Problem Set 9
Quantum Field Theory and Many Body Physics (SoSe2016)

Due: Monday, June 23, 2016 at the beginning of the lecture

In this problem set, we first derive the familiar Stirling's formula for the factorial of a large number. This is a nice example of the method of saddle point integration. We then continue two themes of previous problem sets. The second problem takes a closer look at properties of the perturbation expansion of (interacting) field theories in a particularly simple context, namely when the field theory reduces to a single integral. In the third problem, we look at spinless p-wave superconductors, using the operator approach to the BCS theory which you got to know in a previous problem set. It turns out that we have already encountered the same problem in the beginning of the semester when we discussed the transverse field Ising model. Technically, this problem set is almost a repeat of our considerations then, but the physical interpretation is quite different and has been a central theme of condensed matter physics in recent years.

Problem 1: Stirling's formula as an example for saddle-point integration (10 + 10 + 5 points)

(a) Show that

$$n! = \int_0^\infty \mathrm{d}x \; x^n \mathrm{e}^{-x}$$

Introduce a new integration variable t through $x = nt$. Explain in which sense the integral is controlled by the function

$$V(t) = t - \ln(t)$$

in the vicinity of its minimum, when considering large $n \to \infty$.

(b) Show that the $t-$integration can be approximated by a Gaussian integral in the limit $n \to \infty$ and derive Stirling's formula

$$n! = \sqrt{2\pi n}\, n^n \mathrm{e}^{-n}$$

(c) Find the magnitude of the leading correction to Stirling's formula.

Problem 2: The nature of perturbation theory (10 + 10 + 5 points)

As a toy model of a field theory, consider the integral

$$Z(g) = \int \frac{\mathrm{d}x}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2} - \frac{g}{4}x^4}$$

(a) Expand $Z(g)$ in powers of g,

$$Z(g) = \sum_{n=0}^{\infty} g^n Z_n$$

and perform the $x-$integral to compute Z_n explicitly. Show that

$$g^n Z_n = \frac{(-g)^n (4n)!}{n! 16^n (2n)!}$$

and find an asymptotic expression for Z_n for $n \to \infty$.

(b) Plot $g^n Z_n$ vs. n for $n \leq 40$ and $g = 0.2$, 0.05 and 0.01. Is this perturbative expansion of $Z(g)$ convergent or divergent? Give an argument, why one should have expected the result. (Consider what happens when g changes sign.)

(c) Show that the truncated series $\sum_{m=0}^{n} g^m Z_m$ approximates $Z(g)$ in the sense that

$$\left| Z(g) - \sum_{m=0}^{n} g^m Z_m \right| \leq g^{n+1} |Z_{n+1}| \ .$$

From the asymptotic expression for Z_n derived in (a), derive the accuracy of this approximation at the optimal choice of n.

Problem 3: Spinless p-wave superconductors and Majorana fermions (5+5+5+5+5+5 points)

There is currently enormous interest in realizing topological superconductors in experiment. This effort is partly motivated by Alexei Kitaev's insight that these topological superconductors have Majorana excitations – fermions which are their own antiparticles as introduced by Majorana in the 30's – and that these Majorana excitations might be useful in the context of topological quantum computation. Specifically, the entity that experimentalists are hunting are Majorana bound states which have zero energy and obey a novel type of quantum statistics, distinct from fermionic or bosonic statistics, which is referred to as non-abelian statistics. In this problem set, we want to discuss a simple model which exhibits these excitations, namely a spinless p-wave superconductor in one dimension (Kitaev chain).

At first sight, it might seem problematic that spinless p-wave superconductors do not exist in nature. However, there are now several proposals in the literature which effecively realize this model (or models which are adiabatically connected to this model and exhibit the same essential physics) and which can be implemented in the laboratory. The underlying idea is to induce effective p-wave correlations in a one-dimensional electron system by proximity coupling to a conventional s-wave superconductor. This can be done by judiciously exploiting spin polarization by applied magnetic fields and spin-orbit coupling of the material. Two of the prominent experiments pursuing these ideas and reporting success (which, however, is still under discussion) are: V. Mourik et al., Science **336**, 1003 (2012) and S. Nadj-Perge et al. Science **3**46, 602 (2014).

Incidentally, you should also notice that the Kitaev chain is closely related to the transverse field Ising model in 1d which we discussed previously. In fact, we already showed that the transverse field Ising model maps to the Kitaev chain for specific parameters values. At the time, we only solved specific limiting cases. In this problem set, you will effectively also solve this model for general parameters. It is also worthwhile to understand that the existence of the Jordan-Wigner mapping between the two models does not mean that there are no important differences between the physics of the two models. In fact, the transverse field Ising model exhibits a regular symmetry-breaking quantum phase transition. In contrast, this transition maps to a topological quantum phase transition in the Kitaev chain.

In a previous problem, you outlined the mean-field (BCS) theory of superconductors. Here, we want to discuss the so called Kitaev chain

$$H = -\mu \sum_{i=1}^{N} c_i^\dagger c_i - t \sum_{i=1}^{N} \left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i \right) - \Delta \sum_{i=1}^{N} \left(c_i c_{i+1} + c_{i+1}^\dagger c_i^\dagger \right) . \tag{1}$$

This model describes a p-wave superconductor in mean-field theory as it contains pairing terms consisting of two creation or annihilation operators. This is a p-wave superconductor as the fermion operators in these pairing terms correspond to neighboring sites.

Assuming a chain of N sites with periodic boundary conditions, we identify $c_{N+1} = c_1$. Transforming to momentum space

$$c_j = \frac{1}{\sqrt{N}} \sum_k e^{ikj} a_k \tag{2}$$

and introducing the Nambu spinor

$$\phi_k^T = \left(a_k,\, a_{-k}^\dagger\right) \tag{3}$$

leads to the Hamiltonian

$$H = \sum_{k>0} \phi_k^\dagger \begin{pmatrix} \xi_k & 2\mathrm{i}\Delta \sin k \\ -2\mathrm{i}\Delta \sin k & -\xi_k \end{pmatrix} \phi_k, \tag{4}$$

which we want to study.

(a) Show that the excitation spectrum of the Kitaev chain is

$$E_k = \sqrt{\xi_k^2 + 4\Delta^2 \sin^2 k} \tag{5}$$

(b) In the parameter space spanned by the chemical potential μ and the gap Δ, draw the line(s) where the gap (i.e., the smallest E_k for any k) vanishes. Note, that the gap is nonzero on *both* sides of this line. The two sides correspond to different phases of the model as characterized by different topological indices, and the line marks a *topological* quantum phase transition.

(c) One can distinguish the two different topological phases by considering the model with open boundary conditions. To this end, we consider the same Hamiltonian as in Eq. (4), but for a finite and non-perodic chain with N sites. For this finite-length chain, introduce Majorana fermion operators γ_{Aj} and γ_{Bj} through

$$c_j = \frac{1}{2}\left(\gamma_{Bj} + \mathrm{i}\gamma_{Aj}\right) \qquad \text{with} \qquad \gamma_{\alpha i}^\dagger = \gamma_{\alpha i} \tag{6}$$

(similar to separation a complex number $z = x + \mathrm{i}y$ into real and imaginary part). Show that the fermion anticommutation relations

$$\{c_i, c_j\} = \left\{c_i^\dagger, c_j^\dagger\right\} = 0 \qquad \left\{c_i, c_j^\dagger\right\} = \delta_{ij} \tag{7}$$

imply for the real or Majorana fermions

$$\{\gamma_{\alpha i}, \gamma_{\beta j}\} = 2\delta_{\alpha\beta}\delta_{ij}\,. \tag{8}$$

Physically, the relation $\gamma_{\alpha i} = \gamma_{\alpha i}^\dagger$ reflects that Majorana fermions are their own antiparticles.

(d) Now consider the Hamiltonian in Eq. (4) in the special case $\mu = 0$ and $t = \Delta$ and show that it can then be written as

$$H = -\mathrm{i}t \sum_{i=1}^{N-1} \gamma_{Bi}\gamma_{Ai+1}\,. \tag{9}$$

Note that γ_{A1} and γ_{BN} are *not* contained in H!

(e) Now introduce new (conventional) fermion operators

$$d_i = \frac{1}{2}\left(\gamma_{Bi} - \mathrm{i}\gamma_{Ai+1}\right) \qquad i = 1, ..., N-1 \tag{10}$$

(Check that the d_i satisfy the appropriate commutation relations.) Note that there were N c_i-operators, but that there are only $(N-1)$ d_i-operators. Show that in terms of the d operators

$$H = 2t \sum_{i=1}^{N-1} \left(d_i^\dagger d_i - \frac{1}{2}\right) \tag{11}$$

Also express the d_i in terms of the original fermion operators c_i and $c_i^\dagger$.

(f) Discuss the eigenstates and the spectrum of H, paying particular attention to the degeneracy of the ground state. Argue (without explicitly redoing the derivations) why the degeneracy is stable when varying the parameters of the model away from $\mu = 0$, $\Delta = t$.