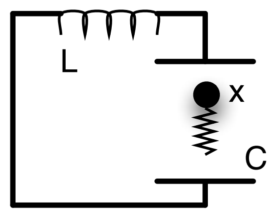
Problem Set 7
Quantum Field Theory and Many Body Physics (SoSe2016)

Due: Thursday, June 9, 2016 at the beginning of the lecture

In this problem set, we study examples of effective field theories.

Problem 1: Effective action of an LC circuit (25 points)

To illustrate the concept of an effective "field" theory, consider an LC circuit coupled to a harmonically bound charge e:



The charge with coordinate x has a Hamiltonian

$$H_d = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0 x^2 - eEx, \tag{1}$$

where E is the electric field of the capacitor. As familiar from elementary physics, the LC circuit is also a harmonic oscillator, as reflected in its energy

$$H_{\text{LC}} = \frac{1}{2}I\dot{Q}^2 + \frac{Q^2}{2C}. \tag{2}$$

This can also be written in terms of the electric field $E = Q/Cd$ in the capacitor (d is the distance between the capacitor plates),

$$H_{\text{LC}} = \frac{1}{2g}\left(\dot{E}^2 + \omega_{\text{LC}}^2 E^2\right). \tag{3}$$

Here $\omega_{\text{LC}} = 1/LC$ is the resonance frequency of the LC circuit and $g = 1/C^2Ld^2$. Thus, we can express the partition function of this system as

$$Z = \int [dE][dx]\, \exp\left[-\int_0^\beta d\tau\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0 x^2 - eEx + \frac{1}{2g}\left(\dot{E}^2 + \omega_{\text{LC}}^2 E^2\right)\right)\right]. \tag{4}$$

Now consider the limit $\omega_{\text{LC}} \ll \omega_0$ and derive an effective action for the LC circuit by tracing out the charge coordinate x (i.e. perform the integral over $[dx]$). Show that to leading order in this limit, the effective action is again a harmonic-oscillator action with renormalized parameters. You should find that to leading order, only the frequency becomes renormalized and the "mass" prefactor of $\dot{E}^2$ remains unchanged.

Problem 2: Friction in quantum mechanics (5+5+5+5+5 points)

Friction is an important phenomenon in everyday life but cannot be described within a Hamiltonian language. This makes it difficult to describe at the quantum level. In this problem, we show that friction can be captured in quantum mechanics within the language of effective "field" theories.

First consider a classical particle subject to a frictional force $-\gamma\dot{x}$ and driving force $F(t)$. The equation of motion is

$$m\ddot{x} = -\gamma\dot{x} + F. \tag{5}$$

In frequency space we can define the response function χ through

$$x(\omega) = \chi(\omega) F(\omega), \tag{6}$$

and read off from the equation of motion that

$$\chi(\omega) = \frac{1}{-m\omega^2 - i\omega\gamma}. \tag{7}$$

Let us now reproduce this from a quantum perspective.
Consider a quantum particle subject to a uniform force F, coupled to an environment consisting of (many) harmonic oscillators,

$$H = \frac{1}{2}m\dot{x}^2 - Fx + \sum_i \left(\frac{1}{2}\mu_i \dot{x}_i^2 + \frac{1}{2}\mu_i \left(\omega_i x_i - g_i x\right)^2\right). \tag{8}$$

The harmonic oscillators are meant to model the environmental degrees of freedom which dissipate the energy of the particle. Define the spectral density of the environmental oscillators

$$n(\omega) = \sum_i \delta(\omega - \omega_i), \tag{9}$$

and the coupling function

$$g^2(\omega) = \frac{1}{n(\omega)} \sum_i \mu_i g_i^2 \delta(\omega - \omega_i). \tag{10}$$

The system can also be described by the imaginary-time action

$$S = \int \mathrm{d}\tau \left(\frac{1}{2}m\dot{x}^2 - Fx + \sum_i \left(\frac{1}{2}\mu_i \dot{x}_i^2 + \frac{1}{2}\mu_i \left(\omega_i x_i - g_i x\right)^2\right)\right). \tag{11}$$

(a) Integrate out the harmonic-oscillator environment to obtain the effective action for x (setting $F = 0$),

$$S_{\text{eff}} = \int \mathrm{d}\tau \, \frac{1}{2}m\dot{x}^2 + \int \mathrm{d}\tau \mathrm{d}\tau' \frac{1}{4}\mathcal{G}(\tau - \tau') \left(x(\tau) - x(\tau')\right)^2, \tag{12}$$

where $\mathcal{G}(\tau - \tau')$ is the (Matsubara) Fourier transform of

$$\mathcal{G}(i\omega) = \sum_i \frac{\mu_i \omega_i^2 g_i^2}{\omega^2 + \omega_i^2}. \tag{13}$$

(b) Use this result to show that the response function defined by

$$x(\tau) = \int d\tau' \, \chi(\tau - \tau') F(\tau') \tag{14}$$

is given by

$$\chi(\omega + i\eta) = \frac{1}{-m(\omega + i\eta)^2 - \left(\mathcal{G}(\omega + i\eta) - \mathcal{G}(0 + i\eta)\right)}. \tag{15}$$

By comparison with the classical response function, we can thus identify the friction coefficient

$$\gamma = \mathrm{Im} \frac{\mathcal{G}(\omega + i\eta) - \mathcal{G}(0 + i\eta)}{\omega}, \tag{16}$$

and the mass renormalization

$$m^* = m + \mathrm{Re} \frac{\mathcal{G}(\omega + i\eta) - \mathcal{G}(0 + i\eta)}{\omega^2}. \tag{17}$$

(c) Show that

$$\gamma = \frac{\pi}{2} n(|\omega|) g^2(|\omega|). \tag{18}$$

You might find the identity $\frac{1}{x-i\eta} = \mathcal{P}\left(\frac{1}{x}\right) + i\pi\delta(x)$ useful. Show also that

$$m^* = m + \mathcal{P}\sum_i \mu_i g_i^2 \frac{1}{\omega_i^2 - \omega^2}, \tag{19}$$

where $\mathcal{P}$ denotes the principal value.

(d) Choose $n(\omega)g^2(\omega) = n_0 g_0 \theta(\Omega - \omega)$, where Ω is an upper frequency (ultraviolet) cutoff of the phonon spectrum, and the step function is given by $\theta(x) = 1$ ($\theta(x) = 0$) for $x > 0$ ($x < 0$). Then

$$\gamma = \frac{\pi}{2} n_0 g_0^2. \tag{20}$$

Show that the associated mass renormalization is given by

$$m^* = m - \frac{n_0 g_0^2}{\Omega}. \tag{21}$$

(e) Now, assume $n(\omega)g^2(\omega) = n_0 g_0^2 e^{-\frac{\omega^2}{\Omega^2}}$, i.e., a phonon spectrum with a smooth cutoff. Show that

$$\mathcal{G}(\tau - \tau') = \int d\omega' \frac{\omega}{2} n(\omega) g^2(\omega) e^{-\omega|\tau|}, \tag{22}$$

and perform the frequency integral to obtain the popular effective action

$$S_{\text{eff}} = \int d\tau \, \frac{1}{2m}\dot{x}^2 + \int d\tau \, d\tau' \, \frac{\gamma\,(x(\tau) - x(\tau'))^2}{4\pi(\tau-\tau')^2}. \tag{23}$$

Problem 3: Equation of motion approach to Hartree-Fock (10+10+5 points)

The Hartree-Fock approximation describes interacting systems in terms of an approximate non-interacting one. In this problem, we want to formulate the Hartree-Fock approximation in the framework of of the equation of motion for the single-particle Green function,

$$(\partial_\tau + H_0)\mathcal{G}(\mathbf{r}\tau, \mathbf{r}'\tau') + \int \mathrm{d}\mathbf{r}_1 \mathrm{d}\tau_1 v(\mathbf{r} - \mathbf{r}_1, \tau - \tau_1)\langle \mathrm{T}_\tau \psi(\mathbf{r}_1, \tau_1)\psi(\mathbf{r}, \tau)\psi^\dagger(\mathbf{r}_1, \tau_1^+)\psi^\dagger(\mathbf{r}', \tau')\rangle = \delta(\mathbf{r} - \mathbf{r}')\delta(\tau - \tau'). \tag{24}$$

Here, τ_1^+ is infinitesimally later than τ_1 and we consider a system with a Hamiltonian of the form

$$H = \int d\mathbf{r}\, \psi^\dagger(\mathbf{r})\left(-\frac{\nabla^2}{2m} + U(\mathbf{r})\right)\psi(\mathbf{r}) + \frac{1}{2}\int d\mathbf{r} d\mathbf{r}' \psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')v(\mathbf{r} - \mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r}), \tag{25}$$

whose non-interacting part is denoted as H_0. We also defined $v(\mathbf{r} - \mathbf{r}_1, \tau - \tau_1) = v(\mathbf{r} - \mathbf{r}_1)\delta(\tau - \tau_1)$.

(a) To close the equation of motion, we have to approximate

$$\langle \mathrm{T}_\tau \psi(\mathbf{r}_1, \tau_1)\psi(\mathbf{r}, \tau)\psi^\dagger(\mathbf{r}_1, \tau_1^+)\psi^\dagger(\mathbf{r}', \tau')\rangle \tag{26}$$

in terms of the single-particle Green function $\mathcal{G}(\mathbf{r}\tau, \mathbf{r}'\tau')$. We can do that by neglecting the two-body interaction $v(\mathbf{r} - \mathbf{r}')$ in evaluating this correlator. Explain why this approximation yields

$$\langle \mathrm{T}_\tau \psi(\mathbf{r}_1, \tau_1)\psi(\mathbf{r}, \tau)\psi^\dagger(\mathbf{r}_1, \tau_1^+)\psi^\dagger(\mathbf{r}', \tau')\rangle \simeq \pm\mathcal{G}(\mathbf{r}\tau, \mathbf{r}'\tau')\mathcal{G}(\mathbf{r}_1\tau_1, \mathbf{r}_1\tau_1^+) + \mathcal{G}(\mathbf{r}\tau, \mathbf{r}_1\tau_1)\mathcal{G}(\mathbf{r}_1\tau_1, \mathbf{r}'\tau'). \tag{27}$$

(b) In the Hartree approximation, one keeps only the first of the two terms on the right hand side of the last equation. This yields the equation of motion

$$(\partial_\tau + H_0 + V_H)\mathcal{G}(\mathbf{r}\tau, \mathbf{r}'\tau') = \delta(\mathbf{r}-\mathbf{r}')\delta(\tau-\tau'), \tag{28}$$

where we defined the Hartree potential

$$V_H(\mathbf{r}) = \pm \int \mathrm{d}\mathbf{r}_1 v(\mathbf{r}-\mathbf{r}_1)\mathcal{G}(\mathbf{r}_1\tau, \mathbf{r}_1\tau^+). \tag{29}$$

Express the Hartree potential in terms of the eigenfunctions and eigenenergies of $H_0 + V_H$,

$$(H_0 + V_H)\phi_\alpha(\mathbf{r}) = \epsilon_\alpha \phi_\alpha(\mathbf{r}), \tag{30}$$

and find

$$V_H(\mathbf{r}) = \int \mathrm{d}\mathbf{r}_1 v(\mathbf{r}-\mathbf{r}_1)\sum_\alpha |\phi_\alpha(\mathbf{r}_1)|^2 n(\epsilon_\alpha). \tag{31}$$

Here, $n(\epsilon)$ denotes the Bose or Fermi function, respectively.

(c) Now consider also the second term in Eq. (27) which introduces the nonlocal Fock potential in addition,

$$(\partial_\tau + H_0 + V_H)\mathcal{G}(\mathbf{r}\tau, \mathbf{r}'\tau') \pm \int \mathrm{d}\mathbf{r}_1 V_F(\mathbf{r}, \mathbf{r}_1)\mathcal{G}(\mathbf{r}_1\tau, \mathbf{r}'\tau') = \delta(\mathbf{r}-\mathbf{r}')\delta(\tau-\tau'). \tag{32}$$

Also express the Fock potential in terms of the effective single-particle eigenfunctions and eigenenergies in Hartree-Fock approximation.

(d) **Bonus Problem** (You gain an additional 10 points and important insights): Consider the Hartree approximation and redo the derivation of the polarization operator. Note that the Hartree potential is a functional of the electron density. Show that this reproduces the RPA approximation discussed on an earlier problem set. If you are even more adventurous, you may want to try to understand the response function derived within the full Hartree-Fock approximation (or ask your tutor).