\kappa\sigma}\right)+\delta^{\kappa\mu}\left(\delta^{\sigma\nu}\delta^{\lambda\tau}+\delta^{\lambda\nu}\delta^{\tau\sigma}+\delta^{\tau\nu}\delta^{\sigma\lambda}\right)\right.\\
&+\;\left.\delta^{\sigma\mu}\left(\delta^{\tau\nu}\delta^{\lambda\kappa}+\delta^{\lambda\nu}\delta^{\kappa\tau}+\delta^{\kappa\nu}\delta^{\tau\lambda}\right)+\delta^{\tau\mu}\left(\delta^{\lambda\nu}\delta^{\kappa\sigma}+\delta^{\kappa\nu}\delta^{\sigma\lambda}+\delta^{\sigma\nu}\delta^{\lambda\kappa}\right)\right]\\
&\quad\times\langle\xi\,\xi\,\rangle\left\langle \xi\,\dot\xi\,\right\rangle\left\langle \xi\,\dot\xi\right\rangle.\end{aligned} \tag{10.483}$$

Inserting these into the expectation value of (10.479) and performing the tensor contractions, we obtain

$$\begin{aligned}\left\langle \mathcal{A}^{\rm int}_{\rm tot}[\xi]\right\rangle &= \int_0^\beta d\tau\Big\{\,\eta\,\frac{1}{6}\bar R\,\left[-\langle \xi\,\xi\rangle\left\langle \dot\xi\,\dot\xi\right\rangle+\left\langle \xi\,\dot\xi\right\rangle\left\langle \xi\,\dot\xi\right\rangle+\,\eta\,\delta(0)\,\langle\xi\,\xi\rangle\right]\\
&+\;\eta^2\frac{1}{45}\left[\left(\bar R^2_{\mu\nu}+\bar R_{\mu\nu\lambda\kappa}\bar R^{\mu\nu\lambda\kappa}+\bar R_{\mu\nu\lambda\kappa}\bar R^{\lambda\nu\mu\kappa}\right)\left(\langle\xi\,\xi\rangle^2\left\langle\dot\xi\,\dot\xi\right\rangle-\langle\xi\,\xi\,\rangle\left\langle\xi\,\dot\xi\,\right\rangle^2\right)\right]\\
&+\;\eta^2\frac{1}{180}\delta(0)\left(\bar R^2_{\mu\nu}+\bar R_{\mu\nu\lambda\kappa}\bar R^{\mu\nu\lambda\kappa}+\bar R_{\mu\nu\lambda\kappa}\bar R^{\lambda\nu\mu\kappa}\right)\langle\xi\,\xi\rangle\,\langle\xi\,\xi\,\rangle\Big\}.\end{aligned} \tag{10.484}$$

Individually, the four tensors in the brackets of (10.483) contribute the tensor contractions, using the antisymmetry of $\bar R_{\mu\nu\lambda\kappa}$ in $\mu\nu$ and the contraction to the Ricci tensor $\bar R_{\lambda\mu\kappa}{}^\delta\,\delta^{\mu\kappa}=\bar R_\lambda{}^\delta$:

$$\begin{aligned}&\bar R_{\lambda\mu\kappa}{}^\delta\bar R_{\sigma\nu\tau\delta}\,\delta^{\lambda\mu}\left(\delta^{\kappa\nu}\delta^{\sigma\tau}+\delta^{\sigma\nu}\delta^{\tau\kappa}+\delta^{\tau\nu}\delta^{\kappa\sigma}\right)=0,\\
&\bar R_{\lambda\mu\kappa}{}^\delta\bar R_{\sigma\nu\tau\delta}\,\delta^{\kappa\mu}\left(\delta^{\sigma\nu}\delta^{\lambda\tau}+\delta^{\lambda\nu}\delta^{\tau\sigma}+\delta^{\tau\nu}\delta^{\sigma\lambda}\right)=\bar R_\lambda{}^\delta\left(-\bar R^\lambda{}_\delta+\bar R^\lambda{}_\delta\right)=0,\\
&\bar R_{\lambda\mu\kappa}{}^\delta\bar R_{\sigma\nu\tau\delta}\,\delta^{\sigma\mu}\left(\delta^{\tau\nu}\delta^{\lambda\kappa}+\delta^{\lambda\nu}\delta^{\kappa\tau}+\delta^{\kappa\nu}\delta^{\tau\lambda}\right)=\bar R_{\lambda\mu\kappa}{}^\delta\left(\bar R^\sigma{}_\delta\,\delta^{\lambda\kappa}+\bar R_{\sigma\lambda\kappa\delta}+\bar R_{\sigma\kappa\lambda\delta}\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad=-\left(\bar R^2_{\mu\nu}+\bar R_{\mu\nu\lambda\kappa}\bar R^{\mu\nu\lambda\kappa}+\bar R_{\mu\nu\lambda\kappa}\bar R^{\lambda\nu\mu\kappa}\right)\\
&\bar R_{\lambda\mu\kappa}{}^\delta\bar R_{\sigma\nu\tau\delta}\,\delta^{\tau\mu}\left(\delta^{\lambda\nu}\delta^{\kappa\sigma}+\delta^{\kappa\nu}\delta^{\sigma\lambda}+\delta^{\sigma\nu}\delta^{\lambda\kappa}\right)=\bar R_{\lambda\tau\kappa}{}^\delta\left(R_{\kappa\lambda\tau\delta}+R_{\lambda\kappa\tau\delta}\right)=0.\end{aligned} \tag{10.485}$$

We now use the *fundamental identity* of Riemannian spaces

$$\bar R_{\mu\nu\lambda\kappa}+\bar R_{\mu\lambda\kappa\nu}+\bar R_{\mu\kappa\nu\lambda}=0. \tag{10.486}$$

By expressing the curvature tensor (10.31) in Riemannian space in terms of the Christoffel symbol (1.70) as

$$R_{\mu\nu\lambda\kappa}=\frac{1}{2}\left(\partial_\mu\partial_\lambda g_{\nu\kappa}-\partial_\mu\partial_\kappa g_{\nu\lambda}-\partial_\nu\partial_\lambda g_{\mu\kappa}+\partial_\mu\partial_\kappa g_{\mu\lambda}\right)-[\bar\Gamma_\mu,\bar\Gamma_\nu]_{\lambda\kappa}, \tag{10.487}$$

we see that the identity (10.486) is a consequence of the symmetry of the metric and the single-valuedness of the metric expressed by the integrability condition[8] $(\partial_\lambda\partial_\kappa-\partial_\kappa\partial_\lambda)g_{\mu\nu}=0$. Indeed, due to the symmetry of $g_{\mu\nu}$ we find

$$\bar R_{\mu\nu\lambda\kappa}+\bar R_{\mu\lambda\kappa\nu}+\bar R_{\mu\kappa\nu\lambda}=\frac{1}{2}[\left(\partial_\nu\partial_\kappa-\partial_\kappa\partial_\nu\right)g_{\mu\lambda}-\left(\partial_\nu\partial_\lambda-\partial_\lambda\partial_\nu\right)g_{\mu\kappa}-\left(\partial_\lambda\partial_\kappa-\partial_\kappa\partial_\lambda\right)g_{\mu\nu}]=0.$$

The integrability has also the consequence that

$$R_{\mu\nu\lambda\kappa}=-R_{\mu\nu\kappa\lambda},\quad R_{\mu\nu\lambda\kappa}=R_{\lambda\kappa\mu\nu}. \tag{10.488}$$

[8]For the derivation see p. 1353 in the textbook [2].

Using (10.486) and (10.488) we find that

$$\bar{R}_{\mu\nu\lambda\kappa}\bar{R}^{\lambda\nu\mu\kappa} = \frac{1}{2}\bar{R}_{\mu\nu\lambda\kappa}\bar{R}^{\mu\nu\lambda\kappa}, \tag{10.489}$$

so that the contracted curvature tensors in the parentheses of (10.484) can be replaced by $\bar{R}_{\mu\nu}^2 + \frac{3}{2}\bar{R}_{\mu\nu\lambda\kappa}\bar{R}^{\mu\nu\lambda\kappa}$.

We now calculate explicitly the contribution of the first-order diagrams in (10.484) [compare (10.427)]:

$\beta f_1 = \frac{1}{6}\bar{R}\Big[-\eta$  $+\eta$  $+\eta\,\delta(0)$  $\Big].$ (10.490)

corresponding to the analytic expression [compare (10.433)]:

$$\beta f_1 = -\eta\frac{1}{6}\bar{R}\int_0^\beta d\tau\left[\Delta(\tau,\tau)\,{}^{\cdot}\Delta^{\cdot}(\tau,\tau) - {}^{\cdot}\Delta^2(\tau,\tau) - \delta(0)\Delta(\tau,\tau)\right]. \tag{10.491}$$

Note that the combination of propagators in the brackets is different from the previous one in (10.433). Using the integrals (10.436) we find, setting $\eta = 1$:

$$\beta f_1 = -\frac{1}{6}\bar{R}\int_0^\beta d\tau\left[-\frac{1}{\beta}\Delta(\tau,\tau) - {}^{\cdot}\Delta^2(\tau,\tau)\right]. \tag{10.492}$$

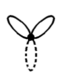
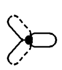
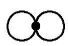
Using Eq. (10.436), this becomes

$$\beta f_1 = \frac{1}{6}\bar{R}\int_0^\beta d\tau\,\frac{3}{2\beta}\Delta(\tau,\tau) = \frac{\beta}{24}\bar{R}. \tag{10.493}$$

Adding to this the similar contribution coming from the nonholonomically transformed measure (10.475), we obtain the first-order expansion of the imaginary-time evolution amplitude

$$(q_a\,\beta|q_a\,0) = \frac{1}{\sqrt{2\pi\beta}^D}\,\exp\left[\frac{\beta}{12}\bar{R}(q_a) + \ldots\right]. \tag{10.494}$$

We now turn to the second-order contributions in η. The sum of the local diagrams (10.429) reads now

$\beta f_2^{(1)} = \eta^2\frac{1}{45}\left(\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + \frac{3}{2}\bar{R}_{\mu\nu\lambda\kappa}\bar{R}^{\mu\nu\lambda\kappa}\right)\Big[$  $-$  $+\frac{1}{4}\delta(0)$  $\Big].$ (10.495)

In terms of the Feynman integrals, the brackets are equal to [compare (10.437)]

$$\int_0^\beta d\tau\left[\Delta^2(\tau,\tau)\,{}^{\cdot}\Delta^{\cdot}(\tau,\tau) - \Delta(\tau,\tau)\,{}^{\cdot}\Delta^{2}(\tau,\tau) + \frac{1}{4}\delta(0)\Delta^2(\tau,\tau)\right]. \tag{10.496}$$

Inserting the equation of motion (10.394) and the right-hand equation (10.435), this becomes

$$\int_0^\beta d\tau\left[-\frac{5}{4\beta}\Delta^2(\tau,\tau) + \frac{5}{4}\delta(0)\Delta^2(\tau,\tau)\right] = \frac{5}{4}\frac{1}{30}\left[1 - \delta(0)\right]. \tag{10.497}$$

Thus we find

$$\beta f_2^{(1)} = -\eta^2\frac{\beta^2}{1080}\left(\bar{R}_{\mu\nu}^2 + \frac{3}{2}\bar{R}_{\mu\nu\kappa\lambda}^2\right)\left[1 - \delta(0)\right]. \tag{10.498}$$

Next we calculate the nonlocal contributions of order $\hbar^2$ to βf coming from the cumulant

$$
\begin{aligned}
-\frac{1}{2}\left\langle \mathcal{A}_{\mathrm{tot}}^{\mathrm{int}\,2}\right\rangle_c = -\frac{\eta^2}{2}\frac{1}{36}\int_0^\beta d\tau\int_0^\beta d\tau'\Bigg\langle\Big[&\delta^2(0)\bar R_{\mu\nu}\,\bar R_{\mu'\nu'}\,\xi^\mu(\tau)\xi^\nu(\tau)\;\xi^{\mu'}(\tau')\xi^{\nu'}(\tau')\\
&+2\,\delta(0)\bar R_{\mu\lambda\nu\kappa}\,\bar R_{\mu'\nu'}\,\xi^\lambda(\tau)\xi^\kappa(\tau)\dot\xi^\mu(\tau)\dot\xi^\nu(\tau)\;\xi^{\mu'}(\tau')\xi^{\nu'}(\tau')\\
&+\bar R_{\mu\lambda\nu\kappa}\,\bar R_{\mu'\lambda'\nu'\kappa'}\,\xi^\lambda(\tau)\xi^\kappa(\tau)\dot\xi^\mu(\tau)\dot\xi^\nu(\tau)\;\xi^{\lambda'}(\tau')\xi^{\kappa'}(\tau')\dot\xi^{\mu'}(\tau')\dot\xi^{\nu'}(\tau')\Big]\Bigg\rangle_c .
\end{aligned}
\tag{10.499}
$$

The first two terms yield the connected diagrams [compare (10.430)]

$$
\beta f_2^{(2)} = -\frac{\eta^2}{2}\frac{\bar R_{\mu\nu}\bar R^{\mu\nu}}{36}\Big\{2\,\delta^2(0)
$$

$$
-\,4\,\delta(0)\,[
$$

$$
-\,2
$$

$$
+
$$

$$
]\Big\}, \tag{10.500}
$$

and the analytic expression diagrams [compare (10.439)]

$$
\begin{aligned}
\beta f_2^{(2)} = -\frac{\eta^2}{2}\frac{\bar R_{\mu\nu}\bar R^{\mu\nu}}{36}\int_0^\beta\int_0^\beta d\tau\,d\tau'\,&\big\{2\delta^2(0)\Delta^2(\tau,\tau')\\
-4\,\delta(0)\,\big[\Delta(\tau,\tau)\,{}^{\cdot}\Delta^{\cdot\,2}(\tau,\tau') - 2\,{}^{\cdot}\Delta(\tau,\tau)\Delta(\tau,\tau')\,{}^{\cdot}\Delta(\tau,\tau') &+ \Delta^2(\tau,\tau')\,{}^{\cdot}\Delta^{\cdot}(\tau,\tau)\big]\big\}\,.
\end{aligned}
\tag{10.501}
$$

The third term in (10.499) leads to the three-bubble diagrams [compare (10.439)]

$$
\beta f_2^{(3)} = -\frac{\eta^2}{2}\frac{\bar R_{\mu\nu}\bar R^{\mu\nu}}{36}\Big[4
$$

$$
+2
$$

$$
-8
$$

$$
+4
$$

$$
+4
$$

$$
+2
$$

$$
-8
$$

$$
\Big]. \tag{10.502}
$$

The analytic expression for the diagrams 1,2,3 is [compare (10.442)]

$$
\begin{aligned}
\beta f_2^{(3)}\Big|_{1,2,3} = -\frac{\eta^2}{2}\frac{\bar R_{\mu\nu}\bar R^{\mu\nu}}{36}\int_0^\beta\int_0^\beta d\tau\,d\tau'\,&\big\{4\,\Delta(\tau,\tau)\,{}^{\cdot}\Delta^{\cdot\,2}(\tau,\tau')\,{}^{\cdot}\Delta^{\cdot}(\tau',\tau')\\
+\;2\,{}^{\cdot}\Delta^{\cdot}(\tau,\tau)\Delta^2(\tau,\tau')\,{}^{\cdot}\Delta^{\cdot}(\tau',\tau') - 8\,{}^{\cdot}\Delta(\tau,\tau)\Delta(\tau,\tau')\,{}^{\cdot}\Delta(\tau,\tau')\,{}^{\cdot}\Delta^{\cdot}(\tau',\tau')&\big\}\,.
\end{aligned}
\tag{10.503}
$$

and for 4 and 5 [compare (10.448)] :

$$
\begin{aligned}
\beta f_2^{(3)}\Big|_{4,5} = -\frac{\eta^2}{2}\frac{\bar R_{\mu\nu}\bar R^{\mu\nu}}{36}(I_4+I_5) &= -\frac{\eta^2}{2}\frac{\bar R_{\mu\nu}\bar R^{\mu\nu}}{36}\int_0^\beta\int_0^\beta d\tau\,d\tau'\,\frac{4}{\beta}\,{}^{\cdot}\Delta(\tau,\tau)\,{}^{\cdot}\Delta(\tau,\tau')\Delta(\tau,\tau')\\
&= -\frac{\eta^2}{2}\frac{\bar R_{\mu\nu}\bar R^{\mu\nu}}{36}\frac{1}{45}\beta^2.
\end{aligned}
\tag{10.504}
$$

For the diagrams 6 and 7, finally, we obtain [compare (10.458)]

$$
\beta f_2^{(3)}\Big|_{6,7} = -\frac{\eta^2}{2}\frac{\bar R_{\mu\nu}\bar R^{\mu\nu}}{36}(2I_6-8I_7) = -\frac{\eta^2}{2}\frac{\bar R_{\mu\nu}\bar R^{\mu\nu}}{36}\left[2\delta(0)\frac{\beta^3}{30}-\frac{\beta^2}{6}\right]. \tag{10.505}
$$

The sum of all bubbles diagrams (10.501) and (10.503) is therefore

$$
\beta f_2^{(2)} + \beta f_2^{(3)} = \eta^2\frac{\beta^2}{432}\bar R_{\mu\nu}^2 - \eta^2\,\delta(0)\,\frac{\beta^3}{1080}\,\bar R_{\mu\nu}^2\,. \tag{10.506}
$$

This compensates exactly the $\delta(0)$-term proportional to $\bar R_{\mu\nu}^2$ in Eq. (10.498), leaving only a finite second-order term

$$
\beta f_1^{(2)} + \beta f_2^{(2)} + \beta f_2^{(3)} = \eta^2\frac{\beta^2}{720}\left(\bar R_{\mu\nu}^2 - \bar R_{\mu\nu\kappa\lambda}^2\right) + \eta^2\delta(0)\frac{\beta^3}{1080}\bar R_{\mu\nu\kappa\lambda}^2. \tag{10.507}
$$

Finally we calculate the second-order watermelon diagrams (10.432) which contain the initially ambiguous Feynman integrals we make the following observation. Their sum is [compare (10.432)]

$$\beta f_2^{(4)} = -\frac{\eta^2}{2} 2 \frac{1}{36} \left(\bar{R}_{\mu\nu\lambda\kappa} \bar{R}^{\mu\nu\lambda\kappa} + \bar{R}_{\mu\nu\lambda\kappa} \bar{R}^{\mu\lambda\nu\kappa} \right) \Bigg[$$  $$-\,2$$  $$+$$  $$\Bigg], \tag{10.508}$$

corresponding to the analytic expression

$$\begin{aligned}
\beta f_2^{(3)} &= -\frac{\eta^2}{2} 2 \frac{3}{2} \frac{1}{36} \bar{R}^2_{\mu\nu\kappa\lambda} \int_0^\beta \int_0^\beta d\tau d\tau' \left[\Delta^2(\tau,\tau') {}^{\cdot}\Delta^{\cdot 2}(\tau,\tau') \right. \\
&\qquad \left. -2\, \Delta(\tau,\tau')\, {}^{\cdot}\Delta\, (\tau,\tau')\, \Delta^{\cdot}(\tau,\tau') {}^{\cdot}\Delta^{\cdot}(\tau,\tau') + {}^{\cdot}\Delta^{\,2}(\tau,\tau')\, \Delta^{\cdot 2}(\tau,\tau') \right] \\
&= -\frac{\eta^2}{24} \bar{R}^2_{\mu\nu\kappa\lambda} \left(I_8 - 2I_9 + I_{10} \right),
\end{aligned} \tag{10.509}$$

where the integrals I_8, I_9, and I_{10} were evaluated before in Eqs. (10.469), (10.468), (10.465), and (10.460). Substituting the results into Eq. (10.509) and using the rules (10.369) and (10.375), we obtain

$$\beta f_2^{(3)} = -\frac{\eta^2}{24} \bar{R}^2_{\mu\nu\kappa\lambda} \int_0^\beta \int_0^\beta d\tau\, d\tau' \Delta^2(\tau,\tau) \delta^2(\tau-\tau') = -\eta^2 \frac{\beta^3}{720} \bar{R}^2_{\mu\nu\kappa\lambda}\, \delta(0). \tag{10.510}$$

Thus the only role of the watermelon diagrams is to cancel the remaining $\delta(0)$-term proportional to $\bar{R}^2_{\mu\nu\kappa\lambda}$ in Eq. (10.507). It gives no finite contribution.

The remaining total sum of all second-order contribution in Eq. (10.507). changes the diagonal time evolution amplitude (10.494) to

$$(q_a\, \beta | q_a\, 0) = \frac{1}{\sqrt{2\pi\beta}^D} \exp\left[\frac{\beta}{12} \bar{R}(q_a) + \frac{\beta^2}{720} \left(\bar{R}^2_{\mu\nu\kappa\lambda} - \bar{R}^2_{\mu\nu} \right) + \dots \right]. \tag{10.511}$$

In Chapter 11 we shall see that this expression agrees with what has been derived in Schrödinger quantum mechanics from a Hamiltonian operator $\hat{H} = -\Delta/2$ which contains only is the Laplace-Beltrami operator $\Delta = g^{-1/2}\partial_\mu g^{1/2} g^{\mu\nu}(q)\partial_\nu$ of Eq. (1.381) and no extra R-term:

$$\begin{aligned}
(q_a\, \beta \mid q_a\, 0) &\equiv \left\langle e^{\beta\Delta/2} \right\rangle = (q_a \mid e^{\beta\Delta/2} \mid q_a) \qquad\qquad (10.512) \\
&= \frac{1}{\sqrt{2\pi\beta}^D} \left\{ 1 + \frac{\beta}{12}\bar{R} + \frac{\beta^2}{2} \left[\frac{1}{144}\bar{R}^2 + \frac{1}{360} \left(\bar{R}^{\mu\nu\kappa\lambda}\bar{R}_{\mu\nu\kappa\lambda} - \bar{R}^{\mu\nu}\bar{R}_{\mu\nu} \right) \right] + \dots \right\}.
\end{aligned}$$

This expansion due to DeWitt and Seeley will be derived in Section 11.6, the relevant equation being (11.110).

Summarizing the results we have found that for one-dimensional q-space as well as for a D-dimensional curved space in normal coordinates, our calculation procedure on a one-dimensional τ-axis yields unique results. The procedure uses only the essence of the d-dimensional extension, together with the rules (10.369) and (10.375). The results guarantee the coordinate independence of path integrals. They also agree with the DeWitt-Seeley expansion of the short-time amplitude to be derived in Eq. (11.110). The agreement is ensured by the initially ambiguous integrals I_8 and I_9 satisfying the equations

$$I_8^R + 4I_9 + I_{10} = -\frac{\beta^2}{120}, \tag{10.513}$$

$$I_8^R - 2I_9 + I_{10} = 0, \tag{10.514}$$

as we can see from Eqs. (10.470) and (10.509). Since the integral $I_{10} = \beta^2/90$ is unique, we must have $I_9 = -\beta^2/720$ and $I_8^R = -\beta^2/72$, and this is indeed what we found from our integration rules.

The main role of the d-dimensional extension of the τ-axis is, in this context, to forbid the application of the equation of motion (10.394) to correlation functions ${}^{\cdot}\Delta^{\cdot}(\tau,\tau')$. This would fix immediately the finite part of the integral I_8 to the wrong value $I_8^R = -\beta^2/18$, leaving the integral I_9 which fixes the integral over distributions (10.375). In this way, however, we could only satisfy one of the equations (10.513) and (10.514), the other would always be violated. Thus, any regularization different from ours will ruin immediately coordinate independence.

10.11.6 Covariant Results for Arbitrary Coordinates

It must be noted that if we were to use arbitrary rather than Riemann normal coordinates, we would find ambiguous integrals already at the two-loop level:

 :

$$I_{14} = \int_0^\beta \int_0^\beta d\tau\, d\tau' \,{}^{\cdot}\Delta(\tau,\tau')\, \Delta^{\cdot}(\tau,\tau')\,{}^{\cdot}\Delta^{\cdot}(\tau,\tau'), \tag{10.515}$$

 :

$$I_{15} = \int_0^\beta \int_0^\beta d\tau\, d\tau' \Delta(\tau,\tau')\,{}^{\cdot}\Delta^{\cdot 2}(\tau,\tau'). \tag{10.516}$$

Let us show that coordinate independence requires these integrals to have the values

$$I_{14} = \beta/24, \qquad I_{15}^R = -\beta/8, \tag{10.517}$$

where the superscript R denotes the finite part of an integral. We study first the ambiguities arising in one dimension. Without dimensional extension, the values (10.517) would be incompatible with partial integration and the equation of motion (10.393). In the integral (10.515), we use the symmetry ${}^{\cdot\cdot}\Delta(\tau,\tau') = \Delta^{\cdot\cdot}(\tau,\tau')$, apply partial integration twice taking care of nonzero boundary terms, and obtain on the one hand

$$\begin{aligned} I_{14} &= \frac{1}{2}\int_0^\beta \int_0^\beta d\tau\, d\tau'\, {}^{\cdot}\Delta(\tau,\tau') \frac{d}{d\tau}\left[\Delta^{\cdot 2}(\tau,\tau')\right] = -\frac{1}{2}\int_0^\beta \int_0^\beta d\tau\, d\tau'\, \Delta^{\cdot 2}(\tau.\tau')\, {}^{\cdot\cdot}\Delta(\tau,\tau') \\ &= -\frac{1}{6}\int_0^\beta \int_0^\beta d\tau\, d\tau' \frac{d}{d\tau'}\left[\Delta^{\cdot 3}(\tau,\tau')\right] = \frac{1}{6}\int_0^\beta d\tau \left[\Delta^{\cdot 3}(\tau,0) - \Delta^{\cdot 3}(\tau,\beta)\right] = \frac{\beta}{12}. \end{aligned} \tag{10.518}$$

On the other hand, we apply Eq. (10.394) and perform two regular integrals, reducing I_{14} to a form containing an undefined integral over a product of distributions:

$$\begin{aligned} I_{14} &= \int_0^\beta \int_0^\beta d\tau\, d\tau'\, {}^{\cdot}\Delta(\tau,\tau')\, \Delta^{\cdot}(\tau,\tau')\delta(\tau-\tau') - \frac{1}{\beta}\int_0^\beta \int_0^\beta d\tau\, d\tau'\, {}^{\cdot}\Delta(\tau,\tau')\, \Delta^{\cdot}(\tau,\tau') \\ &= \int_0^\beta \int_0^\beta d\tau\, d\tau' \left[-\frac{1}{4}\epsilon^2(\tau-\tau')\delta(\tau-\tau')\right] + \int_0^\beta d\tau\, {}^{\cdot}\Delta^2(\tau,\tau) + \frac{\beta}{12} \\ &= \beta\left[-\frac{1}{4}\int d\tau\, \epsilon^2(\tau)\delta(\tau) + \frac{1}{6}\right]. \end{aligned} \tag{10.519}$$

A third, mixed way of evaluating I_{14} employs one partial integration as in the first line of Eq. (10.518), then the equation of motion (10.393) to reduce I_{14} to yet another form

$$\begin{aligned} I_{14} &= \frac{1}{2}\int_0^\beta \int_0^\beta d\tau\, d\tau'\, \Delta^{\cdot 2}(\tau,\tau')\delta(\tau-\tau') = \\ &= \frac{1}{8}\int_0^\beta \int_0^\beta d\tau\, d\tau' \epsilon^2(\tau-\tau')\delta(\tau-\tau') + \frac{1}{2}\int_0^\beta d\tau\, {}^{\cdot}\Delta^2(\tau,\tau) \\ &= \beta\left[\frac{1}{8}\int d\tau\, \epsilon^2(\tau)\delta(\tau) + \frac{1}{24}\right]. \end{aligned} \tag{10.520}$$

We now see that if we set [compare with the correct equation (10.375)]

$$\int d\tau [\epsilon(\tau)]^2\, \delta(\tau) \equiv \frac{1}{3}, \qquad \text{(false)}, \tag{10.521}$$

the last two results (10.520) and (10.519) coincide with the first in Eq. (10.518). The definition (10.521) would obviously be consistent with partial integration if we insert $\delta(\tau) = \dot{\epsilon}(\tau)/2$:

$$\int d\tau [\epsilon(\tau)]^2 \, \delta(\tau) = \frac{1}{2} \int d\tau [\epsilon(\tau)]^2 \, \dot{\epsilon}(\tau) = \frac{1}{6} \int d\tau \frac{d}{d\tau} [\epsilon(\tau)]^3 = \frac{1}{3}. \tag{10.522}$$

In spite of this consistency with partial integration and the equation of motion, Eq. (10.521) is incompatible with the requirement of coordinate independence. This can be seen from the discrepancy between the resulting value $I_{14} = \beta/12$ and the necessary (10.517). In earlier work on the subject by other authors [31]– [36], this discrepancy was compensated by adding the above-mentioned (on p. 817) noncovariant term to the classical action, in violation of Feynman's construction rules for path integrals.

A similar problem appears with the other Feynman integral (10.516). Applying first Eq. (10.394) we obtain

$$I_{15} = \int_0^\beta \int_0^\beta d\tau \, d\tau' \Delta(\tau, \tau') \delta^2(\tau - \tau') - \frac{2}{\beta} \int_0^\beta d\tau \Delta(\tau, \tau) + \frac{1}{\beta^2} \int_0^\beta \int_0^\beta d\tau d\tau' \Delta(\tau, \tau'). \tag{10.523}$$

For the integral containing the square of the δ-function we must postulate the integration rule (10.369) to obtain a divergent term

$$I_{15}^{\rm div} = \delta(0) \int_0^\beta d\tau \, \Delta(\tau, \tau) = \delta(0) \frac{\beta^2}{6}, \tag{10.524}$$

which is proportional to $\delta(0)$, and compensates a similar term from the measure. The remaining integrals in (10.523) are finite and yield the regular part $I_{15}^R = -\beta/4$, which we shall see to be inconsistent with coordinate invariance. In another calculation of I_{15}, we first add and subtract the UV-divergent term, writing

$$I_{15} = \int_0^\beta \int_0^\beta d\tau \, d\tau' \Delta(\tau, \tau') \left[{}^{\cdot}\Delta^{\cdot 2}(\tau, \tau') - \delta^2(\tau - \tau') \right] + \delta(0) \frac{\beta^2}{6}. \tag{10.525}$$

Replacing $\delta^2(\tau - \tau')$ by the square of the left-hand side of the equation of motion (10.393), and integrating the terms in brackets by parts, we obtain

$$\begin{aligned}
I_{15}^R &= \int_0^\beta \int_0^\beta d\tau \, d\tau' \Delta(\tau, \tau') \left[{}^{\cdot}\Delta^{\cdot 2}(\tau, \tau') - \Delta^{\cdot\cdot\, 2}(\tau, \tau') \right] \\
&= \int_0^\beta \int_0^\beta d\tau \, d\tau' \left[- {}^{\cdot}\Delta\,(\tau, \tau') \, \Delta^{\cdot}(\tau, \tau') \, {}^{\cdot}\Delta^{\cdot}(\tau, \tau') - \Delta(\tau, \tau') \, \Delta^{\cdot}(\tau, \tau') \, {}^{\cdot\cdot}\Delta \, d(\tau, \tau') \right] \\
&- \int_0^\beta \int_0^\beta d\tau \, d\tau' \left[- \Delta^{\cdot 2}(\tau, \tau') \, \Delta^{\cdot\cdot}(\tau, \tau') - \Delta(\tau, \tau') \, \Delta^{\cdot}(\tau, \tau') \, \Delta^{\cdots}(\tau, \tau') \right] \\
&= -I_{14} + \int_0^\beta \int_0^\beta d\tau \, d\tau' \, \Delta^{\cdot 2}(\tau, \tau') \, \Delta^{\cdot\cdot}(\tau, \tau') = -I_{14} - \beta/6.
\end{aligned} \tag{10.526}$$

The value of the last integral follows from partial integration.

For a third evaluation of I_{15} we insert the equation of motion (10.393) and bring the last integral in the fourth line of (10.526) to

$$-\int_0^\beta \int_0^\beta d\tau \, d\tau' \, \Delta^{\cdot 2}(\tau, \tau') \delta(\tau - \tau') = -\beta \left[\frac{1}{4} \int d\tau \, \epsilon^2(\tau) \delta(\tau) + \frac{1}{12} \right]. \tag{10.527}$$

All three ways of calculation lead, with the assignment (10.521) to the singular integral, to the same result $I_{15}^R = -\beta/4$ using the rule (10.521). This, however, is again in disagreement with the coordinate-independent value in Eq. (10.517). Note that both integrals I_{14} and I_{15}^R are too large by a factor 2 with respect to the necessary (10.517) for coordinate independence.

How can we save coordinate independence while maintaining the equation of motion and partial integration? The direction in which the answer lies is suggested by the last line of Eq. (10.520): we must find a consistent way to have an integral $\int d\tau\, [\epsilon(\tau)]^2\, \delta(\tau) = 0$, as in Eq. (10.375), instead of the false value (10.521), which means that we need a reason for forbidding the application of partial integration to this singular integral. For the calculation at the infinite time interval, this problem was solved in Refs. [23]–[25] with the help of dimensional regularization.

In dimensional regularization, we would write the Feynman integral (10.515) in d dimensions as

$$I_{14}^d = \int\int d^d x\, d^d x'\, {}_\mu\Delta(x,x')\Delta_\nu(x,x')\, {}_\mu\Delta_\nu(x,x'), \tag{10.528}$$

and see that the different derivatives on ${}_\mu\Delta_\nu(x,x')$ prevent us from applying the field equation (10.418), in contrast to the one-dimensional calculation. We can, however, apply partial integration as in the first line of Eq. (10.518), and arrive at

$$I_{14}^d \;\; = \;\; -\frac{1}{2}\int\int d^d x\, d^d x'\, \Delta_\nu^2(x,x')\Delta_{\mu\mu}(x,x'). \tag{10.529}$$

In contrast to the one-dimensional expression (10.518), a further partial integration is impossible. Instead, we may apply the field equation (10.418), go back to one dimension, and apply the integration rule (10.375) as in Eq. (10.520) to obtain the correct result $I_{14} = \beta/24$ guaranteeing coordinate independence.

The Feynman integral (10.516) for I_{15} is treated likewise. Its d-dimensional extension is

$$I_{15}^d = \int\int d^d x\, d^d x'\, \Delta(x,x')\left[{}_\mu\Delta_\nu(x,x')\right]^2 . \tag{10.530}$$

The different derivatives on ${}_\mu\Delta_\nu(x,x')$ make it impossible to apply a dimensionally extended version of equation (10.394) as in Eq. (10.523). We can, however, extract the UV-divergence as in Eq. (10.525), and perform a partial integration on the finite part which brings it to a dimensionally extended version of Eq. (10.526):

$$I_{15}^R = -I_{14} + \int d^d x\, d^d x'\, \Delta_\nu^2(x,x')\Delta_{\mu\mu}(x,x'). \tag{10.531}$$

On the right-hand side we use the field equation (10.418), as in Eq. (10.527), return to $d=1$, and use the rule (10.375) to obtain the result $I_{15}^R = -I_{14} - \beta/12 = -\beta/8$, again guaranteeing coordinate independence.

Thus, by keeping only track of a few essential properties of the theory in d dimensions we indeed obtain a simple consistent procedure for calculating singular Feynman integrals. All results obtained in this way ensure coordinate independence. They agree with what we would obtain using the one-dimensional integration rule (10.375) for the product of two ϵ- and one δ-distribution.

Our procedure gives us unique rules telling us where we are allowed to apply partial integration and the equation of motion in one-dimensional expressions. Ultimately, all integrals are brought to a regular form, which can be continued back to one time dimension for a direct evaluation. This procedure is obviously much simpler than the previous explicit calculations in d dimensions with the limit $d \to 1$ taken at the end.

The coordinate independence would require the equations (10.517). Thus, although the calculation in normal coordinates are simpler and can be carried more easily to higher orders, the perturbation in arbitrary coordinates help to fix more ambiguous integrals.

Let us see how the integrals I_{14} and I_{15} arise in the perturbation expansion of the time evolution amplitude in arbitrary coordinates up to the order η, and that the values in (10.517) are necessary to guarantee a covariant result. We use arbitrary coordinates and expand the metric around the origin. Dropping the increment symbol δ in front of δq^μ, we write:

$$g_{\mu\nu}(q) \;=\; \delta_{\mu\nu} + \sqrt{\eta}(\partial_\lambda g_{\mu\nu})q^\lambda + \eta\,\frac{1}{2}(\partial_\lambda\partial_\kappa g_{\mu\nu})q^\lambda q^\kappa, \tag{10.532}$$

with the expansion parameter η keeping track of the orders of the perturbation series. At the end it will be set equal to unity. The determinant has the expansion to order η:

$$\log g(q) \;=\; \sqrt{\eta}\, g^{\mu\nu}(\partial_\lambda g_{\mu\nu})q^\lambda + \eta\,\frac{1}{2}\,g^{\mu\nu}\left[(\partial_\lambda\partial_\kappa g_{\mu\nu}) - g^{\sigma\tau}(\partial_\lambda g_{\mu\sigma})(\partial_\kappa g_{\nu\tau})\right]q^\lambda q^\kappa. \tag{10.533}$$

The total interaction (10.410) becomes

$$\begin{aligned}\mathcal{A}^{\rm int}_{\rm tot}[q] \;=\; &\int_0^\beta d\tau\Big\{\Big[\frac{1}{2}\sqrt{\eta}\,(\partial_\kappa g_{\mu\nu})q^\kappa + \frac{1}{4}\eta\,(\partial_\lambda\partial_\kappa g_{\mu\nu})q^\lambda q^\kappa\Big]\dot q^\mu\dot q^\nu \qquad (10.534)\\ &- \frac{1}{2}\sqrt{\eta}\,\delta(0)g^{\mu\nu}(\partial_\kappa g_{\mu\nu})q^\kappa - \frac{1}{4}\eta\,\delta(0)\,g^{\mu\nu}\Big[(\partial_\lambda\partial_\kappa g_{\mu\nu}) - g^{\sigma\tau}\,(\partial_\lambda g_{\mu\sigma})(\partial_\kappa g_{\nu\tau})\Big]q^\lambda q^\kappa\Big\}.\end{aligned}$$

Using the relations following directly from the definition of the Christoffel symbols (1.70) and (1.71),

$$\begin{aligned}\partial_\kappa g_{\mu\nu} &= -g_{\mu\sigma}g_{\nu\tau}\partial_\kappa g^{\sigma\tau} = \Gamma_{\kappa\mu\nu} + \Gamma_{\kappa\nu\mu} = 2\Gamma_{\kappa\{\mu\nu\}}, \qquad g^{\mu\nu}\,(\partial_\kappa g_{\mu\nu}) = 2\Gamma_{\kappa\mu}{}^{\mu},\\ \partial_\lambda\partial_\kappa g_{\mu\nu} &= \partial_\lambda\Gamma_{\kappa\mu\nu} + \partial_\lambda\Gamma_{\kappa\nu\mu} = \partial_\lambda\Gamma_{\kappa\{\mu\nu\}},\\ g^{\mu\nu}\,(\partial_\tau\partial_\lambda g_{\mu\nu}) &= 2g^{\mu\nu}\partial_\kappa\Gamma_{\sigma\mu\nu} = 2\partial_\kappa\,(g^{\mu\nu}\,\Gamma_{\sigma\mu\nu}) - 2\Gamma_{\sigma\mu\nu}\partial_\kappa g^{\mu\nu}\\ &= 2\left(\partial_\kappa\Gamma_{\sigma\mu}{}^{\mu} + g^{\mu\nu}\Gamma_{\sigma\mu\tau}\Gamma_{\lambda\nu}{}^{\tau} + \Gamma_{\sigma\mu}{}^{\nu}\Gamma_{\lambda\nu}{}^{\mu}\right),\end{aligned} \tag{10.535}$$

this becomes

$$\begin{aligned}\mathcal{A}^{\rm int}_{\rm tot}[q] = &\int_0^\beta d\tau\Big\{\Big[\sqrt{\eta}\,\Gamma_{\kappa\mu\nu}q^\kappa + \frac{\eta}{2}\,\partial_\lambda\Gamma_{\kappa\mu\nu}\,q^\lambda q^\kappa\Big]\dot q^\mu\dot q^\nu\\ &- \delta(0)\,\Big[\sqrt{\eta}\,\Gamma_{\mu\kappa}{}^{\mu}q^\kappa + \frac{\eta}{2}\,\partial_\lambda\Gamma_{\tau\mu}{}^{\mu}\,q^\lambda q^\tau\Big]\Big\}.\end{aligned} \tag{10.536}$$

The derivative of the Christoffel symbol in the last term can also be written differently using the identity $\partial_\lambda g^{\mu\nu} = -g^{\mu\sigma}g^{\nu\tau}\partial_\lambda g_{\sigma\tau}$ as follows:

$$\begin{aligned}\partial_\lambda\Gamma_{\tau\mu}{}^{\mu} &= \partial_\lambda g^{\mu\nu}\Gamma_{\tau\mu\nu} = g^{\mu\nu}\partial_\lambda\Gamma_{\tau\mu\nu} = g^{\mu\nu}\partial_\lambda\Gamma_{\tau\mu\nu} - g^{\mu\sigma}g^{\nu\tau}\,(\partial_\lambda g_{\sigma\tau})\,\Gamma_{\tau\mu\nu}\\ &= \partial_\lambda\Gamma_{\tau\mu\nu} - (\Gamma_{\lambda\mu\nu} + \Gamma_{\lambda\nu\mu})\,\Gamma_\tau{}^{\mu\nu}.\end{aligned} \tag{10.537}$$

To first order in η, we obtain from the first cumulant $\left\langle\mathcal{A}^{\rm int}_{\rm tot}[q]\right\rangle_c$:

$$\beta f_1^{(1)} = \eta\int_0^\beta d\tau\,\Big\langle\,\frac{1}{2}\partial_\lambda\Gamma_{\kappa\mu\nu}q^\lambda q^\kappa\dot q^\mu\dot q^\nu - \delta(0)\frac{1}{2}\,\partial_\lambda\Gamma_{\tau\mu}{}^{\mu}q^\lambda q^\tau\,\Big\rangle_c, \tag{10.538}$$

the diagrams (10.490) corresponding to the analytic expression [compare (10.491)]

$$\begin{aligned}\beta f_1^{(1)} = &\frac{\eta}{2}\partial_\lambda\Gamma_{\kappa\{\mu\nu\}}\int_0^\beta d\tau\Big\{g^{\mu\nu}g^{\kappa\lambda}\,{}^{\cdot}\Delta^{\cdot}(\tau,\tau)\Delta(\tau,\tau) + 2g^{\mu\kappa}g^{\nu\lambda}\,{}^{\cdot}\Delta^2(\tau,\tau) - \delta(0)g^{\mu\nu}g^{\kappa\lambda}\Delta(\tau,\tau)\Big\}\\ &- \frac{\eta}{2}g^{\lambda\kappa}\,(\Gamma_{\lambda\nu}{}^{\mu}\,\Gamma_{\mu\kappa}{}^{\nu} + g^{\tau\mu}\,\Gamma_{\tau\kappa}{}^{\nu}\,\Gamma_{\mu\lambda\nu})\,\delta(0)\int_0^\beta d\tau\,\Delta(\tau,\tau).\end{aligned} \tag{10.539}$$

Replacing ${}^{\cdot}\Delta^{\cdot}(\tau,\tau)$ by $\delta(0) - 1/\beta$ according to (10.394), and using the integrals (10.436), the $\delta(0)$-terms in the first integral cancel and we obtain

$$\beta f_1^{(1)} = -\beta\frac{\eta}{4}\Big[\frac{g^{\mu\nu}g^{\kappa\lambda}}{6}(\partial_\lambda\Gamma_{\mu\kappa\nu} - \partial_\mu\Gamma_{\kappa\lambda\nu}) - \frac{g^{\lambda\kappa}}{3}\,(\Gamma_{\lambda\nu}{}^{\mu}\,\Gamma_{\mu\kappa}{}^{\nu} + g^{\tau\mu}\Gamma_{\tau\kappa}{}^{\nu}\,\Gamma_{\mu\lambda\nu})\delta(0)\Big]. \tag{10.540}$$

In addition, there are contributions of order η from the second cumulant

$$-\frac{\eta}{2!}\int_0^\beta d\tau\int_0^\beta d\tau' \Big\langle \left[\Gamma_{\kappa\mu\nu}q^\kappa(\tau)\dot q^\mu(\tau)\dot q^\nu(\tau)+\delta(0)\Gamma_{\mu\kappa}{}^\mu q^\kappa(\tau)\right] \\ \times\left[\Gamma_{\kappa\mu\nu}q^\kappa(\tau')\dot q^\mu(\tau')\dot q^\nu(\tau')+\delta(0)\Gamma_{\mu\kappa}{}^\mu q^\kappa(\tau')\right]\Big\rangle_c, \tag{10.541}$$

These add to the free energy

$$\beta f_1^{(2)} = -\frac{\eta}{2}g^{\lambda\kappa}\,\Gamma_{\lambda\mu}{}^\mu\,\Gamma_{\kappa\nu}{}^\nu\Big[$$  $$-\,2\delta(0)$$  $$+\,\delta^2(0)$$ $$\Big],$$

$$\beta f_1^{(3)} = -\,\eta\Gamma_{\lambda\mu}{}^\mu\,(g^{\lambda\kappa}\,\Gamma_{\kappa\nu}{}^\nu+g^{\nu\kappa}\,\Gamma_{\nu\kappa}{}^\lambda)\Big[$$  $$-\,\delta(0)$$  $$\Big],$$

$$\beta f_1^{(4)} = -\,\frac{\eta}{2}(g^{\mu\lambda}\,g^{\kappa\tau}\,\Gamma_{\mu\lambda}{}^\nu\,\Gamma_{\kappa\tau\nu}+g^{\mu\nu}\,\Gamma_{\mu\kappa}{}^\kappa\,\Gamma_{\nu\lambda}{}^\lambda+2g^{\mu\nu}\,\Gamma_{\mu\nu}{}^\kappa\,\Gamma_{\kappa\lambda}{}^\lambda)$$  ,

$$\beta f_1^{(5)} = -\,\frac{\eta}{2}(g^{\mu\kappa}\,g^{\nu\lambda}\,\Gamma_{\mu\lambda}{}^\tau\,\Gamma_{\kappa\beta}+3g^{\mu\kappa}\,\Gamma_{\mu\lambda}{}^\tau\,\Gamma_{\tau\kappa}{}^\lambda)$$  ,

$$\beta f_1^{(6)} = -\,\frac{\eta}{2}g^{\lambda\kappa}\,(\Gamma_{\lambda\nu}{}^\mu\,\Gamma_{\mu\kappa}{}^\nu+g^{\mu\tau}\,\Gamma_{\tau\kappa}{}^\nu\,\Gamma_{\mu\lambda\nu})$$  . (10.542)

The Feynman integrals associated with the diagrams in the first and second lines are

$$I_{11}=\int\!\!\int d\tau\,d\tau'\,\left\{{}^{\cdot}\Delta^{\cdot}(\tau,\tau)\Delta(\tau,\tau')\,{}^{\cdot}\Delta^{\cdot}(\tau',\tau')-2\delta(0)\,{}^{\cdot}\Delta^{\cdot}(\tau,\tau)\,\Delta(\tau,\tau')+\delta^2(0)\,\Delta(\tau,\tau')\right\} \tag{10.543}$$

and

$$I_{12}=\int\!\!\int d\tau\,d\tau'\,\left\{{}^{\cdot}\Delta\,(\tau,\tau)\,{}^{\cdot}\Delta\,(\tau,\tau')\,{}^{\cdot}\Delta^{\cdot}(\tau',\tau')-\delta(0)\,{}^{\cdot}\Delta\,(\tau,\tau)\,{}^{\cdot}\Delta\,(\tau,\tau')\right\}, \tag{10.544}$$

respectively. Replacing in Eqs. (10.543) and (10.544) ${}^{\cdot}\Delta^{\cdot}(\tau,\tau)$ and ${}^{\cdot}\Delta^{\cdot}(\tau',\tau')$ by $\delta(0)-1/\beta$ leads to cancellation of the infinite factors $\delta(0)$ and $\delta^2(0)$ from the measure, such that we are left with

$$I_{11}=\frac{1}{\beta^2}\int_0^\beta d\tau\int_0^\beta d\tau'\,\Delta(\tau,\tau')=\frac{\beta}{12} \tag{10.545}$$

and

$$I_{12}=-\frac{1}{\beta}\int_0^\beta d\tau\int_0^\beta d\tau'\,{}^{\cdot}\Delta\,(\tau,\tau)\,{}^{\cdot}\Delta\,(\tau,\tau')=-\frac{\beta}{12}. \tag{10.546}$$

The Feynman integral of the diagram in the third line of Eq. (10.542) has d-dimensional extension

$$I_{13}\;=\;\int\!\!\int d\tau\,d\tau'\,{}^{\cdot}\Delta\,(\tau,\tau)\,\Delta^{\cdot}(\tau',\tau')\,{}^{\cdot}\Delta^{\cdot}(\tau,\tau') \\ \to\int\!\!\int d^dx\,d^dx'{}_\mu\Delta(x,x)\Delta_\nu(x',x')_\mu\Delta_\nu(x,x'). \tag{10.547}$$

Integrating this partially yields

$$I_{13}=\frac{1}{\beta}\int\!\!\int d\tau\,d\tau'\,\Delta^{\cdot}(\tau,\tau')\,{}^{\cdot}\Delta\,(\tau',\tau')=\frac{1}{\beta}\int_0^\beta d\tau\int_0^\beta d\tau'\,{}^{\cdot}\Delta\,(\tau,\tau)\,{}^{\cdot}\Delta\,(\tau,\tau')=\frac{\beta}{12}, \tag{10.548}$$

where we have interchanged the order of integration $\tau \leftrightarrow \tau'$ in the second line of Eq. (10.548) and used $d[\dot{\Delta}(\tau,\tau)]/d\tau = -1/\beta$. Multiplying the integrals (10.545), (10.546), and (10.548) by corresponding vertices in Eq. (10.542) and adding them together, we obtain

$$\beta f_1^{(2)} + \beta f_1^{(3)} + \beta f_1^{(4)} = -\frac{\eta\beta}{24}\, g^{\mu\nu} g^{\kappa\lambda}\, \Gamma_{\mu\nu}{}^{\tau}\, \Gamma_{\kappa\lambda\tau}. \tag{10.549}$$

The contributions of the last three diagrams in $\beta f_1^{(5)}$ and $\beta f_1^{(6)}$ of (10.542) are determined by the initially ambiguous integrals (10.515) and (10.516) to be equal to $I_{14} = -\beta/24$ and $I_{15} = -\beta/8 + \delta(0)\beta^2/6$, respectively. Moreover, the $\delta(0)$-part in the latter, when inserted into the last line of Eq. (10.542) for $f_1^{(6)}$, is canceled by the contribution of the local diagram with the factor $\delta(0)$ in $f_1^{(1)}$ of (10.540). We see here an example that with general coordinates, the divergences containing powers of $\delta(0)$ no longer cancel order by order in $\hbar$, but do so at the end.

Thus only the finite part $I_{15}^R = -\beta/24$ remains and we find

$$\begin{aligned}\beta f_1^{(5)} + \beta f_1^{(6)R} &= -\frac{\eta}{2}\Big\{ g^{\mu\kappa} g^{\nu\lambda} \Gamma_{\mu\lambda}{}^{\tau} \Gamma_{\kappa\nu\tau}\left(I_{14} + I_{15}^R\right) + g^{\lambda\kappa} \Gamma_{\lambda\nu}{}^{\mu} \Gamma_{\mu\kappa}{}^{\nu}\left(3I_{14} + I_{15}^R\right)\Big\} \\ &= \frac{\eta\beta}{24}\, g^{\mu\nu} g^{\kappa\lambda}\, \Gamma_{\mu\kappa}{}^{\sigma}\, \Gamma_{\nu\lambda\sigma}.\end{aligned} \tag{10.550}$$

By adding this to (10.549), we find the sum of all diagrams in (10.542) as follows

$$\sum_{i=2}^{6} \beta f_1^{(i)} = -\frac{\eta\beta}{24}\, g^{\mu\nu} g^{\kappa\lambda} \left(\Gamma_{\mu\nu}{}^{\sigma} \Gamma_{\kappa\lambda\sigma} - \Gamma_{\mu\kappa}{}^{\sigma} \Gamma_{jl,\,n}\right). \tag{10.551}$$

Together with the regular part of (10.540) in the first line, this yields the sum of all first-order diagrams

$$\sum_{i=1}^{6} \beta f_1^{(i)} = -\frac{\eta\beta}{24}\, g^{\mu\nu} g^{\kappa\lambda}\, R_{\lambda\mu\kappa\nu} = \frac{\eta\beta}{24}\, \bar{R}. \tag{10.552}$$

The result is covariant and agrees, of course, with Eq. (10.492) derived with normal coordinate. Note that to obtain this covariant result, the initially ambiguous integrals (10.515) and (10.516) over distributions appearing ni Eq. (10.550) mast satisfy

$$\begin{aligned} I_{14} + I_{15}^R &= -\frac{\beta}{12}, \\ 3I_{14} + I_{15}^R &= 0, \end{aligned} \tag{10.553}$$

which leaves only the values (10.517).

10.12 Effective Classical Potential in Curved Space

In Chapter 5 we have seen that the partition function of a quantum statistical system in flat space can always be written as an integral over a classical Boltzmann factor $\exp[-\beta V^{\rm eff\,cl}(\mathbf{x}_0)]$, where $B(\mathbf{x}_0) = V^{\rm eff\,cl}(\mathbf{x}_0)$ is the so-called *effective classical potential* containing the effects of all quantum fluctuations. The variable of integration is the temporal path average $\mathbf{x}_0 \equiv \beta^{-1}\int_0^\beta d\tau\, \mathbf{x}(\tau)$. In this section we generalize this concept to curved space, and show how to calculate perturbatively the high-temperature expansion of $V^{\rm eff\,cl}(q_0)$. The requirement of independence under coordinate transformations $q^\mu(\tau) \to q'^\mu(\tau)$ introduces subtleties into the definition and treatment of the path average q_0^μ, and covariance is achieved only with the help of a procedure invented by Faddeev and Popov [49] to deal with gauge freedoms in quantum field theory.

In the literature, attempts to introduce an effective classical potential in curved space around a fixed temporal average $q_0 \equiv \bar{q}(\tau) \equiv \beta^{-1} \int_0^\beta d\tau q(\tau)$ have so far failed and produced a two-loop perturbative result for $V^{\mathrm{eff\,cl}}(q_0)$ which turned out to deviate from the covariant one by a noncovariant total derivative [34], in contrast to the covariant result (10.494) obtained with Dirichlet boundary conditions. For this reason, perturbatively defined path integrals with periodic boundary conditions in curved space have been of limited use in the presently popular first-quantized worldline approach to quantum field theory (also called the string-inspired approach reviewed in Ref. [50]). In particular, is has so far been impossible to calculate with periodic boundary conditions interesting quantities such as curved-space effective actions, gravitational anomalies, and index densities, all results having been reproduced with Dirichlet boundary conditions [46, 52].

The development in this chapter cures the problems by exhibiting a manifestly covariant integration procedure for periodic paths [51]. It is an adaption of similar procedures used before in the effective action formalism of two-dimensional sigma-models [46]. Covariance is achieved by expanding the fluctuations in the neighborhood of any given point in powers of geodesic coordinates, and by a covariant definition of a path average different from the naive temporal average. As a result, we shall find the same locally covariant perturbation expansion of the effective classical potential as in Eq. (10.494) calculated with Dirichlet boundary conditions.

All problems encountered in the literature occur in the first correction terms linear in β in the time evolution amplitude. It will therefore be sufficient to consider only to lowest-order perturbation expansion. For this reason we shall from now on drop the parameter of smallness η used before.

10.12.1 Covariant Fluctuation Expansion

We want to calculate the partition function from the functional integral over all periodic paths

$$Z \;=\; \oint \mathcal{D}^D q \sqrt{g(q)} e^{-\mathcal{A}[q]}, \tag{10.554}$$

where the symbol $\oint$ indicates the periodicity of the paths. By analogy with (2.443), we split the paths into a time-independent and a time-dependent part:

$$q^\mu = q_0^\mu + \eta^\mu(\tau), \tag{10.555}$$

with the goal to express the partition function as in Eq. (3.811) by an ordinary integral over an effective classical partition function

$$Z = \int \frac{d^D q_0}{\sqrt{2\pi\beta}^D} \sqrt{g(q_0)}\, e^{-\beta V^{\mathrm{eff\,cl}}(q_0)}, \tag{10.556}$$

where $V^{\rm eff\,cl}(q_0)$ is the curved-space version of the effective classical partition function. For a covariant treatment, we parametrize the small fluctuations $\eta^\mu(\tau)$ in terms of the prepoint normal coordinates $\Delta\xi^\mu(\tau)$ of the point q_0^μ introduced in Eq. (10.98), which are here geodesic due to the absence of torsion. Omitting the smallness symbols Δ, there will be some *nonlinear* decomposition

$$q^\mu(\tau) = q_0^\mu + \eta^\mu(q_0,\xi), \tag{10.557}$$

where $\eta^\mu(q_0,\xi)=0$ for $\xi^\mu=0$. Inverting the relation (10.98) we obtain

$$\eta^\mu(q_0,\xi) = \xi^\mu - \frac{1}{2}\bar\Gamma_{\sigma\tau}{}^\mu(q_0)\xi^\sigma\xi^\tau - \frac{1}{6}\bar\Gamma_{\sigma\tau\kappa}{}^\mu(q_0)\xi^\sigma\xi^\tau\xi^\kappa - \dots, \tag{10.558}$$

where the coefficients $\bar\Gamma_{\sigma\tau\dots\kappa}{}^\mu(q_0)$ with more than two subscripts are defined similarly to covariant derivatives with respect to lower indices (they are not covariant quantities):

$$\bar\Gamma_{\sigma\tau\kappa}{}^\mu(q_0) = \nabla_\kappa\bar\Gamma_{\sigma\tau}{}^\mu = \partial_\kappa\bar\Gamma_{\sigma\tau}{}^\mu - 2\bar\Gamma_{\kappa\sigma}{}^\nu\bar\Gamma_{\nu\tau}{}^\mu\ ,\ \dots\ . \tag{10.559}$$

If the initial coordinates q^μ are themselves geodesic at q_0^μ, all coefficients $\bar\Gamma_{\sigma\tau\dots\kappa}{}^\mu(q_0)$ in Eq. (10.558) are zero, so that $\eta^\mu(\tau)=\xi^\mu(\tau)$, and the decomposition (10.557) is linear. In arbitrary coordinates, however, $\eta^\mu(\tau)$ does not transform like a vector under coordinate transformations, and we must use the nonlinear decomposition (10.557).

We now transform the path integral (10.554) to the new coordinates $\xi^\mu(\tau)$ using Eqs. (10.557)–(10.559). The perturbation expansion for the transformed path integral over $\xi^\mu(\tau)$ is constructed for any chosen q_0^μ by expanding the total action (10.408) including the measure factor (10.406) in powers of small linear fluctuations $\xi^\mu(\tau)$. Being interested only in the lowest-order contributions we shall from now on drop the parameter of smallness η counting the orders in the earlier perturbation expansions. This is also useful since the similar symbol $\eta^\mu(\tau)$ is used here to describe the path fluctuations. The action relevant for the terms to be calculated here consists of a free action, which we write after a partial integration as

$$\mathcal{A}^{(0)}[q_0,\xi] = g_{\mu\nu}(q_0)\int_0^\beta d\tau\frac{1}{2}\,\xi^\mu(\tau)(-\partial_\tau^2)\xi^\nu(\tau), \tag{10.560}$$

and an interaction which contains only the leading terms in (10.479):

$$\mathcal{A}^{\rm int}_{\rm tot}[q_0;\xi] \ = \ \int_0^\beta d\tau\left[\frac{1}{6}\bar R_{\mu\lambda\nu\kappa}\,\xi^\lambda\xi^\kappa\,\dot\xi^\mu\dot\xi^\nu + \frac{1}{6}\delta(0)\bar R_{\mu\nu}\,\xi^\mu\xi^\nu\right]. \tag{10.561}$$

The partition function (10.554) in terms of the coordinates $\xi^\mu(\tau)$ is obtained from the perturbation expansion

$$Z = \oint\mathcal{D}^D\xi(\tau)\sqrt{g(q_0)}\,e^{-\mathcal{A}^{(0)}[q_0,\xi]-\mathcal{A}^{\rm int}_{\rm tot}[q_0,\xi]}. \tag{10.562}$$

The path integral (10.562) cannot immediately be calculated perturbatively in the standard way, since the quadratic form of the free action (10.560) is degenerate. The spectrum of the operator $-\partial_\tau^2$ in the space of periodic functions $\xi^\mu(\tau)$ has a zero mode. The zero mode is associated with the fluctuations of the temporal average of $\xi^\mu(\tau)$:

$$\xi_0^\mu = \bar{\xi}^\mu \equiv \beta^{-1}\int_0^\beta d\tau\, \xi^\mu(\tau). \tag{10.563}$$

Small fluctuations of ξ_0^μ have the effect of moving the path as a whole infinitesimally through the manifold. The same movement can be achieved by changing q_0^μ infinitesimally. Thus we can replace the integral over the path average ξ_0^μ by an integral over q_0^μ, provided that we properly account for the change of measure arising from such a variable transformation.

Anticipating such a change, the path average (10.563) can be set equal to zero eliminating the zero mode in the fluctuation spectrum. The basic free correlation function $\langle\xi^\mu(\tau)\xi^\nu(\tau')\rangle$ can then easily be found from its spectral representation as shown in Eq. (3.254). The result is

$$\langle\xi^\mu(\tau)\xi^\nu(\tau')\rangle^{q_0} = g^{\mu\nu}(q_0)(-\partial_\tau^2)^{-1}\,\delta(\tau-\tau') = g^{\mu\nu}(q_0)\bar{\Delta}(\tau,\tau'), \tag{10.564}$$

where $\bar{\Delta}(\tau,\tau')$ is a short notation for the translationally invariant periodic Green function $G^{\mathrm{p}\prime}_{0,\mathrm{e}}(\tau)$ of the operator $-\partial_\tau^2$ without the zero mode in Eq. (3.254) (for a plot see Fig. 3.4):

$$\bar{\Delta}(\tau,\tau') = \bar{\Delta}(\tau-\tau') \equiv \frac{(\tau-\tau')^2}{2\beta} - \frac{|\tau-\tau'|}{2} + \frac{\beta}{12}\,, \quad \tau,\tau'\in[0,\hbar\beta]. \tag{10.565}$$

This notation is useful since we shall have to calculate Feynman integrals of precisely the same form as previously with the Dirichlet-type correlation function Eq. (10.392). In contrast to (10.393) and (10.394) for $\Delta(\tau,\tau')$, the translational invariance of the periodic correlation function implies that $\bar{\Delta}(\tau,\tau') = \bar{\Delta}(\tau-\tau')$, so that the first time derivatives of $\bar{\Delta}(\tau,\tau')$ have opposite signs:

$${}^{\cdot}\bar{\Delta}(\tau,\tau') = -\bar{\Delta}^{\cdot}(\tau,\tau') \equiv \frac{\tau-\tau'}{\beta} - \frac{\epsilon(\tau-\tau')}{2}\,, \quad \tau,\tau'\in[0,\hbar\beta], \tag{10.566}$$

and the three possible double time derivatives are equal, up tp a sign:

$$-{}^{\cdot\cdot}\bar{\Delta}(\tau,\tau') = -\bar{\Delta}^{\cdot\cdot}(\tau,\tau') = {}^{\cdot}\bar{\Delta}^{\cdot}(\tau,\tau') = \delta(\tau-\tau') - 1/\beta. \tag{10.567}$$

The right-hand side contains an extra term on the right-hand side due to the missing zero eigenmode in the spectral representation of the δ-function:

$$\frac{1}{\beta}\sum_{m\neq 0} e^{-i\omega_m(\tau-\tau')} = \delta(\tau-\tau') - \frac{1}{\beta}\,. \tag{10.568}$$

The third equation in (10.567) happens to coincide with the differential equation (10.394) in the Dirichlet case. All three equations have the same right-hand side due to the translational invariance of $\bar{\Delta}(\tau,\tau')$.

10.12.2 Arbitrariness of q_0^μ

We now take advantage of an important property of the perturbation expansion of the partition function (10.562) around $q^\mu(\tau) = q_0^\mu$: the *independence* of the choice of q_0^μ. The separation (10.557) into a constant q_0^μ and a time-dependent $\xi^\mu(\tau)$ paths must lead to the same result for any nearby constant $q_0'^\mu$ on the manifold. The result must therefore be invariant under an arbitrary infinitesimal displacement

$$q_0^\mu \to q_{0\varepsilon}^\mu = q_0^\mu + \varepsilon^\mu, \quad |\varepsilon| \ll 1. \tag{10.569}$$

In the path integral, this will be compensated by some translation of fluctuation coordinates $\xi^\mu(\tau)$, which will have the general nonlinear form

$$\xi^\mu \to \xi_\varepsilon^\mu = \xi^\mu - \varepsilon^\nu Q_\nu^\mu(q_0, \xi). \tag{10.570}$$

The transformation matrix $Q_\nu^\mu(q_0,\xi)$ satisfies the obvious initial condition $Q_\nu^\mu(q_0,0) = \delta_\nu^\mu$. The path $q^\mu(\tau) = q^\mu(q_0,\xi(\tau))$ must remain invariant under simultaneous transformations (10.569) and (10.570), which implies that

$$\delta q^\mu \equiv q_\varepsilon^\mu - q^\mu = \varepsilon^\nu D_\nu q^\mu(q_0,\xi) = 0\,, \tag{10.571}$$

where D_μ is the infinitesimal transition operator

$$D_\mu = \frac{\partial}{\partial q_0^\mu} - Q_\mu^\nu(q_0,\xi)\frac{\partial}{\partial \xi^\nu}\,. \tag{10.572}$$

Geometrically, the matrix $Q_\nu^\mu(q_0,\xi)$ plays the role of a locally flat nonlinear connection [46]. It can be calculated as follows. We express the vector $q^\mu(q_0,\xi)$ in terms of the geodesic coordinates ξ^μ using Eqs. (10.557), (10.558), and (10.559), and substitute this into Eq. (10.571). The coefficients of ε^ν yield the equations

$$\delta_\nu^\mu + \frac{\partial \eta^\mu(q_0,\xi)}{\partial q_0^\nu} - Q_\nu^\kappa(q_0,\xi)\frac{\partial \eta^\mu(q_0,\xi)}{\partial \xi^\kappa} = 0, \tag{10.573}$$

where by Eq. (10.558):

$$\frac{\partial \eta^\mu(q_0,\xi)}{\partial q_0^\nu} = -\frac{1}{2}\partial_\nu \bar{\Gamma}_{(\sigma\tau)}{}^\mu(q_0)\xi^\sigma\xi^\tau - \dots\ , \tag{10.574}$$

and

$$\begin{aligned}\frac{\partial \eta^\mu(q_0,\xi)}{\partial \xi^\nu} &= \delta_\nu^\mu - \bar{\Gamma}_{(\nu\sigma)}{}^\mu(q_0)\xi^\sigma - \frac{1}{2}\bar{\Gamma}_{(\nu\sigma\tau)}{}^\mu(q_0)\xi^\sigma\xi^\tau - \dots \\ &= \delta_\nu^\mu - \bar{\Gamma}_{\nu\sigma}{}^\mu\xi^\sigma - \frac{1}{3}\left(\partial_\sigma\bar{\Gamma}_{\nu\tau}^\mu + \frac{1}{2}\partial_\nu\bar{\Gamma}_{\sigma\tau}{}^\mu - 2\bar{\Gamma}_{\tau\nu}{}^\kappa\bar{\Gamma}_{\kappa\sigma}{}^\mu - \bar{\Gamma}_{\tau\sigma}{}^\kappa\bar{\Gamma}_{\kappa\nu}{}^\mu\right)\xi^\sigma\xi^\tau - \dots\,.\end{aligned} \tag{10.575}$$

To find $Q_\nu^\mu(q_0,\xi)$, we invert the expansion (10.575) to

$$\begin{aligned}\left[\left(\frac{\partial \eta(q_0,\xi)}{\partial \xi}\right)^{-1}\right]^\mu_\nu &= \delta_\nu^\mu + \bar{\Gamma}_{\nu\sigma}^\mu\xi^\sigma + \frac{1}{3}\left(\partial_\sigma\bar{\Gamma}_{\nu\tau}^\mu + \frac{1}{2}\partial_\nu\bar{\Gamma}_{\sigma\tau}^\mu + \bar{\Gamma}_{\tau\nu}^\kappa\bar{\Gamma}_{\kappa\sigma}^\mu - \bar{\Gamma}_{\tau\sigma}^\kappa\bar{\Gamma}_{\kappa\nu}^\mu\right)\xi^\sigma\xi^\tau + \dots \\ &= \left(\frac{\partial \xi^\mu(q_0,\eta)}{\partial \eta^\nu}\right)_{\eta=\eta(q_0,\xi)}\,,\end{aligned} \tag{10.576}$$

the last equality indicating that the result (10.576) can also be obtained from the original expansion (10.98) in the present notation [compare (10.558)]:

$$\xi^\mu(q_0,\eta) = \eta^\mu + \frac{1}{2}\tilde{\Gamma}_{\sigma\tau}{}^\mu(q_0)\eta^\sigma\eta^\tau + \frac{1}{6}\tilde{\Gamma}_{\sigma\tau\kappa}{}^\mu(q_0)\eta^\sigma\eta^\tau\eta^\kappa + \dots, \tag{10.577}$$

with coefficients

$$\begin{aligned} \tilde{\Gamma}_{\sigma\tau}{}^\mu(q_0) &= \bar{\Gamma}_{\sigma\tau}{}^\mu, \\ \tilde{\Gamma}_{\sigma\tau\kappa}{}^\mu(q_0) &= \bar{\Gamma}_{\sigma\tau\kappa}{}^\mu + 3\bar{\Gamma}_{\kappa\sigma}{}^\nu\bar{\Gamma}_{\nu\tau}{}^\mu = \partial_\kappa\bar{\Gamma}_{\sigma\tau}{}^\mu + \bar{\Gamma}_{\kappa\sigma}{}^\nu\bar{\Gamma}_{\nu\tau}{}^\mu, \\ &\vdots \quad . \end{aligned} \tag{10.578}$$

Indeed, differentiating (10.577) with respect to η^ν, and re-expressing the result in terms of ξ^μ via Eq. (10.558), we find once more (10.576).

Multiplying both sides of Eq. (10.573) by (10.576), we express the nonlinear connection $Q^\mu_\nu(q_0,\xi)$ by means of geodesic coordinates $\xi^\mu(\tau)$ as

$$Q^\mu_\nu(q_0,\xi) = \delta^\mu_\nu + \bar{\Gamma}_{\nu\sigma}{}^\mu(q_0)\xi^\sigma + \frac{1}{3}\bar{R}_{\sigma\nu\tau}{}^\mu(q_0)\xi^\sigma\xi^\tau + \dots \quad . \tag{10.579}$$

The effect of simultaneous transformations (10.569), (10.570) upon the fluctuation function $\eta^\mu = \eta^\mu(q_0,\xi)$ in Eq. (10.558) is

$$\eta^\mu \to \eta'^\mu = \eta^\mu - \varepsilon^\nu\bar{Q}^\mu_\nu(q_0,\eta), \quad \bar{Q}^\mu_\nu(q_0,0) = \delta^\mu_\nu, \tag{10.580}$$

where the matrix $\bar{Q}^\mu_\nu(q_0,\eta)$ is related to $Q^\mu_\nu(q_0,\xi)$ as follows

$$\bar{Q}^\mu_\nu(q_0,\eta) = \left[Q^\kappa_\nu(q_0,\xi)\frac{\partial\eta^\mu(q_0,\xi)}{\partial\xi^\kappa} - \frac{\partial\eta^\mu(q_0,\xi)}{\partial q_0^\nu}\right]_{\xi=\xi(q_0,\eta)} . \tag{10.581}$$

Applying Eq. (10.573) to the right-hand side of Eq. (10.581) yields $\bar{Q}^\mu_\nu(q_0,\eta) = \delta^\mu_\nu$, as it should to compensate the translation (10.569).

The above independence of q_0^μ will be essential for constructing the correct perturbation expansion for the path integral (10.562). For some special cases of the Riemannian manifold, such as a surface of sphere in $D+1$ dimensions which forms a homogeneous space $O(D)/O(D-1)$, all points are equivalent, and the local independence becomes global. This will be discussed further in Section 10.12.6.

10.12.3 Zero-Mode Properties

We are now prepared to eliminate the zero mode by the condition of vanishing average $\bar{\xi}^\mu = 0$. As mentioned before, the vanishing fluctuation $\xi^\mu(\tau) = 0$ is obviously a classical saddle-point for the path integral (10.562). In addition, because of the symmetry (10.570) there exist other equivalent extrema $\xi^\mu_\varepsilon(\tau) = -\varepsilon^\mu = \text{const}$. The D components of ε^μ correspond to D zero modes which we shall eliminate in favor of a change of q_0^μ. The proper way of doing this is provided by the Faddeev-Popov procedure. We insert into the path integral (10.562) the trivial unit integral, rewritten with the help of (10.569):

$$1 = \int d^D q_0 \, \delta^{(D)}(q_{0\varepsilon} - q_0) = \int d^D q_0 \, \delta^{(D)}(\varepsilon), \tag{10.582}$$

and decompose the measure of path integration over all periodic paths $\xi^\mu(\tau)$ into a product of an ordinary integral over the temporal average $\xi_0^\mu = \bar{\xi}^\mu$, and a remainder containing only nonzero Fourier components [recall (2.448)]

$$\oint \mathcal{D}^D \xi = \int \frac{d^D \xi_0}{\sqrt{2\pi\beta}^D} \oint \mathcal{D}'^D \xi \, . \tag{10.583}$$

According to Eq. (10.570), the path average $\bar{\xi}^\mu$ is translated under ε^μ as follows

$$\bar{\xi}^\mu \to \bar{\xi}_\varepsilon^\mu = \bar{\xi}^\mu - \varepsilon^\nu \frac{1}{\beta} \int_0^\beta d\tau \, Q_\nu^\mu(q_0, \xi(\tau)). \tag{10.584}$$

Thus we can replace

$$\int \frac{d^D \xi_0}{\sqrt{2\pi\beta}^D} \to \int \frac{d^D \varepsilon}{\sqrt{2\pi\beta}^D} \det \left[\frac{1}{\beta} \int_0^\beta d\tau \, Q_\nu^\mu(q_0, \xi(\tau)) \right]. \tag{10.585}$$

Performing this replacement in (10.583) and performing the integral over ε^μ in the inserted unity (10.582), we obtain the measure of path integration in terms of q_0^μ and geodesic coordinates of zero temporal average

$$\begin{aligned} \oint \mathcal{D}^D \xi &= \int d^D q_0 \oint \frac{d^D \xi_0}{\sqrt{2\pi\beta}^D} \delta^{(D)}(\varepsilon) \oint \mathcal{D}'^D \xi \\ &= \int \frac{d^D q_0}{\sqrt{2\pi\beta}^D} \oint \mathcal{D}'^D \xi \, \det \left[\frac{1}{\beta} \int_0^\beta d\tau \, Q_\nu^\mu(q_0, \xi(\tau)) \right]. \end{aligned} \tag{10.586}$$

The factor on the right-hand side is the Faddeev-Popov determinant $\Delta[q_0, \xi]$ for the change from ξ_0^μ to q_0^μ. We shall write it as an exponential:

$$\Delta[q_0, \xi] = \det \left[\frac{1}{\beta} \int_0^\beta d\tau \, Q_\nu^\mu(q_0, \xi) \right] = e^{-\mathcal{A}^{\rm FP}[q_0, \xi]} \, , \tag{10.587}$$

where $\mathcal{A}^{\rm FP}[q_0, \xi]$ is an auxiliary action accounting for the Faddeev-Popov determinant

$$\mathcal{A}^{\rm FP}[q_0, \xi] \equiv -\mathrm{tr} \log \left[\frac{1}{\beta} \int_0^\beta \mathrm{d}\tau \, \mathrm{Q}_\nu^\mu(\mathrm{q}_0, \xi) \right] , \tag{10.588}$$

which must be included into the interaction (10.561). Inserting (10.579) into Eq. (10.588), we find explictly

$$\begin{aligned} \mathcal{A}^{\rm FP}[q_0, \xi] &= -\mathrm{tr} \log \left[\delta_\nu^\mu + (3\beta)^{-1} \int_0^\beta d\tau \, \bar{R}_{\sigma\nu\tau}{}^\mu(q_0) \xi^\sigma(\tau) \xi^\tau(\tau) + \dots \right] \\ &= \frac{1}{3\beta} \int_0^\beta d\tau \, \bar{R}_{\mu\nu}(q_0) \xi^\mu \xi^\nu + \dots \ . \end{aligned} \tag{10.589}$$

The contribution of this action will crucial for obtaining the correct perturbation expansion of the path integral (10.562).

With the new interaction

$$\mathcal{A}^{\rm int}_{\rm tot,FP}[q_0,\xi]=\mathcal{A}^{\rm int}_{\rm tot}[q_0,\xi]+\mathcal{A}^{\rm FP}[q_0,\xi] \tag{10.590}$$

the partition function (10.562) can be written as a classical partition function

$$Z=\int\frac{d^Dq_0}{\sqrt{2\pi\beta}^D}\sqrt{g(q_0)}\,e^{-\beta V^{\rm eff\,cl}(q_0)}\,, \tag{10.591}$$

where $V^{\rm eff\,cl}(q_0)$ is the curved-space version of the effective classical partition function of Ref. [53]. The effective classical Boltzmann factor

$$B(q_0)\equiv e^{-\beta V^{\rm eff\,cl}(q_0)} \tag{10.592}$$

is given by the path integral

$$B(q_0)=\oint\mathcal{D}'^D\xi\,\sqrt{g(q_0)}e^{-\mathcal{A}^{(0)}[q_0,\xi]-\mathcal{A}^{\rm int}_{\rm tot,FP}[q_0,\xi]}. \tag{10.593}$$

Since the zero mode is absent in the fluctuations on the right-hand side, the perturbation expansion is now straightforward. We expand the path integral (10.593) in powers of the interaction (10.590) around the free Boltzmann factor

$$B_0(q_0)=\oint\mathcal{D}'^D\xi\,\sqrt{g(q_0)}e^{-\int_0^\beta d\tau\,\frac{1}{2}g_{\mu\nu}(q_0)\dot\xi^\mu\dot\xi^\nu} \tag{10.594}$$

as follows:

$$B(q_0)=B_0(q_0)\left[1-\left\langle\mathcal{A}^{\rm int}_{\rm tot,FP}[q_0,\xi]\right\rangle^{q_0}+\frac{1}{2}\left\langle\mathcal{A}^{\rm int}_{\rm tot,FP}[q_0,\xi]^2\right\rangle^{q_0}-\dots\right], \tag{10.595}$$

where the q_0-dependent correlation functions are defined by the Gaussian path integrals

$$\langle\dots\rangle^{q_0}=B^{-1}(q_0)\oint\mathcal{D}'^D\xi\;[\dots]^{q_0}\,e^{-\mathcal{A}^{(0)}[q_0,\xi]}. \tag{10.596}$$

By taking the logarithm of (10.594), we obtain directly a cumulant expansion for the effective classical potential $V^{\rm eff\,cl}(q_0)$.

For a proper normalization of the Gaussian path integral (10.594) we diagonalize the free action in the exponent by going back to the orthonormal components Δx^i in (10.97). Omitting again the smallness symbols Δ, the measure of the path integral becomes simply:

$$\oint\mathcal{D}'^D\xi^\mu\,\sqrt{g(q_0)}=\oint\mathcal{D}'^Dx^i\,, \tag{10.597}$$

and we find

$$B_0(q_0)=\oint\mathcal{D}'^Dx^i\;e^{-\int_0^\beta d\tau\,\frac{1}{2}\dot\xi^2}. \tag{10.598}$$

If we expand the fluctuations $x^i(\tau)$ into the eigenfunctions $e^{-i\omega_m\tau}$ of the operator $-\partial_\tau^2$ for periodic boundary conditions $x^i(0) = x^i(\beta)$,

$$x^i(\tau) = \sum_m x^i_m u_m(\tau) = x^i_0 + \sum_{m\neq 0} x^i_m u_m(\tau), \quad x^i_{-m} = x^{i\,*}_m, \quad m > 0, \tag{10.599}$$

and substitute this into the path integral (10.598), the exponent becomes

$$-\frac{1}{2}\int_0^\beta d\tau \, \left[x^i(\tau)\right]^2 = -\frac{\beta}{2}\sum_{m\neq 0}\omega_m^2 \; x^i_{-m}x^i_m = -\beta\sum_{m>0}\omega_m^2 \; x^{i\,*}_m x^i_m \,. \tag{10.600}$$

Remembering the explicit form of the measure (2.447) the Gaussian integrals in (10.598) yield the free-particle Boltzmann factor

$$B_0(q_0) = 1, \tag{10.601}$$

corresponding to a vanishing effective classical potential in Eq. (10.592).

The perturbation expansion (10.595) becomes therefore simply

$$B(q_0) = 1 - \left\langle \mathcal{A}^{\rm int}_{\rm tot,FP}[q_0,\xi]\right\rangle^{q_0} + \frac{1}{2}\left\langle \mathcal{A}^{\rm int}_{\rm tot,FP}[q_0,\xi]^2\right\rangle^{q_0} - \dots \,. \tag{10.602}$$

The expectation values on the right-hand side are to be calculated with the help of Wick contractions involving the basic correlation function of $\xi^a(\tau)$ associated with the unperturbed action in (10.598):

$$\left\langle x^i(\tau)x^j(\tau')\right\rangle^{q_0} = \delta^{ij}\bar{\Delta}(\tau,\tau')\,, \tag{10.603}$$

which is, of course, consistent with (10.564) via Eq. (10.97).

10.12.4 Covariant Perturbation Expansion

We now perform all possible Wick contractions of the fluctuations $\xi^\mu(\tau)$ in the expectation values (10.602) using the correlation function (10.564). We restrict our attention to the lowest-order terms only, since all problems of previous treatments arise already there. Making use of Eqs. (10.565) and (10.567), we find for the interaction (10.561):

$$\begin{aligned}\left\langle \mathcal{A}^{\rm int}_{\rm tot}[q_0,\xi]\right\rangle^{q_0} &= \int_0^\beta d\tau\,\frac{1}{6}\left[\bar{R}_{\mu\lambda\nu\kappa}(q_0)\left\langle \xi^\lambda\xi^\kappa\dot{\xi}^\mu\dot{\xi}^\nu\right\rangle^{q_0} + \delta(0)\bar{R}_{\mu\nu}(q_0)\left\langle \xi^\mu\xi^\nu\right\rangle^{q_0}\right]\\ &= \frac{1}{72}\bar{R}(q_0)\beta,\end{aligned} \tag{10.604}$$

and for (10.589):

$$\left\langle \mathcal{A}^{\rm FP}[q_0,\xi]\right\rangle^{q_0} = \int_0^\beta d\tau\,\frac{1}{3\beta}\bar{R}_{\mu\nu}(q_0)\left\langle \xi^\mu\xi^\nu\right\rangle^{q_0} = \frac{1}{36}\bar{R}(q_0)\beta. \tag{10.605}$$

The sum of the two contributions yields the manifestly covariant high-temperature expansion up to two loops:

$$B(q_0) = 1 - \left\langle \mathcal{A}^{\rm int}_{\rm tot,FP}[q_0,\xi] \right\rangle^{q_0} + \ldots = 1 - \frac{1}{24}\bar{R}(q_0)\beta + \ldots \tag{10.606}$$

in agreement with the partition function density (10.494) calculated from Dirichlet boundary conditions. The associated partition function

$$Z^{\rm P} = \int \frac{d^D q_0}{\sqrt{2\pi\beta}^D}\sqrt{g(q_0)}\; B(q_0) \tag{10.607}$$

coincides with the partition function obtained by integrating over the partition function density (10.494). Note the crucial role of the action (10.589) coming from the Faddeev-Popov determinant in obtaining the correct two-loop coefficient in Eq. (10.606) and the normalization in Eq. (10.607).

The intermediate transformation to the geodesic coordinates $\xi^\mu(\tau)$ has made our calculations rather lengthy if the action is given in arbitrary coordinates, but it guarantees complete independence of the coordinates in the result (10.606). The entire derivation simplifies, of course, drastically if we choose from the outset geodesic coordinates to parametrize the curved space.

10.12.5 Covariant Result from Noncovariant Expansion

Having found the proper way of calculating the Boltzmann factor $B(q_0)$ we can easily set up a procedure for calculating the same covariant result without the use of the geodesic fluctuations $\xi^\mu(\tau)$. Thus we would like to evaluate the path integral (10.594) by a direct expansion of the action in powers of the noncovariant fluctuations $\eta^\mu(\tau)$ in Eq. (10.557). In order to make q_0^μ equal to the path average, $\bar{q}(\tau)$, we now require $\eta^\mu(\tau)$ to have a vanishing temporal average $\eta_0^\mu = \bar{\eta}^\mu = 0$.

Instead of (10.560), the free action reads now

$$\mathcal{A}^{(0)}[q_0,\eta] = g_{\mu\nu}(q_0)\int_0^\beta d\tau \frac{1}{2}\eta^\mu(\tau)(-\partial_\tau^2)\eta^\nu(\tau)\,, \tag{10.608}$$

and the small-β behavior of the path integral (10.607) is governed by the interaction $\mathcal{A}^{\rm int}_{\rm tot}[q]$ fo Eq. (10.536) with unit smallness parameter η. In a notation as in (10.561), the interaction (10.536) reads

$$\begin{aligned}\mathcal{A}^{\rm int}_{\rm tot}[q_0;\eta] = \int_0^\beta d\tau\bigg\{&\left[\Gamma_{\kappa\mu\nu}\eta^\kappa + \frac{1}{2}\,\partial_\lambda\Gamma_{\kappa\mu\nu}\,\eta^\lambda\eta^\kappa\right]\dot{\eta}^\mu\dot{\eta}^\nu\\ &-\delta(0)\left[\Gamma_{\mu\kappa}{}^{\mu}\eta^\kappa + \frac{1}{2}\,\partial_\lambda\Gamma_{\tau\mu}{}^{\mu}\eta^\lambda\eta^\tau\right]\bigg\}.\end{aligned} \tag{10.609}$$

We must deduce the measure of functional integration over η-fluctuations without zero mode $\eta_0^\mu = \bar{\eta}^\mu$ from the proper measure in (10.586) of ξ-fluctuations without zero mode:

$$\oint \mathcal{D}'^D\xi\, J(q_0,\xi)\Delta^{\rm FP}[q_0,\xi] \equiv \oint \mathcal{D}^D\xi(\tau) J(q_0,\xi)\delta^{(D)}(\xi_0)\Delta^{\rm FP}[q_0,\xi]. \tag{10.610}$$

This is transformed to coordinates $\eta^\mu(\tau)$ via Eqs. (10.577) and (10.578) yielding

$$\oint \mathcal{D}'^D\xi\, J(q_0,\xi)\Delta^{\rm FP}[q_0,\xi] = \oint \mathcal{D}'^D\eta\,\bar{\Delta}^{\rm FP}[q_0,\eta]\,, \tag{10.611}$$

where $\bar{\Delta}^{\rm FP}[q_0,\eta]$ is obtained from the Faddeev-Popov determinant $\Delta^{\rm FP}[q_0,\xi]$ of (10.587) by expressed the coordinates $\eta^\mu(\tau)$ in terms of $\xi^\mu(\tau)$, and multiplying the result by a Jacobian accounting for the change of the δ-function of ξ_0 to a δ-function of η_0 via the transformation Eq. (10.577):

$$\bar{\Delta}^{\rm FP}[q_0,\eta] \;=\; \Delta^{\rm FP}[q_0,\xi(q_0,\eta)]\times\det\left(\frac{\partial\bar{\eta}^\mu(q_0,\xi)}{\partial\xi^\nu}\right)_{\xi=\xi(q_0,\eta)}. \tag{10.612}$$

The last determinant has the exponential form

$$\det\left(\frac{\partial\bar{\eta}^\mu(q_0,\xi)}{\partial\xi^\nu}\right)_{\xi=\xi(q_0,\eta)}=\exp\left\{\operatorname{tr}\log\left[\frac{1}{\beta}\int_0^\beta d\tau\,\left(\frac{\partial\eta^\mu(q_0,\xi)}{\partial\xi^\nu}\right)_{\xi=\xi(q_0,\eta)}\right]\right\}, \tag{10.613}$$

where the matrix in the exponent has small-η expansion

$$\begin{aligned}\left(\frac{\partial\eta^\mu(q_0,\xi)}{\partial\xi^\nu}\right)_{\xi=\xi(q_0,\eta)} \;=&\;\; \delta^\mu_\nu-\bar{\Gamma}_{\nu\sigma}{}^\mu\eta^\sigma\\ &-\frac{1}{3}\left(\partial_\sigma\bar{\Gamma}_{\nu\tau}{}^\mu+\frac{1}{2}\partial_\nu\bar{\Gamma}_{\sigma\tau}{}^\mu-2\bar{\Gamma}_{\tau\nu}{}^\kappa\bar{\Gamma}_{\kappa\sigma}{}^\mu+\frac{1}{2}\bar{\Gamma}_{\tau\sigma}{}^\kappa\bar{\Gamma}_{\kappa\nu}{}^\mu\right)\eta^\sigma\eta^\tau+\dots\,.\end{aligned} \tag{10.614}$$

The factor (10.612) in (10.611) leads to a new contribution to the interaction (10.608), if we rewrite it as

$$\bar{\Delta}^{\rm FP}[q_0,\eta]=e^{-\bar{\mathcal{A}}^{\rm FP}[q_0,\eta]}\,. \tag{10.615}$$

Combining Eqs. (10.587) and (10.613), we find a new Faddeev-Popov type action for η^μ-fluctuations at vanishing η_0^μ:

$$\begin{aligned}\bar{\mathcal{A}}^{\rm FP}[q_0,\eta] \;=&\;\; \mathcal{A}^{\rm FP}[q_0,\xi(q_0,\eta)]-\operatorname{tr}\log\left[\frac{1}{\beta}\int_0^\beta d\tau\,\left(\frac{\partial\eta^\mu(q_0,\xi)}{\partial\xi^\nu}\right)_{\xi=\xi(q_0,\eta)}\right]\\ =&\;\;\frac{1}{2\beta}\int_0^\beta d\tau\;T_{\sigma\tau}(q_0)\eta^\sigma\eta^\tau+\dots\,,\end{aligned} \tag{10.616}$$

where

$$T_{\sigma\tau}(q_0)=\left(\partial_\mu\bar{\Gamma}_{\sigma\tau}{}^\mu-2\bar{\Gamma}_{\sigma\kappa}{}^\mu\bar{\Gamma}_{\mu\tau}{}^\kappa+\bar{\Gamma}_{\kappa\mu}{}^\mu\bar{\Gamma}_{\sigma\tau}{}^\kappa\right). \tag{10.617}$$

The unperturbed correlation functions associated with the action (10.608) are:

$$\langle\eta^\mu(\tau)\eta^\nu(\tau')\rangle^{q_0}=g^{\mu\nu}(q_0)\bar{\Delta}(\tau,\tau') \tag{10.618}$$

and the free Boltzmann factor is the same as in Eq. (10.601). The perturbation expansion of the interacting Boltzmann factor is to be calculated from an expansion like (10.602):

$$B(q_0)=1-\left\langle\mathcal{A}^{\rm int}_{\rm tot,FP}[q_0,\eta]\right\rangle^{q_0}+\frac{1}{2}\left\langle\left(\mathcal{A}^{\rm int}_{\rm tot,FP}[q_0,\eta]\right)^2\right\rangle^{q_0}-\dots\,, \tag{10.619}$$

where the interaction is now

$$\mathcal{A}^{\rm int}_{\rm tot,FP}[q_0,\eta]=\mathcal{A}^{\rm int}_{\rm tot}[q_0,\eta]+\bar{\mathcal{A}}^{\rm FP}[q_0,\eta]\,. \tag{10.620}$$

As before in the Dirichlet case, the divergences containing powers of $\delta(0)$ no longer cancel order by order, but do so at the end.

The calculations proceed as in the Dirichlet case, except that the correlation functions are now given by (10.565) which depend only on the difference of their arguments.

The first expectation value contributing to (10.619) is given again by $f_1^{(1)}$ of Eq. (10.539), except that the integrals have to be evaluated with the periodic correlation function $\bar{\Delta}(\tau,\tau)$ of Eq. (10.565), which has the properties

$$\bar{\Delta}(\tau,\tau)=\frac{1}{12}\,,\quad {}^{\cdot}\bar{\Delta}(\tau,\tau)=\bar{\Delta}^{\cdot}(\tau,\tau)=0\,. \tag{10.621}$$

Using further the common property ${}^{\cdot}\bar{\Delta}^{\cdot}(\tau,\tau')=\delta(\tau-\tau')-1/\beta$ of Eq. (10.567), we find directly from (10.539):

$$\begin{aligned}\bar{f}_1^{(1)}=\left\langle\mathcal{A}_{\rm tot}^{\rm int}[q_0,\eta]\right\rangle^{q_0} &= \frac{\beta}{24}g^{\sigma\tau}\left(\partial_\sigma\bar{\Gamma}_{\tau\mu}{}^{\mu}+g^{\mu\nu}\bar{\Gamma}_{\tau\mu\kappa}\bar{\Gamma}_{\sigma\nu}{}^{\kappa}+\bar{\Gamma}_{\tau\nu}{}^{\mu}\bar{\Gamma}_{\sigma\mu}{}^{\nu}\right)\\ &- \frac{\beta^2}{24}\delta(0)\,g^{\sigma\tau}\left(g^{\mu\nu}\bar{\Gamma}_{\tau\mu\kappa}\bar{\Gamma}_{\sigma\nu}{}^{\kappa}+\bar{\Gamma}_{\tau\mu}{}^{\nu}\bar{\Gamma}_{\sigma\nu}{}^{\mu}\right).\end{aligned} \tag{10.622}$$

To this we must the expectation value of the Faddeev-Popov action:

$$\left\langle\bar{\mathcal{A}}^{\rm FP}[q_0,\eta]\right\rangle^{q_0} = -\frac{\beta}{24}g^{\sigma\tau}\left(\partial_\mu\bar{\Gamma}_{\sigma\tau}{}^{\mu}-2\bar{\Gamma}_{\sigma\nu}{}^{\mu}\bar{\Gamma}_{\mu\tau}{}^{\nu}+\bar{\Gamma}_{\mu\kappa}{}^{\mu}\bar{\Gamma}_{\sigma\tau}{}^{\kappa}\right). \tag{10.623}$$

The divergent term with the factor $\delta(0)$ in (10.622) is canceled by the same expression in the second-order contribution to (10.602) which we calculate now, evaluating the second cumulant (10.541) with the periodic correlation function $\bar{\Delta}(\tau,\tau)$ of Eq. (10.565). The diagrams are the same as in (10.543), but their evaluation is much simpler. Due to the absence of the zero modes in $\eta^\mu(\tau)$, all one-particle reducible diagrams vanish, so that the analogs of $f_1^{(2)}$, $f_1^{(3)}$, and $f_1^{(4)}$ in (10.542) are all zero. Only those of $f_1^{(5)}$ and $f_1^{(6)}$ survive, which involve now the Feynman integrals I_{14} and I_{15} evaluated with $\bar{\Delta}(\tau,\tau)$ which are

$$: \quad \bar{I}_{14}=\int_0^\beta\int_0^\beta d\tau\,d\tau'\;{}^{\cdot}\bar{\Delta}(\tau,\tau')\,\bar{\Delta}^{\cdot}(\tau,\tau')\,{}^{\cdot}\bar{\Delta}^{\cdot}(\tau,\tau')=-\frac{\beta}{24}+\delta(0)\frac{\beta^2}{12}, \tag{10.624}$$

$$: \quad \bar{I}_{15}=\int_0^\beta\int_0^\beta d\tau\,d\tau'\;\bar{\Delta}(\tau,\tau')\,{}^{\cdot}\bar{\Delta}^{\cdot 2}(\tau,\tau')=-\frac{\beta}{24}. \tag{10.625}$$

This leads to the second cumulant

$$\begin{aligned}\frac{1}{2}\left\langle\left(\mathcal{A}_{\rm tot,FP}^{\rm int}[q_0,\eta]\right)^2\right\rangle_c^{q_0} &= -\frac{\beta}{24}g^{\sigma\tau}\left(g^{\mu\nu}\bar{\Gamma}_{\tau\mu\kappa}\bar{\Gamma}_{\sigma\nu}{}^{\kappa}+2\,\bar{\Gamma}_{\tau\nu}{}^{\mu}\bar{\Gamma}_{\sigma\mu}{}^{\nu}\right)\\ &+\frac{\beta^2}{24}\delta(0)\,g^{\sigma\tau}\left(g^{\mu\nu}\bar{\Gamma}_{\tau\mu\kappa}\bar{\Gamma}_{\sigma\nu}{}^{\kappa}+\bar{\Gamma}_{\tau\mu}{}^{\nu}\bar{\Gamma}_{\sigma\nu}{}^{\mu}\right).\end{aligned} \tag{10.626}$$

The sum of Eqs. (10.623) and (10.626) is finite and yields the same covariant result $\beta\bar{R}/24$ as in Eq. (10.552), so that we re-obtain the same covariant perturbation expansion of the effective classical Boltzmann factor as before in Eq. (10.606). Note the importance of the contribution (10.623) from the Faddeev-Popov determinant in producing the curvature scalar. Neglecting this, as done by other authors in Ref. [34], will produce in the effective classical Boltzmann factor (10.619) an additional noncovariant term $g^{\sigma\tau}T_{\sigma\tau}(q_0)/24$. This may be rewritten as a covariant divergence of a nonvectorial quantity

$$g^{\sigma\tau}T_{\sigma\tau}=\nabla_\mu V^\mu,\quad V^\mu(q_0)=g^{\sigma\tau}(q_0)\bar{\Gamma}^\mu_{\sigma\tau}(q_0). \tag{10.627}$$

As such it does not contribute to the integral over q_0^μ in Eq. (10.607), but it is nevertheless a wrong noncovariant result for the Boltzmann factor (10.606).

The appearance of a noncovariant term in a treatment where q_0^μ is the path average of $q^\mu(\tau)$ is not surprising. If the time dependence of a path shows an acceleration, the average of a path is not an invariant concept even for an infinitesimal time. One may covariantly impose the condition of a vanishing temporal average only upon fluctuation coordinates which have no acceleration. This is the case of geodesic coordinates $\xi^a(\tau)$ since their equation of motion at q_0^μ is $\ddot{\xi}^a(\tau)=0$.

10.12.6 Particle on Unit Sphere

A special treatment exists for particle in homogeneous spaces. As an example, consider a quantum particle moving on a unit sphere in $D+1$ dimensions. The partition function is defined by Eq. (10.554) with the Euclidean action (10.408) and the invariant measure (10.406), where the metric and its determinant are

$$g_{\mu\nu}(q) = \delta_{\mu\nu} + \frac{q_\mu q_\nu}{1-q^2}, \qquad g(q) = \frac{1}{1-q^2}. \tag{10.628}$$

It is, of course, possible to calculate the Boltzmann factor $B(q_0)$ with the procedure of Section 10.12.3. Instead of doing this we shall, however, exploit the homogeneity of the sphere. The invariance under reparametrizations of general Riemannian space becomes here an isometry of the metric (10.628). Consequently, the Boltzmann factor $B(q_0)$ in Eq. (10.607) becomes *independent* of the choice of q_0^μ, and the integral over q_0^μ in (10.591) yields simply the total surface of the sphere times the Boltzmann factor $B(q_0)$. The homogeneity of the space allows us to treat paths $q^\mu(\tau)$ themselves as small quantum fluctuations around the origin $q_0^\mu = 0$, which extremizes the path integral (10.554). The possibility of this expansion is due to the fact that $\Gamma_{\sigma\tau}^{\mu} = q^\mu g_{\sigma\tau}$ vanishes at $q^\mu(\tau) = 0$, so that the movement is at this point free of acceleration, this being similar to the situation in geodesic coordinates. As before we now take account of the fact that there are other equivalent saddle-points due to isometries of the metric (10.628) on the sphere (see, e.g., [54]). The infinitesimal transformations of a small vector q^μ:

$$q_\varepsilon^\mu = q^\mu + \varepsilon^\mu\sqrt{1-q^2}, \quad \varepsilon^\mu = \text{const}, \ \mu = 1, \dots, D \tag{10.629}$$

move the origin $q_0^\mu = 0$ by a small amount on the surface of the sphere. Due to the rotational symmetry of the system in the D-dimensional space, these fluctuations have a vanishing action. There are also $D(D-1)/2$ more isometries consisting of the rotations around the origin $q^\mu(\tau) = 0$ on the surface of the sphere. These are, however, irrelevant in the present context since they leave the origin unchanged.

The transformations (10.629) of the origin may be eliminated from the path integral (10.554) by including a factor $\delta^{(D)}(\bar{q})$ to enforce the vanishing of the temporal path average $\bar{q} = \beta^{-1}\int_0^\beta d\tau q(\tau)$. The associated Faddeev-Popov determinant $\Delta^{\rm FP}[q]$ is determined by the integral

$$\Delta^{\rm FP}[q] \int d^D\varepsilon\, \delta^{(D)}\,(\bar{q}_\varepsilon) = \Delta^{\rm FP}[q] \int d^D\varepsilon\, \delta^{(D)} \left(\varepsilon^\mu \frac{1}{\beta}\int_0^\beta d\tau\, \sqrt{1-q^2}\right) = 1\,. \tag{10.630}$$

The result has the exponential form

$$\Delta^{\rm FP}[q] = \left(\frac{1}{\beta}\int_0^\beta d\tau\, \sqrt{1-q^2}\right)^D = e^{-\mathcal{A}^{\rm FP}[q]}\,, \tag{10.631}$$

where $\mathcal{A}^{\rm FP}[q]$ must be added to the action (10.408):

$$\mathcal{A}^{\rm FP}[q] = -D\log\left(\frac{1}{\beta}\int_0^\beta d\tau\, \sqrt{1-q^2}\right). \tag{10.632}$$

The Boltzmann factor $B(q_0) \equiv B$ is then given by the path integral without zero modes

$$
\begin{aligned}
B &= \oint \prod_{\mu,\tau} \left[dq^\mu(\tau) \sqrt{g(q(\tau))} \right] \delta^{(D)}(\bar{q}) \Delta^{\rm FP}[q] e^{-\mathcal{A}[q]} \\
&= \oint \mathcal{D}'^D q \sqrt{g(q(\tau))} \Delta^{\rm FP}[q] e^{-\mathcal{A}[q]} ,
\end{aligned}
\tag{10.633}
$$

where the measure $\mathcal{D}'^D q$ is defined as in Eq. (2.447). This can also be written as

$$
B = \oint \mathcal{D}'^D q \, e^{-\mathcal{A}[q]-\mathcal{A}^J[q]-\mathcal{A}^{\rm FP}[q]} ,
\tag{10.634}
$$

where $\mathcal{A}^J[q]$ is a contribution to the action (10.408) coming from the product

$$
\prod_\tau \sqrt{g(q(\tau))} \equiv e^{-\mathcal{A}^J[q]}.
\tag{10.635}
$$

By inserting (10.628), this becomes

$$
\mathcal{A}^J[q] = -\int_0^\beta d\tau \, \frac{1}{2} \delta(0) \log g(q) = \int_0^\beta d\tau \, \frac{1}{2} \delta(0) \log(1-q^2) \, .
\tag{10.636}
$$

The total partition function is, of course, obtained from B by multiplication with the surface of the unit sphere in $D+1$ dimensions $2\pi^{(D+1)/2}/\Gamma(D+1)/2)$. To calculate B from (10.634), we now expand $\mathcal{A}[q]$, $\mathcal{A}^J[q]$ and $\mathcal{A}^{\rm FP}[q]$ in powers of $q^\mu(\tau)$. The metric $g_{\mu\nu}(q)$ and its determinant $g(q)$ in Eq. (10.628) have the expansions

$$
g_{\mu\nu}(q) = \delta_{\mu\nu} + q_\mu q_\nu + \dots \, , \quad g(q) = 1 + q^2 + \dots \, ,
\tag{10.637}
$$

and the unperturbed action reads

$$
\mathcal{A}^{(0)}[q] = \int_0^\beta d\tau \, \frac{1}{2} \dot{q}^2(\tau).
\tag{10.638}
$$

In the absence of the zero eigenmodes due to the δ-function over $\bar{q}$ in Eq. (10.633), we find as in Eq. (10.601) the free Boltzmann factor

$$
B_0 = 1.
\tag{10.639}
$$

The free correlation function looks similar to (10.603):

$$
\langle q^\mu(\tau) q^\nu(\tau') \rangle = \delta^{\mu\nu} \bar{\Delta}(\tau, \tau').
\tag{10.640}
$$

The interactions coming from the higher expansions terms in Eq. (10.637) begin with

$$
\mathcal{A}^{\rm int}_{\rm tot}[q] = \mathcal{A}^{\rm int}[q] + \mathcal{A}^J[q] = \int_0^\beta d\tau \, \frac{1}{2} \left[(q\dot{q})^2 - \delta(0) q^2 \right] .
\tag{10.641}
$$

To the same order, the Faddeev-Popov interaction (10.632) contributes

$$\mathcal{A}^{\rm FP}[q] = \frac{D}{2\beta}\int_0^\beta d\tau\, q^2. \tag{10.642}$$

This has an important effect upon the two-loop perturbation expansion of the Boltzmann factor

$$B(q_0) = 1 - \left\langle \mathcal{A}^{\rm int}_{\rm tot}[q]\right\rangle^{q_0} - \left\langle \mathcal{A}^{\rm FP}[q]\right\rangle^{q_0} + \dots \;= B(0) \equiv B. \tag{10.643}$$

Performing the Wick contractions with the correlation function (10.640) with the properties (10.565)–(10.567), we find from Eqs. (10.641), (10.642)

$$\begin{aligned}\left\langle \mathcal{A}^{\rm int}_{\rm tot}[q]\right\rangle^{q_0} &= \frac{1}{2}\int_0^\beta d\tau \left\{ D\,{}^{\cdot}\bar\Delta^{\cdot}(\tau,\tau)\bar\Delta(\tau,\tau) + D(D+1)\,{}^{\cdot}\bar\Delta^2(\tau,\tau) - \delta(0)D\bar\Delta(\tau,\tau)\right\}\\ &= \frac{1}{2}\int_0^\beta d\tau \left\{ -\frac{D}{\beta}\bar\Delta(\tau,\tau) + D(D+1)\,{}^{\cdot}\bar\Delta^2(\tau,\tau)\right\} = -\frac{D}{24}\beta,\end{aligned} \tag{10.644}$$

and

$$\left\langle \mathcal{A}^{\rm FP}[q]\right\rangle^{q_0} = \frac{D}{2\beta}\int_0^\beta d\tau\, D\,\bar\Delta(\tau,\tau) = \frac{D^2}{24}\beta. \tag{10.645}$$

Their combination in Eq. (10.643) yields the high-temperature expansion

$$B = 1 - \frac{D(D-1)}{24}\beta + \dots\,. \tag{10.646}$$

This is in perfect agreement with Eqs. (10.494) and (10.606), since the scalar curvature for a unit sphere in $D+1$ dimensions is $\bar R = D(D-1)$. It is remarkable how the contribution (10.645) of the Faddeev-Popov determinant has made the noncovariant result (10.644) covariant.

10.13 Covariant Effective Action for Quantum Particle with Coordinate-Dependent Mass

The classical behavior of a system is completely determined by the extrema of the classical action. The quantum-mechanical properties can be found from the extrema of the effective action (see Subsection 3.22.5). This important quantity can in general only be calculated perturbatively. This will be done here for a particle with a coordinate-dependent mass. The calculation [44] will make use of the background method of Subsection 3.23.6 combined with the techniques developed earlier in this chapter. From the one-particle-irreducible (1PI) Feynman diagrams with no external lines we obtain an expansion in powers of the Planck constant $\hbar$. The result will be applicable to a large variety of interesting physical systems, for instance compound nuclei, where the collective Hamiltonian, commonly derived from a microscopic description via time-dependent Hartree-Fock theory [45], contains coordinate-dependent mass parameters.

10.13.1 Formulating the Problem

Consider a particle with coordinate-dependent mass $m(q)$ moving as in the one-dimensional potential $V(q)$. We shall study the Euclidean version of the system where the paths $q(t)$ are continued to an imaginary times $\tau = -it$ and the Lagrangian for $q(\tau)$ has the form

$$L(q,\dot{q}) = \frac{1}{2}\,m(q)\dot{q}^2 + V(q). \tag{10.647}$$

The dot stands for the derivative with respect to the imaginary time. The q-dependent mass may be written as $mg(q)$ where $g(q)$ plays the role of a one-dimensional dynamical metric. It is the trivial 1×1 Hessian metric of the system [recall the definition (1.12) and Eq. (1.388)]. In D-dimensional configuration space, the kinetic term would read $mg_{\mu\nu}(q)\dot{q}^\mu\dot{q}^\nu/2$, having the same form as in the curved-space action (10.186).

Under an arbitrary single-valued coordinate transformation $q = q(\tilde{q})$, the potential $V(q)$ is assumed to transform like a scalar whereas the metric $m(q)$ is a one-dimensional tensor of rank two:

$$V(q) = V(q(\tilde{q})) \equiv \widetilde{V}(\tilde{q}), \qquad m(q) = \widetilde{m}(\tilde{q})\,[d\tilde{q}(q)/dq]^2. \tag{10.648}$$

This coordinate transformation leaves the Lagrangian (10.647) and thus also the classical action

$$\mathcal{A}[q] = \int_{-\infty}^{\infty} d\tau\, L(q,\dot{q}) \tag{10.649}$$

invariant. Quantum theory has to possess the same invariance, exhibited automatically by Schrödinger theory. It must be manifest in the effective action. This will be achieved by combining the background technique in Subsection 3.23.6 with the techniques of Sections 10.6–10.10. In the background field method [46] we split all paths into $q(\tau) = Q(\tau) + \delta q(\tau)$, where $Q(\tau)$ is the final extremal orbit and δq describes the quantum fluctuations around it. At the one-loop level, the covariant effective action $\Gamma[Q]$ becomes a sum of the classical Lagrangian $L(Q,\dot{Q})$ and a correction term ΔL. It is defined by the path integral [recall (3.776)]

$$e^{-\Gamma[Q]/\hbar} = \int \mathcal{D}\mu(\delta q) e^{-(1/\hbar)\left\{\mathcal{A}[Q+\delta q] - \int d\tau\,\delta q\,\delta\Gamma[Q]/\delta Q\right\}}, \tag{10.650}$$

where the measure of functional integration $\mathcal{D}\mu(\delta q)$ is obtained from the initial invariant measure $\mathcal{D}\mu(q) = Z^{-1}\prod_\tau dq(\tau)\sqrt{m(q)}$ and reads

$$\mathcal{D}\mu(\delta q) = Z^{-1}\prod_\tau d\delta q(\tau)\,\sqrt{m(Q)}\,e^{(1/2)\delta(0)\int d\tau\,\log[m(Q+\delta q)/m(Q)]}, \tag{10.651}$$

with Z being some normalization factor. The generating functional (10.650) possesses the same symmetry under reparametrizations of the configuration space as the classical action (10.649).

We now calculate $\Gamma[Q]$ in Eq. (10.650) perturbatively as a power series in $\hbar$:

$$\Gamma[Q] = \mathcal{A}[Q] + \hbar\Gamma_1[Q] + \hbar^2\Gamma_2[Q] + \dots . \tag{10.652}$$

The quantum corrections to the classical action (10.649) are obtained by expanding $\mathcal{A}[Q + \delta q]$ and the measure (10.651) *covariantly* in powers of δq:

$$\begin{aligned}\mathcal{A}[Q+\delta q] = \mathcal{A}[Q] &+ \int d\tau \frac{D\mathcal{A}}{\delta Q(\tau)}\delta x(\tau) + \frac{1}{2}\int d\tau \int d\tau' \frac{D^2\mathcal{A}}{\delta Q(\tau)\delta Q(\tau')}\,\delta x(\tau)\,\delta x(\tau') \\ &+ \frac{1}{6}\int d\tau \int d\tau' \int d\tau'' \frac{D^3\mathcal{A}}{\delta Q(\tau)\delta Q(\tau')\delta Q(\tau'')}\,\delta x(\tau)\,\delta x(\tau')\,\delta x(\tau'') + \dots . \end{aligned}\tag{10.653}$$

The expansion is of the type (10.101), i.e., the expressions δx are *covariant fluctuations* related to the ordinary variations δq in the same way as the normal coordinates Δx^μ are related to the differences Δq^μ in the expansion (10.98). The symbol $D/\delta Q$ denotes the *covariant* functional derivative in one dimension. To first order, this is the ordinary functional derivative

$$\frac{D\mathcal{A}[Q]}{\delta Q(\tau)} = \frac{\delta\mathcal{A}[Q]}{\delta Q(\tau)} = V'(Q) - \frac{1}{2}\,m'(Q)\,\dot{Q}^2(\tau) - m(Q)\,\ddot{Q}(\tau). \tag{10.654}$$

This vanishes for the classical orbit $Q(\tau)$. The second covariant derivative is [compare (10.100)]

$$\frac{D^2\mathcal{A}[Q]}{\delta Q(\tau)\delta Q(\tau')} = \frac{\delta^2\mathcal{A}[Q]}{\delta Q(\tau)\delta Q(\tau')} - \Gamma(Q(\tau))\,\frac{\delta\mathcal{A}[Q]}{\delta Q(\tau')}, \tag{10.655}$$

where $\Gamma(Q) = m'(Q)/2\,m(Q)$ is the one-dimensional version of the Christoffel symbol for the metric $g_{\mu\nu} = \delta_{\mu\nu}m(Q)$. More explicitly, the result is

$$\frac{\delta^2\mathcal{A}[Q]}{\delta Q(\tau)\delta Q(\tau')} = -\left[m(Q)\partial_\tau^2 + m'(Q)\dot{Q}\partial_\tau + m'(Q)\ddot{Q} + \frac{1}{2}m''(Q)\dot{Q}^2 - V''(Q)\right]\delta(\tau-\tau'). \tag{10.656}$$

The validity of the expansion (10.653) follows from the fact that it is equivalent by a coordinates transformation to an ordinary functional expansion in Riemannian coordinates where the Christoffel symbol vanishes for the particular background coordinates.

The inverse of the functional matrix (10.655) supplies us with the free correlation function $G(\tau, \tau')$ of the fluctuations $\delta x(\tau)$. The higher derivatives define the interactions. The expansion terms $\Gamma_n[Q]$ in (10.652) are found from all one-particle-irreducible vacuum diagrams (3.784) formed with the propagator $G(\tau, \tau')$ and the interaction vertices. The one-loop correction to the effective action is given by the simple harmonic path integral

$$e^{-\Gamma_1[Q]} = \int \mathcal{D}\delta x\,\sqrt{m(Q)}\,e^{-\mathcal{A}^{(2)}[Q,\delta x]}, \tag{10.657}$$

with the quadratic part of the expansion (10.653):

$$\mathcal{A}^{(2)}[Q,\delta x] = \frac{1}{2}\int_{-\infty}^{\infty} d\tau\, d\tau'\, \delta x(\tau)\, \frac{D^2\mathcal{A}[Q]}{\delta Q(\tau)\delta Q(\tau')}\, \delta x(\tau')\,. \tag{10.658}$$

The presence of $m(Q)$ in the free part of the covariant kinetic term (10.658) and in the measure in Eq. (10.657) suggests exchanging the fluctuation δx by the new coordinates $\delta\tilde{x} = h(Q)\delta x$, where $h(Q) \equiv \sqrt{m(Q)}$ is the one-dimensional version of the triad (10.12) $e(Q) = \sqrt{m(Q)}$ associated with the metric $m(Q)$. The fluctuations $\delta\tilde{x}$ correspond to the differences Δx^i in (10.97). The covariant derivative of $e(Q)$ vanishes $D_Q e(Q) = \partial_Q e(Q) - \Gamma(Q)\, e(Q) \equiv 0$ [recall (10.40)]. Then (10.658) becomes

$$\mathcal{A}^{(2)}[Q,\delta x] = \frac{1}{2}\int_{-\infty}^{\infty} d\tau\, d\tau'\, \delta\tilde{x}(\tau)\, \frac{D^2\mathcal{A}[Q]}{\delta\tilde{Q}(\tau)\delta\tilde{Q}(\tau')}\, \delta\tilde{x}(\tau')\,, \tag{10.659}$$

where

$$\frac{D^2\mathcal{A}[Q]}{\delta\tilde{Q}(\tau)\delta\tilde{Q}(\tau')} = e^{-1}(Q)\, \frac{D^2\mathcal{A}[Q]}{\delta Q(\tau)\delta Q(\tau')}\, e^{-1}(Q) = \left[-\frac{d^2}{d\tau^2} + \omega^2(Q(\tau))\right]\delta(\tau-\tau'), \tag{10.660}$$

and

$$\begin{aligned} \omega^2(Q) &= e^{-1}(Q)\, D^2V(Q)\, e^{-1}(Q) = e^{-1}(Q)\, DV'(Q)\, e^{-1}(Q) \\ &= \frac{1}{m(Q)}\left[V''(Q) - \Gamma(Q)\, V'(Q)\right]. \end{aligned} \tag{10.661}$$

Note that this is the one-dimensional version of the Laplace-Beltrami operator (1.381) applied to $V(Q)$:

$$\omega^2(Q) = \Delta V(Q) = \frac{1}{\sqrt{m(Q)}}\frac{d}{dQ}\left[\sqrt{m(Q)}\left(\frac{V'(Q)}{m(Q)}\right)\right]. \tag{10.662}$$

Indeed, $e^{-2}D^2$ is the one-dimensional version of $g^{\mu\nu}D_\mu D_\nu$ [recall (10.38)] Since $V(Q)$ is a scalar, so is $\Delta V(Q)$.

Equation (10.660) shows that the fluctuations $\delta\tilde{x}$ behave like those of a harmonic oscillator with the time-dependent frequency $\omega^2(Q)$. The functional measure of integration in Eq. (10.657) simplifies in terms of $\delta\tilde{x}$:

$$\prod_\tau d\delta x(\tau)\sqrt{m(Q)} = \prod_\tau d\delta\tilde{x}(\tau)\,. \tag{10.663}$$

This allows us to integrate the Gaussian path integral (10.657) trivially to obtain the one-loop quantum correction to the effective action

$$\Gamma_1[Q] = \frac{1}{2}\mathrm{Tr}\log\left[-\partial_\tau^2 + \omega^2(Q(\tau))\right]. \tag{10.664}$$

Due to the τ-dependence of ω^2, this cannot be evaluated explicitly. For sufficiently slow motion of $Q(\tau)$, however, we can resort to a gradient expansion which yields asymptotically a local expression for the effective action.

10.13.2 Gradient Expansion

The gradient expansion of the one-loop effective action (10.664) has the general form

$$\Gamma_1[Q] = \int_{-\infty}^{\infty} d\tau \left[V_1(Q) + \frac{1}{2} Z_1(Q)\, \dot{Q}^2 + \cdots \right]. \tag{10.665}$$

It is found explicitly by recalling the gradient expansion of the trace of the logarithm derived in Eq. (4.314):

$$\mathrm{Tr} \log \left[-\partial_\tau^2 + \omega^2(\tau) \right] \equiv \int_{-\infty}^{\infty} d\tau \left\{ \omega(\tau) + \frac{[\partial_\tau \omega^2(\tau)]^2}{32\omega^5(\tau)} + \ldots \right\}. \tag{10.666}$$

Inserting $\Omega(\tau) = \omega(Q(\tau))$, we identify

$$V_1(Q) = \hbar\omega(Q)/2, \qquad Z_1(Q) = \frac{(D\omega^2)^2(Q)}{32\,\omega^5(Q)}, \tag{10.667}$$

and obtain the effective action to order $\hbar$ for slow motion

$$\Gamma^{\text{eff}}[Q] = \int_{-\infty}^{\infty} d\tau \left[\frac{1}{2} m^{\text{eff}}(Q)\, \dot{Q}^2 + V^{\text{eff}}(Q) \right], \tag{10.668}$$

where the bare metric $m^{\text{eff}}(Q)$ and the potential $V^{\text{eff}}(Q)$ are related to the initial classical expressions by

$$m^{\text{eff}}(Q) = m(Q) + \hbar \frac{(D\omega^2)^2(Q)}{32\,\omega^5(Q)}, \tag{10.669}$$

$$V^{\text{eff}}(Q) = V(Q) + \hbar \frac{\omega(Q)}{2}. \tag{10.670}$$

For systems in which only the mass is Q-independent, the result has also been obtained by Ref. [48].

The range of validity of the expansion is determined by the characteristic time scale $1/\omega$. Within this time, the particle has to move only little.

Appendix 10A Nonholonomic Gauge Transformations in Electromagnetism

To introduce the subject, let us first recall the standard treatment of magnetism. Since there are no magnetic monopoles, a magnetic field $\mathbf{B}(\mathbf{x})$ satisfies the identity $\boldsymbol{\nabla} \cdot \mathbf{B}(\mathbf{x}) = 0$, implying that only two of the three field components of $\mathbf{B}(\mathbf{x})$ are independent. To account for this, one usually expresses a magnetic field $\mathbf{B}(\mathbf{x})$ in terms of a vector potential $\mathbf{A}(\mathbf{x})$, setting $\mathbf{B}(\mathbf{x}) = \boldsymbol{\nabla}\times \mathbf{A}(\mathbf{x})$. Then Ampère's law, which relates the magnetic field to the electric current density $\mathbf{j}(\mathbf{x})$ by $\boldsymbol{\nabla}\times\mathbf{B} = 4\pi\mathbf{j}(\mathbf{x})$, becomes a second-order differential equation for the vector potential $\mathbf{A}(\mathbf{x})$ in terms of an electric current

$$\boldsymbol{\nabla}\times [\boldsymbol{\nabla}\times \mathbf{A}(\mathbf{x})] = \mathbf{j}(\mathbf{x}). \tag{10A.1}$$

The vector potential $\mathbf{A}(\mathbf{x})$ is a *gauge field*. Given $\mathbf{A}(\mathbf{x})$, any locally gauge-transformed field

$$\mathbf{A}(\mathbf{x}) \to \mathbf{A}'(\mathbf{x}) = \mathbf{A}(\mathbf{x}) + \boldsymbol{\nabla}\Lambda(\mathbf{x}) \tag{10A.2}$$

yields the same magnetic field $\mathbf{B}(\mathbf{x})$. This reduces the number of physical degrees of freedom in the gauge field $\mathbf{A}(\mathbf{x})$ to two, just as those in $\mathbf{B}(\mathbf{x})$. In order for this to hold, the transformation function must be single-valued, i.e., it must have commuting derivatives

$$(\partial_i \partial_j - \partial_j \partial_i)\Lambda(\mathbf{x}) = 0. \tag{10A.3}$$

The equation for absence of magnetic monopoles $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ is ensured if the vector potential has commuting derivatives

$$(\partial_i \partial_j - \partial_j \partial_i)\mathbf{A}(\mathbf{x}) = 0. \tag{10A.4}$$

This integrability property makes $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ the *Bianchi identity* in this gauge field representation of the magnetic field.

In order to solve (10A.1), we remove the gauge ambiguity by choosing a particular gauge, for instance the *transverse gauge* $\boldsymbol{\nabla}\cdot \mathbf{A}(\mathbf{x}) = 0$ in which $\boldsymbol{\nabla}\times [\boldsymbol{\nabla}\times \mathbf{A}(\mathbf{x})] = -\boldsymbol{\nabla}^2 \mathbf{A}(\mathbf{x})$, and obtain

$$\mathbf{A}(\mathbf{x}) = \int d^3x' \, \frac{\mathbf{j}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}. \tag{10A.5}$$

The associated magnetic field is

$$\mathbf{B}(\mathbf{x}) = \int d^3x' \, \frac{\mathbf{j}(\mathbf{x}') \times \mathbf{R}'}{R'^3}, \qquad \mathbf{R}' \equiv \mathbf{x}' - \mathbf{x}. \tag{10A.6}$$

This standard representation of magnetic fields is not the only possible one. There exists another one in terms of a scalar potential $\Lambda(\mathbf{x})$, which must, however, be multivalued to account for the two physical degrees of freedom in the magnetic field.

10A.1 Gradient Representation of Magnetic Field of Current Loops

Consider an infinitesimally thin closed wire carrying an electric current I along the line L. It corresponds to a current density

$$\mathbf{j}(\mathbf{x}) = I \, \boldsymbol{\delta}(\mathbf{x}; L), \tag{10A.7}$$

where $\boldsymbol{\delta}(\mathbf{x}; L)$ is the δ-function on the line L:

$$\boldsymbol{\delta}(\mathbf{x}; L) = \int_L d\mathbf{x}' \delta^{(3)}(\mathbf{x} - \mathbf{x}'). \tag{10A.8}$$

For a closed line L, this function has zero divergence:

$$\boldsymbol{\nabla}\cdot \boldsymbol{\delta}(\mathbf{x}; L) = 0. \tag{10A.9}$$

This follows from the property of the δ-function on an arbitrary line L connecting the points $\mathbf{x}_1$ and $\mathbf{x}_2$:

$$\boldsymbol{\nabla}\cdot \boldsymbol{\delta}(\mathbf{x}; L) = \delta(\mathbf{x}_2) - \delta(\mathbf{x}_1). \tag{10A.10}$$

For closed loops, the right-hand side vanishes.

From Eq. (10A.5) we obtain the associated vector potential

$$\mathbf{A}(\mathbf{x}) = I \int_L d\mathbf{x}' \frac{1}{|\mathbf{x} - \mathbf{x}'|}, \tag{10A.11}$$

yielding the magnetic field

$$\mathbf{B}(\mathbf{x}) = -I \int_L \frac{d\mathbf{x}' \times \mathbf{R}'}{R'^3}, \qquad \mathbf{R}' \equiv \mathbf{x}' - \mathbf{x}. \tag{10A.12}$$

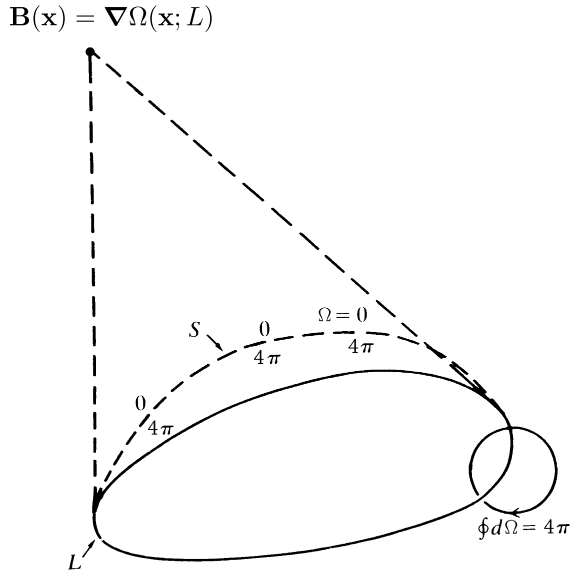


Figure 10.5 Infinitesimally thin closed current loop L. The magnetic field $\mathbf{B}(\mathbf{x})$ at the point $\mathbf{x}$ is proportional to the solid angle $\Omega(\mathbf{x})$ under which the loop is seen from $\mathbf{x}$. In any single-valued definition of $\Omega(\mathbf{x})$, there is some surface S across which $\Omega(\mathbf{x})$ jumps by 4π. In the multivalued definition, this surface is absent.

Let us now derive the same result from a scalar field. Let $\Omega(\mathbf{x}; S)$ be the solid angle under which the current loop L is seen from the point $\mathbf{x}$ (see Fig. 10.5). If S denotes an arbitrary smooth surface enclosed by the loop L, and $d\mathbf{S}'$ a surface element, then $\Omega(\mathbf{x}; S)$ can be calculated from the surface integral

$$\Omega(\mathbf{x}; S) = \int_S \frac{d\mathbf{S}' \cdot \mathbf{R}'}{R'^3}. \tag{10A.13}$$

The argument S in $\Omega(\mathbf{x}; S)$ emphasizes that the definition depends on the choice of the surface S. The range of $\Omega(\mathbf{x}; S)$ is from -2π to 2π, as can be most easily be seen if L lies in the xy-plane and S is chosen to lie in the same place. Then we find for $\Omega(\mathbf{x}; S)$ the value 2π for $\mathbf{x}$ just below S, and -2π just above. Let us calculate from (10A.13) the vector field

$$\mathbf{B}(\mathbf{x}; S) = I\boldsymbol{\nabla}\Omega(\mathbf{x}; S). \tag{10A.14}$$

For this we rewrite

$$\boldsymbol{\nabla}\Omega(\mathbf{x}; S) = \int_S dS'_k \boldsymbol{\nabla} \frac{R'{}_k}{R'^3} = -\int_S dS'_k \boldsymbol{\nabla}' \frac{R'{}_k}{R'^3}, \tag{10A.15}$$

which can be rearranged to

$$\boldsymbol{\nabla}\Omega(\mathbf{x}; S) = -\left[\int_S \left(dS'_k\, \partial'_i \frac{R'_k}{R'^3} - dS'_i\, \partial'_k \frac{R'_k}{R'^3}\right) + \int_S dS'_i\, \partial'_k \frac{R'_k}{R'^3}\right]. \tag{10A.16}$$

With the help of Stokes' theorem

$$\int_S (dS_k \partial_i - dS_i \partial_k) f(\mathbf{x}) = \epsilon_{kil} \int_L dx_l f(\mathbf{x}), \tag{10A.17}$$

and the relation $\partial'_k(R'_k/R'^3) = 4\pi\delta^{(3)}(\mathbf{x} - \mathbf{x}')$, we obtain

$$\boldsymbol{\nabla}\Omega(\mathbf{x}; S) = -\left[\int_L \frac{d\mathbf{x}' \times \mathbf{R}'}{R'^3} + 4\pi \int_S d\mathbf{S}' \delta^{(3)}(\mathbf{x} - \mathbf{x}')\right]. \tag{10A.18}$$

Multiplying the first term by I, we reobtain the magnetic field (10A.12) of the current I. The second term yields the singular magnetic field of an infinitely thin magnetic dipole layer lying on the arbitrarily chosen surface S enclosed by L.

The second term is a consequence of the fact that the solid angle $\Omega(\mathbf{x};S)$ was defined by the surface integral (10A.13). If $\mathbf{x}$ crosses the surface S, the solid angle jumps by 4π.

It is useful to re-express Eq. (10A.15) in a slightly different way. By analogy with (10A.8) we define a δ-function on a surface as

$$\boldsymbol{\delta}(\mathbf{x};S) = \int_S d\mathbf{S}'\,\delta^{(3)}(\mathbf{x}-\mathbf{x}'), \tag{10A.19}$$

and observe that Stokes' theorem (10A.17) can be written as an identity for δ-functions:

$$\boldsymbol{\nabla}\times\boldsymbol{\delta}(\mathbf{x};S) = \boldsymbol{\delta}(\mathbf{x};L), \tag{10A.20}$$

where L is the boundary of the surface S. This equation proves once more the zero divergence (10A.9).

Using the δ-function (10A.19) on the surface S, Eq. (10A.15) can be rewritten as

$$\boldsymbol{\nabla}\Omega(\mathbf{x};S) = -\int d^3x'\,\delta_k(\mathbf{x}';S)\boldsymbol{\nabla}'\frac{R'_k}{R'^3}, \tag{10A.21}$$

and if we used also Eq. (10A.20), we find from (10A.18) a magnetic field

$$\mathbf{B}_i(\mathbf{x};S) = -I\left[\int d^3x'[\nabla\times\boldsymbol{\delta}(\mathbf{x};\mathbf{S})]\times\frac{\mathbf{R}'}{R'^3} + 4\pi\boldsymbol{\delta}(\mathbf{x}';S)\right]. \tag{10A.22}$$

Stokes theorem written in the form (10A.20) displays an important property. If we move the surface S to S' with the same boundary, the δ-function $\delta(\mathbf{x};S)$ changes by

$$\boldsymbol{\delta}(\mathbf{x};S)\to\boldsymbol{\delta}(\mathbf{x};S') = \boldsymbol{\delta}(\mathbf{x};S)+\boldsymbol{\nabla}\delta(\mathbf{x};V), \tag{10A.23}$$

where

$$\delta(\mathbf{x};V)\equiv\int d^3x'\,\delta^{(3)}(\mathbf{x}-\mathbf{x}'), \tag{10A.24}$$

and V is the volume over which the surface has swept. Under this transformation, the curl on the left-hand side of (10A.20) is invariant. Comparing (10A.23) with (10A.2) we identify (10A.23) as a novel type of gauge transformation.[9] The magnetic field in the first term of (10A.22) is invariant under this, the second is not. It is then obvious how to find a gauge-invariant magnetic field: we simply subtract the singular S-dependent term and form

$$\mathbf{B}(\mathbf{x}) = I\left[\boldsymbol{\nabla}\Omega(\mathbf{x};S)+4\pi\boldsymbol{\delta}(\mathbf{x};S)\right]. \tag{10A.25}$$

This field is independent of the choice of S and coincides with the magnetic field (10A.12) derived in the usual gauge theory. Hence the description of the magnetic field as a gradient of field $\Omega(\mathbf{x};S)$ is completely equivalent to the usual gauge field description in terms of the vector potential $\mathbf{A}(\mathbf{x})$. Both are gauge theories, but of a completely different type.

The gauge freedom (10A.23) can be used to move the surface S into a standard configuration. One possibility is the make *gauge fixing*. choose S so that the third component of $\boldsymbol{\delta}(\mathbf{x};S)$ vanishes. This is called the *axial gauge*. If $\boldsymbol{\delta}(\mathbf{x};S)$ does not have this property, we can always shift S by a volume V determined by the equation

$$\delta(V) = -\int_{-\infty}^{z}\delta_z(\mathbf{x};S), \tag{10A.26}$$

[9]For a discussion of this gauge freedom, which is independent of the electromagnetic one, see Ref. [56].

and the transformation (10A.23) will produce a $\boldsymbol{\delta}(\mathbf{x}; S)$ in the axial gauge, to be denoted by $\boldsymbol{\delta}^{\mathrm{ax}}(\mathbf{x}; S)$

Equation (10A.25) suggests defining a solid angle $\Omega(\mathbf{x})$ which is independent of S and depends only on the boundary L of S:

$$\boldsymbol{\nabla}\Omega(\mathbf{x}; L) \equiv \boldsymbol{\nabla}\Omega(\mathbf{x}; S) + 4\pi\boldsymbol{\delta}(\mathbf{x}; S). \tag{10A.27}$$

This is is the analytic continuation of $\Omega(\mathbf{x}; S)$ through the surface S which removes the jump and produces a *multivalued function* $\Omega(\mathbf{x}; L)$ ranging from $-\infty$ to ∞. At each point in space, there are infinitely many Riemann sheets starting from a singularity at L. The values of $\Omega(\mathbf{x}; L)$ on the sheets differ by integer multiples of 4π. From this multivalued function, the magnetic field (10A.12) can be obtained as a simple gradient:

$$\mathbf{B}(\mathbf{x}) = I\, \boldsymbol{\nabla}\Omega(\mathbf{x}; L). \tag{10A.28}$$

Ampère's law (10A.1) implies that the multivalued solid angle $\Omega(\mathbf{x}; L)$ satisfies the equation

$$(\partial_i\partial_j - \partial_j\partial_i)\Omega(\mathbf{x}; L) = 4\pi\epsilon_{ijk}\delta_k(\mathbf{x}; L). \tag{10A.29}$$

Thus, as a consequence of its multivaluedness, $\Omega(\mathbf{x}; L)$ violates the Schwarz integrability condition as the coordinate transformations do in Eq. (10.19). This makes it an unusual mathematical object to deal with. It is, however, perfectly suited to describe the physics.

To see explicitly how Eq. (10A.29) is fulfilled by $\Omega(\mathbf{x}; L)$, let us go to two dimensions where the loop corresponds to two points (in which the loop intersects a plane). For simplicity, we move one of them to infinity, and place the other at the coordinate origin. The role of the solid angle $\Omega(\mathbf{x}; L)$ is now played by the azimuthal angle $\varphi(\mathbf{x})$ of the point $\mathbf{x}$:

$$\varphi(\mathbf{x}) = \arctan\frac{x^2}{x^1}. \tag{10A.30}$$

The function $\arctan(x^2/x^1)$ is usually made unique by cutting the $\mathbf{x}$-plane from the origin along some line C to infinity, preferably along a straight line to $\mathbf{x} = (-\infty, 0)$, and assuming $\varphi(\mathbf{x})$ to jump from π to $-\pi$ when crossing the cut. The cut corresponds to the magnetic dipole surface S in the integral (10A.13). In contrast to this, we shall take $\varphi(\mathbf{x})$ to be the *multivalued* analytic continuation of this function. Then the derivative ∂_i yields

$$\partial_i\varphi(\mathbf{x}) = -\epsilon_{ij}\frac{x_j}{(x^1)^2 + (x^2)^2}. \tag{10A.31}$$

With the single-valued definition of $\partial_i\varphi(\mathbf{x})$, there would have been a δ-function $\epsilon_{ij}\delta_j(C; \mathbf{x})$ across the cut C, corresponding to the second term in (10A.18). When integrating the curl of (10A.31) across the surface s of a small circle c around the origin, we obtain by Stokes' theorem

$$\int_s d^2x(\partial_i\partial_j - \partial_j\partial_i)\varphi(\mathbf{x}) = \int_c dx_i\partial_i\varphi(\mathbf{x}), \tag{10A.32}$$

which is equal to 2π in the multivalued definition of $\varphi(\mathbf{x})$. This result implies the violation of the integrability condition as in (10A.41):

$$(\partial_1\partial_2 - \partial_2\partial_1)\varphi(\mathbf{x}) = 2\pi\delta^{(2)}(\mathbf{x}), \tag{10A.33}$$

whose three-dimensional generalization is (10A.29). In the single-valued definition with the jump by 2π across the cut, the right-hand side of (10A.32) would vanish, making $\varphi(\mathbf{x})$ satisfy the integrability condition (10A.29).

On the basis of Eq. (10A.33) we may construct a Green function for solving the corresponding differential equation with an arbitrary source, which is a superposition of infinitesimally thin line-like currents piercing the two-dimensional space at the points $\mathbf{x}_n$:

$$j(\mathbf{x}) = \sum_n I_n\delta^{(2)}(\mathbf{x} - \mathbf{x}_n), \tag{10A.34}$$

where I_n are currents. We may then easily solve the differential equation

$$(\partial_1\partial_2 - \partial_2\partial_1)f(\mathbf{x}) = j(\mathbf{x}), \tag{10A.35}$$

with the help of the Green function

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{2\pi}\varphi(\mathbf{x} - \mathbf{x}') \tag{10A.36}$$

which satisfies

$$(\partial_1\partial_2 - \partial_2\partial_1)G(\mathbf{x} - \mathbf{x}') = \delta^{(2)}(\mathbf{x} - \mathbf{x}'). \tag{10A.37}$$

The solution of (10A.35) is obviously

$$f(\mathbf{x}) = \int d^2\mathbf{x}'\, G(\mathbf{x}, \mathbf{x}')j(\mathbf{x}). \tag{10A.38}$$

The gradient of $f(\mathbf{x})$ yields the magnetic field of an arbitrary set of line-like currents vertical to the plane under consideration.

It is interesting to realize that the Green function (10A.36) is the imaginary part of the complex function $(1/2\pi)\log(z-z')$ with $z = x^1+ix^2$, whose real part $(1/2\pi)\log|z-z'|$ is the Green function $G_\triangle(\mathbf{x} - \mathbf{x}')$ of the two dimensional Poisson equation:

$$(\partial_1^2 + \partial_2^2)G_\triangle(\mathbf{x} - \mathbf{x}') = \delta^{(2)}(\mathbf{x} - \mathbf{x}'). \tag{10A.39}$$

It is important to point out that the superposition of line-like currents cannot be smeared out into a continuous distribution. The integral (10A.38) yields the superposition of multivalued functions

$$f(\mathbf{x}) = \frac{1}{2\pi}\sum_n I_n \arctan\frac{x^2 - x_n^2}{x^1 - x_n^1}, \tag{10A.40}$$

which is properly defined only if one can clearly continue it analytically into the all parts of the Riemann sheets defined by the endpoints of the cut at the origin. If we were to replace the sum by an integral, this possibility would be lost. Thus it is, strictly speaking, impossible to represent arbitrary continuous magnetic fields as gradients of superpositions of scalar potentials $\Omega(\mathbf{x}; L)$. This, however, is not a severe disadvantage of this representation since any current can be approximated by a superposition of line-like currents with any desired accuracy, and the same will be true for the associated magnetic fields.

The arbitrariness of the shape of the jumping surface is the origin of a further interesting gauge structure which has interesting physical consequences discussed in Subsection 10A.5.

10A.2 Generating Magnetic Fields by Multivalued Gauge Transformations

After this first exercise in multivalued functions, we now turn to another example in magnetism which will lead directly to our intended geometric application. We observed before that the local gauge transformation (10A.2) produces the same magnetic field $\mathbf{B}(\mathbf{x}) = \boldsymbol{\nabla}\times \mathbf{A}(\mathbf{x})$ only, as long as the function $\Lambda(\mathbf{x})$ satisfies the Schwarz integrability criterion (10A.29)

$$(\partial_i\partial_j - \partial_j\partial_i)\Lambda(\mathbf{x}) = 0. \tag{10A.41}$$

Any function $\Lambda(\mathbf{x})$ violating this condition would change the magnetic field by

$$\Delta B_k(\mathbf{x}) = \epsilon_{kij}(\partial_i\partial_j - \partial_j\partial_i)\Lambda(\mathbf{x}), \tag{10A.42}$$

thus being no proper gauge function. The gradient of $\Lambda(\mathbf{x})$

$$\mathbf{A}(\mathbf{x}) = \boldsymbol{\nabla}\Lambda(\mathbf{x}) \tag{10A.43}$$

would be a *nontrivial* vector potential.

By analogy with the multivalued coordinate transformations violating the integrability conditions of Schwarz as in (10A.29), the function $\Lambda(\mathbf{x})$ will be called *nonholonomic gauge function*.

Having just learned how to deal with multivalued functions we may change our attitude towards gauge transformations and decide to generate *all* magnetic fields approximately in a field-free space by such improper gauge transformations $\Lambda(\mathbf{x})$. By choosing for instance

$$\Lambda(\mathbf{x}) = \Phi\Omega(\mathbf{x}; L), \tag{10A.44}$$

we find from (10A.29) that this generates a field

$$B_k(\mathbf{x}) = \epsilon_{kij}(\partial_i\partial_j - \partial_j\partial_i)\Lambda(\mathbf{x}) = \Phi\delta_k(\mathbf{x}; L). \tag{10A.45}$$

This is a magnetic field of total flux Φ inside an infinitesimal tube. By a superposition of such infinitesimally thin flux tubes analogous to (10A.38) we can obviously generate a discrete approximation to any desired magnetic field in a field-free space.

10A.3 Magnetic Monopoles

Multivalued fields have also been used to describe magnetic monopoles [10, 13, 14]. A monopole charge density $\rho_{\rm m}(\mathbf{x})$ is the source of a magnetic field $\mathbf{B}(\mathbf{x})$ as defined by the equation

$$\boldsymbol{\nabla}\cdot \mathbf{B}(\mathbf{x}) = 4\pi\rho_{\rm m}(\mathbf{x}). \tag{10A.46}$$

If $\mathbf{B}(\mathbf{x})$ is expressed in terms of a vector potential $\mathbf{A}(\mathbf{x})$ as $\mathbf{B}(\mathbf{x}) = \boldsymbol{\nabla}\times \mathbf{A}(\mathbf{x})$, equation (10A.46) implies the noncommutativity of derivatives in front of the vector potential $\mathbf{A}(\mathbf{x})$:

$$\frac{1}{2}\epsilon_{ijk}(\partial_i\partial_j - \partial_j\partial_i)A_k(\mathbf{x}) = 4\pi\rho_{\rm m}(\mathbf{x}). \tag{10A.47}$$

Thus $\mathbf{A}(\mathbf{x})$ must be multivalued. Dirac in his famous theory of monopoles [16] made the field single-valued by attaching to the world line of the particle a jumping world surface, whose intersection with a coordinate plane at a fixed time forms the *Dirac string*, along which the magnetic field of the monopole is imported from infinity. This world surface can be made physically irrelevant by quantizing it appropriately with respect to the charge. Its shape in space is just as irrelevant as that of the jumping surface S in Fig. 10.5. The invariance under shape deformations constitute once more a second gauge structure of the type mentioned earlier and discussed in Refs. [2, 6, 7, 10, 12].

Once we allow ourselves to work with multivalued fields, we may easily go one step further and express also $\mathbf{A}(\mathbf{x})$ as a gradient of a scalar field as in (10A.43). Then the condition becomes

$$\epsilon_{ijk}\partial_i\partial_j\partial_k\Lambda(\mathbf{x}) = 4\pi\rho_{\rm m}(\mathbf{x}). \tag{10A.48}$$

Let us construct the field of a magnetic monopole of charge g at a point $\mathbf{x}_0$, which satisfies (10A.46) with $\rho_m(\mathbf{x}) = g\,\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)$. Physically, this can be done only by setting up an infinitely thin solenoid along an arbitrary line $L_0^{\uparrow}$ whose initial point lies at $\mathbf{x}_0$ and the final anywhere at infinity. The superscript $\uparrow$ indicates that the line has a strating point $\mathbf{x}_0$. Inside this solenoid, the magnetic field is infinite, equal to

$$\mathbf{B}_{\rm inside}(\mathbf{x}; L) = 4\pi g\,\boldsymbol{\delta}(\mathbf{x}; L_0^{\uparrow}), \tag{10A.49}$$

where $\boldsymbol{\delta}(\mathbf{x}; L_0^{\uparrow})$ is the open-ended version of (10A.8)

$$\boldsymbol{\delta}(\mathbf{x}; L_0^{\uparrow}) = \int_{L_0^{\uparrow},\mathbf{x}_0} d^3x'\delta^{(3)}(\mathbf{x}-\mathbf{x}'). \tag{10A.50}$$

The divergence of this function is concentrated at the starting point:

$$\boldsymbol{\nabla}\cdot\boldsymbol{\delta}(\mathbf{x};L_0^{\uparrow})=\delta^{(3)}(\mathbf{x}-\mathbf{x}_0). \tag{10A.51}$$

This follows from (10A.10) by moving the end point to infinity. By analogy with the curl relation (10A.20) we observe a further gauge invariance. If we deform the line L, at fixed initial point $\mathbf{x}$, the δ-function (10A.50) changes as follows:

$$\boldsymbol{\delta}(\mathbf{x};L_0^{\uparrow})\to\boldsymbol{\delta}(\mathbf{x};L_0'^{\uparrow})=\boldsymbol{\delta}(\mathbf{x};L_0^{\uparrow})+\boldsymbol{\nabla}\times\boldsymbol{\delta}(\mathbf{x};S), \tag{10A.52}$$

where S is the surface over which $L^{\uparrow}$ has swept on its way to $L'^{\uparrow}$. Under this gauge transformation, the relation (10A.51) is obviously invariant. We shall call this *monopole gauge invariance*. The flux (10A.49) inside the solenoid is therefore a *monopole gauge field*. It is straightforward to construct from it the ordinary gauge field $\mathbf{A}(\mathbf{x})$ of the monopole. First we define the $L_0^{\uparrow}$-dependent field

$$\mathbf{A}(\mathbf{x};L_0^{\uparrow})=-g\int d^3x'\,\frac{\boldsymbol{\nabla}'\times\boldsymbol{\delta}(\mathbf{x}';L_0^{\uparrow})}{R'}=g\int d^3x'\,\boldsymbol{\delta}(\mathbf{x}';L_0^{\uparrow})\times\frac{\mathbf{R}'}{R'^3}. \tag{10A.53}$$

The curl of the first expression is

$$\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x};L_0^{\uparrow})=-g\int d^3x'\,\frac{\boldsymbol{\nabla}'\times[\boldsymbol{\nabla}'\times\boldsymbol{\delta}(\mathbf{x}';L_0^{\uparrow})]}{R'}, \tag{10A.54}$$

and consists of two terms

$$-g\int d^3x'\,\frac{\boldsymbol{\nabla}'[\boldsymbol{\nabla}'\cdot\boldsymbol{\delta}(\mathbf{x}';L_0^{\uparrow})]}{R'}+g\int d^3x'\,\frac{\boldsymbol{\nabla}'^2\boldsymbol{\delta}(\mathbf{x}';L_0^{\uparrow})}{R'}. \tag{10A.55}$$

After an integration by parts, and using (10A.51), the first term is $L_0^{\uparrow}$-independent and reads

$$g\int d^3x'\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)\boldsymbol{\nabla}'\frac{1}{R'}=g\frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|^3}. \tag{10A.56}$$

The second term becomes, after two integration by parts,

$$-4\pi g\,\boldsymbol{\delta}(\mathbf{x}';L_0^{\uparrow}). \tag{10A.57}$$

The first term is the desired magnetic field of the monopole. Its divergence is $\delta(\mathbf{x}-\mathbf{x}_0)$, which we wanted to archive. The second term is the monopole gauge field, the magnetic field inside the solenoid. The total divergence of this field is, of course, zero.

By analogy with (10A.25) we now subtract the latter term and find the magnetic field of the monopole

$$\mathbf{B}(\mathbf{x})=\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x};L_0^{\uparrow})+4\pi g\,\boldsymbol{\delta}(\mathbf{x};L_0^{\uparrow}). \tag{10A.58}$$

This field is independent of the string $L_0^{\uparrow}$. It depends only on the source point $\mathbf{x}_0$ and satisfies $\boldsymbol{\nabla}\cdot\mathbf{B}(\mathbf{x})=4\pi g\,\delta^{(3)}(\mathbf{x}-\mathbf{x}_0)$.

Let us calculate the vector potential for some simple choices of $\mathbf{x}_0$ and $L_0^{\uparrow}$, for instance $\mathbf{x}_0=\mathbf{0}$ and $L_0^{\uparrow}$ along the positive z-axis, so that $\boldsymbol{\delta}(\mathbf{x};L_0^{\uparrow})$ becomes $\hat{\mathbf{z}}\,\Theta(z)\delta(x)\delta(y)$, where $\hat{\mathbf{z}}$ is the unit vector in z-direction. Inserting this into the second expression in (10A.53) yields

$$\begin{aligned}\mathbf{A}^{(g)}(\mathbf{x};L_0^{\uparrow})&=-g\int_0^{\infty}dz'\frac{\hat{\mathbf{z}}\times\mathbf{x}}{\sqrt{x^2+y^2+(z'-z)^2}^{3/2}}\\&=-g\frac{\hat{\mathbf{z}}\times\mathbf{x}}{r(r-z)}=g\frac{(y,-x,0)}{r(r-z)}.\end{aligned} \tag{10A.59}$$

If $L_0^{\uparrow}$ runs to $-\infty$, so that $\boldsymbol{\delta}(\mathbf{x}; L_0^{\uparrow})$ is equal to $-\hat{\mathbf{z}}\,\Theta(-z)\delta(x)\delta(y)$, we obtain

$$\begin{aligned}\mathbf{A}^{(g)}(\mathbf{x}; L_0^{\uparrow}) &= g\int_{-\infty}^{0} dz' \frac{\hat{\mathbf{z}}\times\mathbf{x}}{\sqrt{x^2+y^2+(z'-z)^2}^{3/2}}\\ &= g\frac{\hat{\mathbf{z}}\times\mathbf{x}}{r(r+z)} = -g\frac{(y,-x,0)}{r(r+z)}.\end{aligned} \tag{10A.60}$$

The vector potential has only azimuthal components. If we parametrize (x, y, z) in terms of spherical angles $\theta, \varphi)$ as $r(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, these are

$$A_{\varphi}^{(g)}(\mathbf{x}; L_0^{\uparrow}) = \frac{g}{r\sin\theta}(1-\cos\theta) \quad \text{or} \quad A_{\varphi}^{(g)}(\mathbf{x}; L_0^{\uparrow}) = -\frac{g}{r\sin\theta}(1+\cos\theta), \tag{10A.61}$$

respectively.

The shape of the line $L_0^{\uparrow}$ can be brought to a standard form, which corresponds to fixing a gauge of the field $\boldsymbol{\delta}(\mathbf{x}; L_0^{\uparrow})$. For example, we may always choose $L_0^{\uparrow}$ to run from $\mathbf{x}_0$ into the z-direction.

Also here, there exists an equivalent formulation in terms of a multivalued **A**-field with infinitely many Riemann sheets around the line L. For a detailed discussion of the physics of multivalued fields see Refs. [2, 6, 7, 10, 12].

An interesting observation is the following: If the gauge function $\Lambda(\mathbf{x})$ is considered as a nonholonomic displacement in some fictitious crystal dimension, then the magnetic field of a current loop which gives rise to noncommuting derivatives $(\partial_i\partial_j - \partial_j\partial_i)\Lambda(\mathbf{x}) \neq 0$ is the analog of a dislocaton [compare (10.23)], and thus implies torsion in the crystal. A magnetic monopole, on the other hand, arises from noncommuting derivatives $(\partial_i\partial_j - \partial_j\partial_i)\partial_k\Lambda(\mathbf{x}) \neq 0$ in Eq. (10A.48). It corresponds to a disclination [see (10.57)] and implies curvature. The defects in the multivalued description of magnetism are therefore similar to those in a crystal where dislocations much more abundantly observed than disclinations. They are opposite to those in general relativity which is governed by curvature alone, with no evidence for torsion so far [11].

10A.4 Minimal Magnetic Coupling of Particles from Multivalued Gauge Transformations

Multivalued gauge transformations are the perfect tool to minimally couple electromagnetism to any type of matter. Consider for instance a free nonrelativistic point particle with a Lagrangian

$$L = \frac{M}{2}\dot{\mathbf{x}}^2. \tag{10A.62}$$

The equations of motion are invariant under a gauge transformation

$$L \to L' = L + \boldsymbol{\nabla}\Lambda(\mathbf{x})\,\dot{\mathbf{x}}, \tag{10A.63}$$

since this changes the action $\mathcal{A} = \int_{t_a}^{t_b} dt\, L$ merely by a surface term:

$$\mathcal{A}' \to \mathcal{A} = \mathcal{A} + \Lambda(\mathbf{x}_b) - \Lambda(\mathbf{x}_a). \tag{10A.64}$$

The invariance is absent if we take $\Lambda(\mathbf{x})$ to be a multivalued gauge function. In this case, a nontrivial vector potential $\mathbf{A}(\mathbf{x}) = \boldsymbol{\nabla}\Lambda(\mathbf{x})$ (working in natural units with $e = 1$) is created in the field-free space, and the nonholonomically gauge-transformed Lagrangian corresponding to (10A.63),

$$L' = \frac{M}{2}\dot{\mathbf{x}}^2 + \mathbf{A}(\mathbf{x})\,\dot{\mathbf{x}}, \tag{10A.65}$$

describes correctly the dynamics of a free particle in an external magnetic field.

The coupling derived by multivalued gauge transformations is automatically invariant under additional ordinary single-valued gauge transformations of the vector potential

$$\mathbf{A}(\mathbf{x}) \to \mathbf{A}'(\mathbf{x}) = \mathbf{A}(\mathbf{x}) + \boldsymbol{\nabla}\Lambda(\mathbf{x}), \tag{10A.66}$$

since these add to the Lagrangian (10A.65) once more the same pure derivative term which changes the action by an irrelevant surface term as in (10A.64).

The same procedure leads in quantum mechanics to the minimal coupling of the Schrödinger field $\psi(\mathbf{x})$. The action is $\mathcal{A} = \int dt d^3x\, L$ with a Lagrange density (in natural units with $\hbar = 1$)

$$L = \psi^*(\mathbf{x}) \left(i\partial_t + \frac{1}{2M}\boldsymbol{\nabla}^2 \right) \psi(\mathbf{x}). \tag{10A.67}$$

The physics described by a Schrödinger wave function $\psi(\mathbf{x})$ is invariant under arbitrary local phase changes

$$\psi(\mathbf{x}, t) \to \psi'(\mathbf{x}) = e^{i\Lambda(\mathbf{x})}\psi(\mathbf{x}, t), \tag{10A.68}$$

called local U(1) transformations. This implies that the Lagrange density (10A.67) may equally well be replaced by the gauge-transformed one

$$L = \psi^*(\mathbf{x}, t) \left(i\partial_t + \frac{1}{2M}\mathbf{D}^2 \right) \psi(\mathbf{x}, t), \tag{10A.69}$$

where $-i\mathbf{D} \equiv -i\boldsymbol{\nabla} - \boldsymbol{\nabla}\Lambda(\mathbf{x})$ is the operator of physical momentum.

We may now go over to nonzero magnetic fields by admitting gauge transformations with multivalued $\Lambda(\mathbf{x})$ whose gradient is a nontrivial vector potential $\mathbf{A}(\mathbf{x})$ as in (10A.43). Then $-i\mathbf{D}$ turns into the covariant momentum operator

$$\hat{\mathbf{P}} = -i\mathbf{D} = -i\boldsymbol{\nabla} - \mathbf{A}(\mathbf{x}), \tag{10A.70}$$

and the Lagrange density (10A.69) describes correctly the magnetic coupling in quantum mechanics.

As in the classical case, the coupling derived by multivalued gauge transformations is automatically invariant under ordinary single-valued gauge transformations under which the vector potential $\mathbf{A}(\mathbf{x})$ changes as in (10A.66), whereas the Schrödinger wave function undergoes a local U(1)-transformation (10A.68). This invariance is a direct consequence of the simple transformation behavior of $\mathbf{D}\psi(\mathbf{x}, t)$ under gauge transformations (10A.66) and (10A.68) which is

$$\mathbf{D}\psi(\mathbf{x}, t) \to \mathbf{D}\psi'(\mathbf{x}, t) = e^{i\Lambda(\mathbf{x})}\mathbf{D}\psi(\mathbf{x}, t). \tag{10A.71}$$

Thus $\mathbf{D}\psi(\mathbf{x}, t)$ transforms just like $\psi(\mathbf{x}, t)$ itself, and for this reason, $\mathbf{D}$ is called gauge-covariant derivative. The generation of magnetic fields by a multivalued gauge transformation is the simplest example for the power of the nonholonomic mapping principle.

After this discussion it is quite suggestive to introduce the same mathematics into differential geometry, where the role of gauge transformations is played by reparametrizations of the space coordinates. This is precisely what is done in Subsection (10.2.2).

10A.5 Gauge Field Representation of Current Loops and Monopoles

In the previous subsections we have given examples for the use of multivalued fields in describing magnetic phenomena. The nonholonomic gauge transformations by which we created line-like nonzero field configurations were shown to be the natural origin of the minimal couplings to the classical actions as well as to the Schrödinger equation. It is interesting to observe that there exists a fully fledged theory of magnetism (which is easily generalized to electromagnetism) with these multivalued fields, if we properly handle the freedom in choosing the jumping surfaces S whose boundary represents the physical current loop in Eq. (10A.12).

To understand this we pose ourselves the problem of setting up an action formalism for calculating the magnetic energy of a current loop in the gradient representation of the magnetic field. In this Euclidean field theory, the action is the field energy:

$$\mathcal{E} = \frac{1}{8\pi} \int d^3x \, \mathbf{B}^2(\mathbf{x}). \tag{10A.72}$$

Inserting the gradient representation (10A.28) of the magnetic field, we can write

$$\mathcal{E} = \frac{I^2}{8\pi} \int d^3x \, [\boldsymbol{\nabla}\Omega(\mathbf{x})]^2. \tag{10A.73}$$

This holds for the multivalued solid angle $\Omega(\mathbf{x})$ which is independent of S. In order to perform field theoretic calculations, we must go over to the single-valued representation (10A.25), so that

$$\mathcal{E} = \frac{I^2}{8\pi} \int d^3x \, [\boldsymbol{\nabla}\Omega(\mathbf{x}; S) + 4\pi\boldsymbol{\delta}(\mathbf{x}; S)]^2. \tag{10A.74}$$

The δ-function removes the unphysical field energy on the artificial magnetic dipole layer on S. Let us calculate the magnetic field energy of the current loop from the action (10A.74). For this we rewrite the action (10A.74) in terms of an *auxiliary* vector field $\mathbf{B}(\mathbf{x})$ as

$$\mathcal{E} = \int d^3x \left\{ -\frac{1}{8\pi}\mathbf{B}^2(\mathbf{x}) - \frac{I}{4\pi}\mathbf{B}(\mathbf{x}) \cdot [\boldsymbol{\nabla}\Omega(\mathbf{x}; S) + 4\pi\boldsymbol{\delta}(\mathbf{x}; S)] \right\}. \tag{10A.75}$$

A partial integration brings the second term to

$$\int d^3x \, \frac{I}{4\pi} \boldsymbol{\nabla}\cdot \mathbf{B}(\mathbf{x}) \, \Omega(\mathbf{x}; S).$$

Extremizing this in $\Omega(\mathbf{x})$ yields the equation

$$\boldsymbol{\nabla}\cdot \mathbf{B}(\mathbf{x}) = 0, \tag{10A.76}$$

implying that the field lines of $\mathbf{B}(\mathbf{x})$ form closed loops. This equation may be enforced identically (as a Bianchi identity) by expressing $\mathbf{B}(\mathbf{x})$ as a curl of an auxiliary vector potential $\mathbf{A}(\mathbf{x})$, setting

$$\mathbf{B}(\mathbf{x}) \equiv \boldsymbol{\nabla}\times \mathbf{A}(\mathbf{x}). \tag{10A.77}$$

With this ansatz, the equation which brings the action (10A.75) to the form

$$\mathcal{E} = \int d^3x \left\{ -\frac{1}{8\pi}[\boldsymbol{\nabla}\times \mathbf{A}(\mathbf{x})]^2 - I \, [\boldsymbol{\nabla}\times \mathbf{A}(\mathbf{x})] \cdot \boldsymbol{\delta}(\mathbf{x}; S) \right\}. \tag{10A.78}$$

A further partial integration leads to

$$\mathcal{E} = \int d^3x \left\{ -\frac{1}{8\pi}[\boldsymbol{\nabla}\times \mathbf{A}(\mathbf{x})]^2 - I \, \mathbf{A}(\mathbf{x}) \cdot [\boldsymbol{\nabla}\times \boldsymbol{\delta}(\mathbf{x}; S)] \right\}, \tag{10A.79}$$

and we identify in the linear term in $\mathbf{A}(\mathbf{x})$ the *auxiliary current*

$$\mathbf{j}(\mathbf{x}) \equiv I \, \boldsymbol{\nabla}\times \boldsymbol{\delta}(\mathbf{x}; S) = I \, \boldsymbol{\delta}(\mathbf{x}; L), \tag{10A.80}$$

due to Stoke's law (10A.20). According to Eq. (10A.9), this current is conserved for closed loops L.

By extremizing the action (10A.78), we obtain Ampère's law (10A.1). Thus the auxiliary quantities $\mathbf{B}(\mathbf{x})$, $\mathbf{A}(\mathbf{x})$, and $\mathbf{j}(\mathbf{x})$ coincide with the usual magnetic quantities with the same name. If we insert the explicit solution (10A.5) of Ampère's law into the energy, we obtain the *Biot-Savart* energy for an arbitrary current distribution

$$\mathcal{E} = \frac{1}{2}\int d^3x\, d^3x'\; \mathbf{j}(\mathbf{x})\, \frac{1}{|\mathbf{x}-\mathbf{x}'|}\, \mathbf{j}(\mathbf{x}'). \tag{10A.81}$$

Note that the action (10A.78) is invariant under two mutually dual gauge transformations, the usual magnetic one in (10A.2), by which the vector potential receives a gradient of an arbitrary scalar field, and the gauge transformation (10A.23), by which the irrelevant surface S is moved to another configuration S'.

Thus we have proved the complete equivalence of the gradient representation of the magnetic field to the usual gauge field representation. In the gradient representation, there exists a new type of gauge invariance which expresses the physical irrelevance of the jumping surface appearing when using single-valued solid angles.

The action (10A.79) describes magnetism in terms of a *double gauge theory* [8], in which both the gauge of $\mathbf{A}(\mathbf{x})$ and the shape of S can be changed arbitrarily. By setting up a grand-canonical partition function of many fluctuating surfaces it is possible to describe a large family of phase transitions mediated by the proliferation of line-like defects. Examples are vortex lines in the superfluid-normal transition in helium and dislocation and disclination lines in the melting transition of crystals [2, 6, 7, 10, 12].

Let us now go through the analogous calculation for a gas of monopoles at $\mathbf{x}_n$ from the magnetic energy formed with the field (10A.58):

$$\mathcal{E} = \frac{1}{8\pi}\int d^3x \left[\boldsymbol{\nabla}\times\mathbf{A} + 4\pi g\sum_n \boldsymbol{\delta}(\mathbf{x}; L_n^{\uparrow})\right]^2. \tag{10A.82}$$

As in (10A.75) we introduce an auxiliary magnetic field and rewrite (10A.82) as

$$\mathcal{E} = \int d^3x \left\{-\frac{1}{8\pi}\mathbf{B}^2(\mathbf{x}) - \frac{1}{4\pi}\mathbf{B}(\mathbf{x})\cdot\left[\boldsymbol{\nabla}\times\mathbf{A} + g\sum_n \boldsymbol{\delta}(\mathbf{x}; L_n^{\uparrow})\right]\right\}. \tag{10A.83}$$

Extremizing this in $\mathbf{A}$ yields $\boldsymbol{\nabla}\times\mathbf{B} = 0$, so that we may set $\mathbf{B} = \boldsymbol{\nabla}\Lambda$, and obtain

$$\mathcal{E} = \int d^3x \left\{-\frac{1}{8\pi}\left[\boldsymbol{\nabla}\Lambda(\mathbf{x})\right]^2 + g\Lambda(\mathbf{x})\sum_n \boldsymbol{\nabla}\cdot\boldsymbol{\delta}(\mathbf{x}; L_n^{\uparrow})\right\}. \tag{10A.84}$$

Recalling (10A.51), the extremal Λ field is

$$\Lambda(\mathbf{x}) = -\frac{4\pi g}{\boldsymbol{\nabla}^2}\sum_n \delta(\mathbf{x}-\mathbf{x}_n) = g\sum_n \frac{1}{|\mathbf{x}-\mathbf{x}_n|}, \tag{10A.85}$$

which leads, after reinsertion into (10A.84), to the Coulomb interaction energy

$$\mathcal{E} = \frac{g^2}{2}\sum_{n,n'} \frac{1}{|\mathbf{x}_n - \mathbf{x}_{n'}|}. \tag{10A.86}$$

Appendix 10B Comparison of Multivalued Basis Tetrads and Vierbein Fields

The standard *tetrads* or vierbein fields were introduced a long time ago in gravitational theories of spinning particles both in purely Riemann [16] as well as in Riemann-Cartan spacetimes [17, 18, 19, 20, 2]. Their mathematics is described in detail in the literature [21]. Their purpose was to define at every point a local Lorentz frame by means of another set of coordinate differentials

$$dx^\alpha = h^\alpha{}_\lambda(q)dq^\lambda, \quad \alpha = 0, 1, 2, 3, \tag{10B.1}$$

which can be contracted with Dirac matrices γ^α to form locally Lorentz invariant quantities. Local Lorentz frames are reached by requiring the induced metric in these coordinates to be Minkowskian:

$$g_{\alpha\beta} = h_\alpha{}^\mu(q)h_\beta{}^\nu(q)g_{\mu\nu}(q) = \eta_{\alpha\beta}, \tag{10B.2}$$

where $\eta_{\alpha\beta}$ is the flat Minkowski metric (10.30). Just like $e^i{}_\mu(q)$ in (10.12), these vierbeins possess reciprocals

$$h_\alpha{}^\mu(q) \equiv \eta_{\alpha\beta}g^{\mu\nu}(q)h^\beta{}_\nu(q), \tag{10B.3}$$

and satisfy orthonormality and completeness relations as in (10.13):

$$h_\alpha{}^\mu h^\beta{}_\mu = \delta_\alpha{}^\beta, \qquad h^\alpha{}_\mu h_\alpha{}^\nu = \delta_\mu{}^\nu. \tag{10B.4}$$

They also can be multiplied with each other as in (10.14) to yield the metric

$$g_{\mu\nu}(q) = h^\alpha{}_\mu(q)h^\beta{}_\nu(q)\eta_{\alpha\beta}. \tag{10B.5}$$

Thus they constitute another "square root" of the metric. The relation between these square roots is some linear transformation

$$e^a{}_\mu(q) = e^a{}_\alpha(q)h^\alpha{}_\mu(q), \tag{10B.6}$$

which must necessarily be a local Lorentz transformation

$$\Lambda^a{}_\alpha(q) = e^a{}_\alpha(q), \tag{10B.7}$$

since this matrix connects the two Minkowski metrics (10.30) and (10B.2) with each other:

$$\eta_{ab}\Lambda^a{}_\alpha(q)\Lambda^b{}_\beta(q) = \eta_{\alpha\beta}. \tag{10B.8}$$

The different local Lorentz transformations allow us to choose different local Lorentz frames which distinguish fields with definite spin by the irreducible representations of these transformations. The physical consequences of the theory must be independent of this local choice, and this is the reason why the presence of spinning fields requires the existence of an additional gauge freedom under local Lorentz transformations, in addition to Einstein's invariance under general coordinate transformations. Since the latter may be viewed as local translations, the theory with spinning particles are locally Poincaré invariant.

The vierbein fields $h^\alpha{}_\mu(q)$ have in common with ours that both violate the integrability condition as in (10.22), thus describing nonholonomic coordinates dx^α for which there exists only a differential relation (10B.1) to the physical coordinates q^μ. However, they differ from our multivalued tetrads $e^a\mu(q)$ by being *single-valued* fields satisfying the integrability condition

$$(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)h^\alpha{}_\lambda(q) = 0, \tag{10B.9}$$

in contrast to our multivalued tetrads $e^i{}_\lambda(q)$ in Eq. (10.23).

In the local coordinate system dx^α, curvature arises from a violation of the integrability condition of the local Lorentz transformations (10B.7).

The simple equation (10.24) for the torsion tensor in terms of the multivalued tetrads $e^i{}_\lambda(q)$ must be contrasted with a similar-looking, but geometrically quite different, quantity formed from the vierbein fields $h^\alpha{}_\lambda(q)$ and their reciprocals, the *objects of anholonomy* [21]:

$$\Omega_{\alpha\beta}{}^\gamma(q) = \frac{1}{2}h_\alpha{}^\mu(q)h_\beta{}^\nu(q)\left[\partial_\mu h^\gamma{}_\nu(q) - \partial_\nu h^\gamma{}_\mu(q)\right]. \tag{10B.10}$$

$$\boxed{dx^a = e^a{}_\mu dq^\mu \qquad \Gamma_{ab}{}^c = 0}$$

$$\boxed{dx^\alpha = h^\alpha{}_\mu dq^\mu \quad \Gamma_{\alpha\beta}{}^\gamma = h^\gamma{}_\lambda h_\alpha{}^\mu h_\beta{}^\nu K_{\mu\nu}{}^\lambda - \overset{h}{K}{}_{\alpha\beta}{}^\gamma}$$

$$\boxed{dq^\mu \qquad \Gamma_{\mu\nu}{}^\lambda = e_a{}^\lambda \partial_\mu e^a{}_\nu}$$

Figure 10.6 Coordinate system q^μ and the two sets of local nonholonomic coordinates dx^α and dx^a. The intermediate coordinates dx^α have a Minkowski metric only at the point q, the coordinates dx^a in an entire small neighborhood (at the cost of a closure failure).

A combination of these similar to (10.28),

$$\overset{h}{K}{}_{\alpha\beta}{}^\gamma(q) = \Omega_{\alpha\beta}{}^\gamma(q) - \Omega_\beta{}^\gamma{}_\alpha(q) + \Omega^\gamma{}_{\alpha\beta}(q), \tag{10B.11}$$

appears in the so-called *spin connection*

$$\Gamma_{\alpha\beta}{}^\gamma = h^\gamma{}_\lambda h_\alpha{}^\mu h_\beta{}^\nu (K_{\mu\nu}{}^\lambda - \overset{h}{K}{}_{\mu\nu}{}^\lambda), \tag{10B.12}$$

which is needed to form a covariant derivative of local vectors

$$v_\alpha(q) = v_\mu(q) h_\alpha{}^\mu(q), \qquad v^\alpha(q) = v^\mu(q) h^\alpha{}_\mu(q). \tag{10B.13}$$

These have the form

$$D_\alpha v_\beta(q) = \partial_\alpha v_\beta(q) - \Gamma_{\alpha\beta}{}^\gamma(q) v_\gamma(q), \qquad D_\alpha v^\beta(q) = \partial_\alpha v^\beta(q) + \Gamma_{\alpha\gamma}{}^\beta(q) v^\gamma(q). \tag{10B.14}$$

For details see Ref. [2, 6]. In spite of the similarity between the defining equations (10.24) and (10B.10), the tensor $\Omega_{\alpha\beta}{}^\gamma(q)$ bears no relation to torsion, and $\overset{h}{K}{}_{\alpha\beta}{}^\gamma(q)$ is independent of the contortion $K_{\alpha\beta}{}^\gamma$. In fact, the objects of anholonomy $\Omega_{\alpha\beta}{}^\gamma(q)$ are in general nonzero in the absence of torsion [22], and may even be nonzero in flat spacetime, where the matrices $h^\alpha{}_\mu(q)$ degenerate to local Lorentz transformations. The quantities $\overset{h}{K}{}_{\alpha\beta}{}^\gamma(q)$, and thus the spin connection (10B.12), characterize the orientation of the local Lorentz frames.

The nonholonomic coordinates dx^α transform the metric $g_{\mu\nu}(q)$ to a Minkowskian form η_{ab} at any given point q^μ. They correspond to a small "falling elevator" of Einstein in which the gravitational forces vanish precisely at the center of mass, the neighborhood still being subject to tidal forces. In contrast, the nonholonomic coordinates dx^a flatten the spacetime *in an entire neighborhood* of the point. This is at the expense of producing defects in spacetime (like those produced when flattening an orange peel by stepping on it), as will be explained in Section IV. The affine connection $\Gamma_{ab}{}^c(q)$ in the latter coordinates dx^a vanishes identically.

The difference between our multivalued tetrads and the usual vierbeins is illustrated in the diagram of Fig. 10.6.

Appendix 10C Cancellation of Powers of $\delta(0)$

There is a simple way of proving the cancellation of all UV-divergences $\delta(0)$. Consider a free particle whose mass depends on the time with an action

$$\mathcal{A}_{\rm tot}[q] = \int_0^\beta d\tau \left[\frac{1}{2} Z(\tau) \dot{q}^2(\tau) - \frac{1}{2}\delta(0) \log Z(\tau)\right], \tag{10C.1}$$

where $Z(\tau)$ is some function of τ but independent now of the path $q(\tau)$. The last term is the simplest nontrivial form of the Jacobian action in (10.408). Since it is independent of q, it is conveniently taken out of the path integral as a factor

$$J = e^{(1/2)\delta(0)\int_0^\beta d\tau \log Z(\tau)}. \tag{10C.2}$$

We split the action into a sum of a free and an interacting part

$$\mathcal{A}^{(0)} = \int_0^\beta d\tau\, \frac{1}{2}\dot{q}^2(\tau), \qquad \mathcal{A}^{\rm int} = \int_0^\beta d\tau\, \frac{1}{2}\left[Z(\tau)-1\right]\dot{q}^2(\tau), \tag{10C.3}$$

and calculate the transition amplitude (10.411) as a sum of all connected diagrams in the cumulant expansion

$$\begin{aligned}
(0\,\beta|0\,0) &= J\int \mathcal{D}q(\tau)e^{-\mathcal{A}^{(0)}[q]-\mathcal{A}_{\rm int}[q]} = J\int \mathcal{D}q(\tau)e^{-\mathcal{A}^{(0)}[q]}\left(1-\mathcal{A}_{\rm int}+\frac{1}{2}\mathcal{A}_{\rm int}^2-\dots\right)\\
&= (2\pi\beta)^{-1/2}J\left[1-\langle\mathcal{A}_{\rm int}\rangle+\frac{1}{2}\left\langle\mathcal{A}_{\rm int}^2\right\rangle-\dots\right]\\
&= (2\pi\beta)^{-1/2}J\,e^{-\langle\mathcal{A}_{\rm int}\rangle_c+\frac{1}{2}\left\langle\mathcal{A}_{\rm int}^2\right\rangle_c-\dots}.
\end{aligned} \tag{10C.4}$$

We now show that the infinite series of $\delta(0)$-powers appearing in a Taylor expansion of the exponential (10C.2) is precisely compensated by the sum of all terms in the perturbation expansion (10C.4). Being interested only in these singular terms, we may extend the τ-interval to the entire time axis. Then Eq. (10.394) yields the propagator ${}^{\cdot}\Delta^{\cdot}(\tau,\tau')=\delta(\tau-\tau')$, and we find the first-order expansion term

$$\langle\mathcal{A}_{\rm int}\rangle_c = \int d\tau\,\frac{1}{2}[Z(\tau)-1]\,{}^{\cdot}\Delta^{\cdot}(\tau,\tau) = -\frac{1}{2}\delta(0)\int d\tau\,[1-Z(\tau)]. \tag{10C.5}$$

To second order, divergent integrals appear involving products of distributions, thus requiring an intermediate extension to d dimensions as follows

$$\begin{aligned}
\left\langle\mathcal{A}_{\rm int}^2\right\rangle_c &= \int\!\!\int d\tau_1\,d\tau_2\,\frac{1}{2}(Z-1)_1\,\frac{1}{2}(Z-1)_2\,2\,{}^{\cdot}\Delta^{\cdot}(\tau_1,\tau_2)\,{}^{\cdot}\Delta^{\cdot}(\tau_2,\tau_1)\\
&\to \int\!\!\int d^dx_1\,d^dx_2\,\frac{1}{2}(Z-1)_1\,\frac{1}{2}(Z-1)_2\,2\,{}_{\mu}\Delta_{\nu}(x_1,x_2)\,{}_{\nu}\Delta_{\mu}(x_2,x_1)\\
&= \int\!\!\int d^dx_1\,d^dx_2\,\frac{1}{2}(Z-1)_1\,\frac{1}{2}(Z-1)_2\,2\,\Delta_{\mu\mu}(x_2,x_1)\,\Delta_{\nu\nu}(x_1,x_2)\,,
\end{aligned} \tag{10C.6}$$

the last line following from partial integrations. For brevity, we have abbreviated $[Z(\tau_i)-1]$ by $(Z-1)_i$. Using the field equation (10.418) and going back to one dimension yields, with the further abbreviation $(Z-1)_i\to z_i$:

$$\left\langle\mathcal{A}_{\rm int}^2\right\rangle_c = \frac{1}{2}\int\!\!\int d\tau_1\,d\tau_2\,z_1\,z_2\,\delta^2(\tau_1,\tau_2). \tag{10C.7}$$

To third order we calculate

$$\begin{aligned}
\left\langle\mathcal{A}_{\rm int}^3\right\rangle_c &= \int\!\!\int\!\!\int d\tau_1\,d\tau_2\,d\tau_3\,\frac{1}{2}z_1\,\frac{1}{2}z_2\,\frac{1}{2}z_3\,8\,{}^{\cdot}\Delta^{\cdot}(\tau_1,\tau_2)\,{}^{\cdot}\Delta^{\cdot}(\tau_2,\tau_3)\,{}^{\cdot}\Delta^{\cdot}(\tau_3,\tau_1)\\
&\to \int\!\!\int\!\!\int d^dx_1\,d^dx_2\,d^dx_3\,\frac{1}{2}z_1\,\frac{1}{2}z_2\,\frac{1}{2}z_3\,8\,{}_{\mu}\Delta_{\nu}(x_1,x_2)\,{}_{\nu}\Delta_{\sigma}(x_2,x_3)\,{}_{\sigma}\Delta_{\mu}(x_3,x_1)\\
&= -\int\!\!\int\!\!\int d^dx_1\,d^dx_2\,d^dx_3\,\frac{1}{2}z_1\,\frac{1}{2}z_2\,\frac{1}{2}z_3\,8\,\Delta_{\mu\mu}(x_3,x_1)\,\Delta_{\nu\nu}(x_1,x_2)\,\Delta_{\sigma\sigma}(x_2,x_3).
\end{aligned} \tag{10C.8}$$

Applying again the field equation (10.418) and going back to one dimension, this reduces to

$$\left\langle \mathcal{A}_{\rm int}^3 \right\rangle_c = \int\int\int d\tau_1\, d\tau_2\, d\tau_3\, z_1\, z_2\, z_3 \delta(\tau_1,\tau_2)\,\delta(\tau_2,\tau_3)\,\delta(\tau_3,\tau_1). \tag{10C.9}$$

Continuing to n-order and substituting Eqs. (10C.5), (10C.7), (10C.9), etc. into (10C.4), we obtain in the exponent of Eq. (10C.4) a sum

$$-\left\langle \mathcal{A}_{\rm int} \right\rangle_c + \frac{1}{2}\left\langle \mathcal{A}_{\rm int}^2 \right\rangle_c - \frac{1}{3!}\left\langle \mathcal{A}_{\rm int}^3 \right\rangle_c + \ldots = \frac{1}{2}\sum_1^\infty (-1)^n \frac{c_n}{n}, \tag{10C.10}$$

with

$$c_n = \int d\tau_1 \ldots d\tau_n\, C(\tau_1,\tau_2)\, C(\tau_2,\tau_3)\, \ldots\, C(\tau_n,\tau_1) \tag{10C.11}$$

where

$$C(\tau,\tau') = [Z(\tau)-1]\,\delta(\tau,\tau'). \tag{10C.12}$$

Substituting this into Eq. (10C.11) and using the rule (10.369) yields

$$c_n = \int\int d\tau_1 d\tau_n\, [Z(\tau_1)-1]^n\, \delta^2(\tau_1-\tau_n) = \delta(0)\int d\tau\, [Z(\tau)-1]^n. \tag{10C.13}$$

Inserting these numbers into the expansion (10C.10), we obtain

$$\begin{aligned} -\left\langle \mathcal{A}_{\rm int} \right\rangle_c + \frac{1}{2}\left\langle \mathcal{A}_{\rm int}^2 \right\rangle_c - \frac{1}{3!}\left\langle \mathcal{A}_{\rm int}^3 \right\rangle_c + \ldots \quad &= \quad \frac{1}{2}\,\delta(0)\int d\tau \sum_1^\infty (-1)^n \frac{[Z(\tau)-1]^n}{n} \\ &= \quad -\frac{1}{2}\,\delta(0)\int d\tau\, \log Z(\tau), \end{aligned} \tag{10C.14}$$

which compensates precisely the Jacobian factor J in (10C.4).

Notes and References

Path integrals in spaces with curvature but no torsion have been discussed by
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