Problem Set 2
Theoretical Solid State Physics (SoSe2017)

Due: Due Thursday, May 4, 2017; **at the beginning of class**

Problem 1: Mean field theory for the Ising model

Consider the Ising model

$$H = -J \sum_{\langle ij \rangle} S_i S_j - h \sum_i S_i \tag{1}$$

on a square lattice in D spatial dimensions. Here, $S_i = \pm 1$ denotes spins placed on the vertices of the lattice and $\langle ij \rangle$ denotes the set of nearest neighbors (or the set of bonds of the square lattice). Replace

$$S_i = S + (S_i - S) \tag{2}$$

in the Hamiltonian, where S = $\langle S_i \rangle$ denotes the thermodynamic average of S_i, i.e., the magnetization of the model. Now, make a mean field approximation by assuming that the fluctuations $S_i - S$ about the thermodynamic average are small and need to be carried along in the Hamiltonian only to linear order.

Use this approximation to compute the partition function as well as the free energy of the model. Use the free energy to obtain a self consistent equation for the average magnetization S and discuss its solutions graphically.

Give an expression for the critical temperature for $h = 0$. Expand the free energy per spin in powers of S in the vicinity of the critical temperature and show that its form is consistent with the arguments of Landau theory.

Problem 2: Infinite-range Ising model

It was mentioned in the lecture that the Landau functional should really be considered as an effective Hamiltonian in a functional integral over all order parameter configurations. Perhaps the simplest case in which this can be seen explicitly is the infinite range Ising model in 1D

$$H = -\frac{2J}{N} \sum_{ij} S_i S_j - h \sum_i S_i, \tag{3}$$

where all spins are coupled regardless of distance. To compensate for the fact that each spin is now coupled to $N-1$ instead of two spins, we rescaled the exchange coupling to be $2J/N$. (Note that N is large.) Coupling each spin to all other spins effectively guarantees that mean field theory becomes exact in the thermodynamic limit $N \to \infty$ by virtue of the central limit theorem.

Use the (simple version of) a Hubbard-Stratonovich transformation (prove?)

$$e^{(\beta J/N)(\sum_i S_i)^2} = \left(\frac{N\beta J}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} \mathrm{d}S e^{-N\beta J S^2 + 2\beta J S \sum_i S_i} \tag{4}$$

to rewrite the partition function as an ordinary integral over S. Show that the integral can be evaluated in saddle point approximation and discuss the corresponding saddle point equation. Spell out the analogy with the mean field approximation discussed in Problem 1 in some detail and show in which sense the Landau functional should now be considered as an effective Hamiltonian in the exponent of a partition 'sum'.