Problem Set 7a
Quantum Field Theory and Many Body Physics (SoSe2016)

Due: In class problems for the weeks of June 6 and 13

In this problem set, we study noninteracting fermions in a disordered potential. Approximating the electron liquid as non-interacting fermions is the simplest (but often very successful) description of metals or semiconductors. Incorporating disorder into this theory is clearly important as any real metal or semiconductor will not be a perfect single crystal but rather full of defects. In practice, disorder effects are important to understand the transport properties (such as the conductivity) of metals and semiconductors. Disorder can dramatically change the nature of the eigenstates. While the eigenstates of perfect crystals are Bloch states that are extended throughout the entire crystal, eigenstates in disordered systems can be Anderson localized, with the wavefunction falling off exponentially from their center with a localization length ξ. (P.W. Anderson, *Absence of Diffusion in Certain Random Lattices*, Phys. Rev. **109**, 1492 (1958); Anderson received the Nobel Prize in 1977 for this work.) It is believed that the eigenfunctions are always localized in one and two dimensions, even for arbitrarily weak disorder. Such systems would be insulating at zero temperature (Anderson insulator). In three dimensions, the spectrum can support both extended and localized states which are separated in energy by a so-called mobility edge. Such systems can be either metallic or insulating, depending on whether the Fermi energy falls into the region of localized or extended states. Here, we will not be able to discuss these more advanced problems but in principle, these results can be obtained with the methods described in this problem set.

This problem set contains in class problems and will not be corrected. But you should feel free to attempt to solve it on your own before the class. It teaches you many important concepts of diagrammatic perturbation theory and Feynman diagrams in a simple context.

Problem 1: Feynman diagrams and disordered metals

Consider a non-interacting fermion system subject to a potential $V(\mathbf{r})$,

$$H = \int d\mathbf{r} \, \psi^\dagger(\mathbf{r}) \left(-\frac{\nabla^2}{2m} + V(\mathbf{r}) \right) \psi^\dagger(\mathbf{r}). \tag{1}$$

(a) Show that the thermal Green function

$$\mathcal{G}(\mathbf{r}\tau, \mathbf{r}'\tau') = \langle \mathcal{T}_\tau \, \psi(\mathbf{r}, \tau)\psi^\dagger(\mathbf{r}', \tau')_\pm \rangle \tag{2}$$

satisfies the equation of motion

$$(\partial_t + H) \, \mathcal{G}(\mathbf{r}\tau, \mathbf{r}'\tau') = \delta(\mathbf{r} - \mathbf{r}')\delta(\tau - \tau'), \tag{3}$$

with the single-particle Hamiltonian $H = -\frac{\nabla^2}{2m} + V(\mathbf{r})$.

(b) Consider the thermal Green function $\mathcal{G}_0(\mathbf{r}\tau, \mathbf{r}'\tau')$ of the free particle with $V(\mathbf{r}) = 0$. Show that $\mathcal{G}$ satisfies the Dyson equation

$$\mathcal{G}(\mathbf{r}\tau, \mathbf{r}'\tau') = \mathcal{G}_0(\mathbf{r}\tau, \mathbf{r}'\tau') - \int d\mathbf{r}_1 d\tau_1 \, \mathcal{G}_0(\mathbf{r}\tau, \mathbf{r}_1\tau_1)V(\mathbf{r}_1)\mathcal{G}(\mathbf{r}_1\tau t, \mathbf{r}'\tau'), \tag{4}$$

or

$$\mathcal{G} = \mathcal{G} - \mathcal{G}_0 V \mathcal{G} \tag{5}$$

in matrix notation (with $V(\mathbf{r}\tau, \mathbf{r}'\tau') = V(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}')\delta(\tau - \tau')$).

(c) Solve the Dyson equation iteratively in powers of V. Show that the result can be represented graphically as

where the bold arrow represents the full Green function and the normal arrows stand for the free particle Green function (the internal labels $3, 4, 5, 6$ can be dropped). Give the explicit rule how to associate each of these Feynman diagrams with an analytical expression.

(d) For bosons, rederive the Feynman diagram expansion of (c) starting with the bosonic coherent state functional integral. You should expand the action in powers of the potential V. (It should be clear that this provides a rather general procedure to develop diagrammatic perturbation expansions, even for more complicated interactions in the Hamiltonian.)

(e) Assume that $V(\mathbf{r})$ is a random potential, so that the Hamiltonian might describe a disordered metal in the non-interacting electron approximation. We assume that the random potential is drawn from the (functional) Gaussian probability distribution

$$P[V(\mathbf{r})] = \frac{1}{N} \exp\left[-\frac{1}{2\gamma} \int d\mathbf{r}\, V^2(\mathbf{r})\right], \tag{6}$$

where $N = \int [dV(\mathbf{r})]\, P(V)$ is a normalization constant. Define the generating functional (denoting averages over the disorder potential $V(\mathbf{r})$ by an overbar)

$$W[\rho(\mathbf{r})] = \frac{1}{N} \int [dV(\mathbf{r})] \exp\left[-\frac{1}{2\gamma} \int d\mathbf{r}\, V^2(\mathbf{r}) + \int d\mathbf{r}\, \rho(\mathbf{r})V(\mathbf{r})\right] = \overline{e^{\int d\mathbf{r}\, \rho(\mathbf{r})V(\mathbf{r})}}. \tag{7}$$

Show that

$$W[\rho(\mathbf{r})] = e^{\frac{\gamma}{2} \int d\mathbf{r}\, \rho^2(\mathbf{r})}. \tag{8}$$

Use this result to show that

$$
\begin{align}
\overline{V(\mathbf{r})} &= 0, \tag{9}\\
\overline{V(\mathbf{r})V(\mathbf{r}')} &= \gamma\delta(\mathbf{r} - \mathbf{r}'), \tag{10}\\
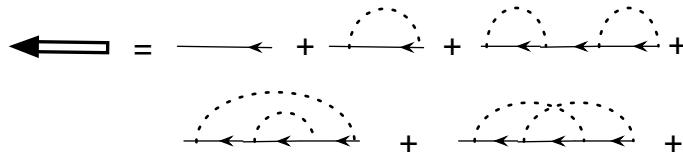\overline{V(\mathbf{r}_1)V(\mathbf{r}_2)V(\mathbf{r}_3)V(\mathbf{r}_4)} &= \gamma\delta(\mathbf{r}_1 - \mathbf{r}_2)\,\gamma\delta(\mathbf{r}_3 - \mathbf{r}_4) + \gamma\delta(\mathbf{r}_1 - \mathbf{r}_3)\,\gamma\delta(\mathbf{r}_2 - \mathbf{r}_4) + \gamma\delta(\mathbf{r}_1 - \mathbf{r}_4)\,\gamma\delta(\mathbf{r}_2 - \mathbf{r}_3) \tag{11}
\end{align}
$$

Generally, all correlators of an even number of $V(\mathbf{r})$ are given as the sum over all possible pairings (Wick's theorem).

(f) Consider the impurity averaged Green function

$$\overline{\mathcal{G}(\mathbf{r}\tau, \mathbf{r}'\tau')} = \int [dV(\mathbf{r})]\, P[V(\mathbf{r})]\, \mathcal{G}(\mathbf{r}\tau, \mathbf{r}'\tau'). \tag{12}$$

explain why the result of (d) implies that $\overline{\mathcal{G}(\mathbf{r}\tau, \mathbf{r}'\tau')}$ can be represented by all distinct diagrams of the type


where the thick arrows stands for the disorder averaged Green function and $\overset{1}{\cdot}\!\cdots\cdots\!\overset{2}{\cdot} = \gamma\delta(\mathbf{r}_1 - \mathbf{r}_2)$. Explain why this is an expansion in powers of γ, i.e., in the strength of the random potential.

(g) An important concept of diagrammatic approach is the self-energy. Consider a general diagram contributing to the average Green function. This diagram either falls apart into separate pieces when cutting an internal Green function line, or not. Diagrams of the first type are called single-particle reducible, e.g.,
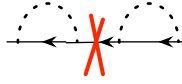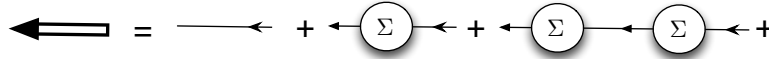
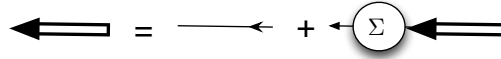diagrams of the second type single-particle irreducible, e.g.,
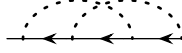
Now, consider the sum of all single-particle irreducible diagrams

$$\Sigma = \quad + \quad + \quad +$$
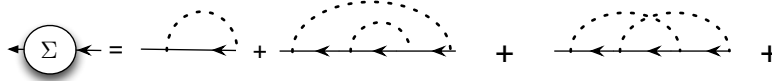
This quantity, with amputated external Green function legs, is the self-energy. Explain why

and thus

As an equation in matrix notation this becomes

$$\overline{\mathcal{G}} = \mathcal{G}_0 - \mathcal{G}_0 \Sigma \overline{\mathcal{G}}. \tag{13}$$

Show that this can be solved as

$$\overline{\mathcal{G}} = \left(\mathcal{G}_0^{-1} + \Sigma \right)^{-1}. \tag{14}$$

(h) Retain only the lowest-order diagram for the self energy of the disorder-averaged Green function and show that

$$\overline{\mathcal{G}}(\mathbf{k}, \omega) = \frac{1}{i\omega_n - (\epsilon_k - \mu) + \frac{i}{2\tau}\mathrm{sgn}\omega_n}. \tag{15}$$

Given an explicit expression for the elastic scattering rate $1/\tau$ and interpret it also in terms of Fermi's Golden Rule.

(i) Consider the polarization operator of a disordered metal and show that it is given by a diffusion pole. (This is a difficult problem and requires summation of a series of ladder diagrams.)

(j) Start with the Kubo formula for the conductivity and evaluate it for a disordered metal. (Also difficult!)