Problem Set 4
Quantum Field Theory and Many Body Physics (SoSe2016)

Due: Thursday, May 19, 2016 before the beginning of the class

In this problem set, we learn how to compute path integrals explicitly. For the most part, the only path integrals that can be computed exactly are those for a free particle and for the harmonic oscillator, and it is these path integrals which we want to do. To do this, we exploit that the semiclassical calculation is exact in these cases. In addition, we will learn how to do path integrals for spin.

Problem 1: Path integral for a free particle (10+10+5 points)

Calculate the path integral for a free particle in one dimension, i.e., compute the propagator $i\,G(xt, x't')$ for $H = p^2/2m$. When written in its discrete form, the path integral for this problem becomes a Gaussian integral which we can compute explicitly.

(a) We start from the configuration-space path integral and write the path as

$$x(\tau) = x_{\rm cl}(\tau) + \delta x(\tau). \tag{1}$$

Here,

$$x_{\rm cl}(\tau) = x' + \frac{x - x'}{t - t'}(\tau - t') \tag{2}$$

is the classical path. By Hamilton's principle, the classical path extremizes the action. Explain why this implies that

$$S = \int_{t'}^{t} d\tau \frac{1}{2} m\dot{x}^2 = S_{\rm cl} + \int_{t'}^{t} d\tau \frac{1}{2} m\delta\dot{x}^2 \tag{3}$$

with the classical action

$$S_{\rm cl} = \frac{1}{2} m \frac{(x - x')^2}{t - t'}. \tag{4}$$

Now, write the path integral in its discrete form and show that

$$iG(xt, x't') = \left(\frac{m}{2\pi i\,\Delta t}\right)^{N/2} \frac{(2\pi)^{(N-1)/2}}{\sqrt{\det[-iM]}} e^{iS_{\rm cl}}, \tag{5}$$

with $\Delta t = \frac{t-t'}{N}$, $S_{\rm cl} = \frac{m}{2}\frac{(x-x')^2}{t-t'}$, and M the $(N-1)\times(N-1)$ matrix

$$M = \frac{m}{\Delta t} \begin{pmatrix} 2 & -1 & 0 & \dots & & \\ -1 & 2 & -1 & 0 & \dots & \\ 0 & -1 & 2 & -1 & & \\ \vdots & 0 & \ddots & \ddots & \ddots & \\ & \vdots & & \ddots & & -1 \\ & & & & -1 & 2 \end{pmatrix}. \tag{6}$$

(b) We now need to compute $\det M$. Let us actually consider the slightly more general $N \times N$ matrix with matrix elements

$$(M_N)_{ij} = \begin{cases} 2\cosh u & i = j \\ -1 & i = j \pm 1 \\ 0 & \text{else} \end{cases}. \tag{7}$$

First show that $\det M_N$ satisfies the recursion relation

$$\begin{aligned}
\det M_N &= 2\cosh u \, \det M_{N-1} - \det M_{N-2} && (8)\\
\det M_1 &= 2\cosh u && (9)\\
\det M_2 &= 4\cosh^2 - 1. && (10)
\end{aligned}$$

Solve this recursion relation with the ansatz $\det M_N = a\, e^{Nn} + b\, e^{-Nn}$, to show that

$$\det M_N = \frac{\sinh(N+1)u}{\sinh u}. \tag{11}$$

(c) Now use the result of (b) to show that

$$iG(xt, x't') = \left(\frac{m}{2\pi i\, \Delta t}\right)^{1/2} e^{iS_{\mathrm{cl}}(xt,x't')}, \tag{12}$$

which is the final result.

Note that this way of solving the path integral relies on a semiclassical approximation, which turns out to be exact in this simple problem. In this approximation, one expands the path about the classical path, $x = x_{\mathrm{cl}} + \delta x$. Since the action is stationary for the classical path, the expansion of the action in δx has no linear term. Moreover, we already argued in the lecture that the vicinity of the classical path gives the dominant contribution. Expanding the action up to the quadratic term in δx, we recover a Gaussian integral which can be performed. Mathematically, this is known as a stationary phase approximation since the exponent becomes stationary for the classical path.

Problem 2: Path integral for the harmonic oscillator (25 points)

Use the same approach as in the previous problem to derive the path integral for the harmonic oscillator,

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2. \tag{13}$$

You should find the result $iG(xt, x'0) = Ae^{iS_{\mathrm{cl}}}$, with

$$A = \left(\frac{m\omega}{2\pi i \sin\omega t}\right)^{1/2}, \text{ and } S_{\mathrm{cl}} = \frac{m\omega}{2\sin\omega t}\left[\left(x^2 + x'^2\right)\cos\omega t - 2xx'\right]. \tag{14}$$

Problem 3: Path integrals for spin (5+5+5+5+5 points)

In this problem we want to derive a path integral representation for a spin-$\frac{1}{2}$ Hamiltonian, e.g.,

$$H = -g\boldsymbol{\sigma} \cdot \mathbf{B}(t), \tag{15}$$

where the components σ_i of $\boldsymbol{\sigma}$ denote the Pauli matrices. Such path integrals are useful to study the quantum mechanics of spin models such as the Heisenberg model.

(a) Consider the unit vector $\hat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ in the radial direction in spherical coordinates. Now, consider the spin-up eigenvectors for a Zeeman field along $\hat{\mathbf{n}}$ as defined by the eigenvalue equation

$$\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}|\hat{\mathbf{n}}\rangle = |\hat{\mathbf{n}}\rangle. \tag{16}$$

Show that when written in the usual spin basis with the quantization axis along the z axis, this is solved by the eigenvector

$$|\hat{\mathbf{n}}\rangle = \begin{pmatrix} e^{-i\phi}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}. \tag{17}$$

Note also that the eigenvalue equation defines $|\hat{\mathbf{n}}\rangle$ only up to an overall phase.

(b) Show that the $|\hat{\mathbf{n}}\rangle$ define an overcomplete set of states, i.e., that

$$\mathbf{1} = \int \frac{d^2\hat{\mathbf{n}}}{2\pi} |\hat{\mathbf{n}}\rangle\langle\hat{\mathbf{n}}|, \tag{18}$$

where $d^2\hat{\mathbf{n}} = \sin\theta\, d\theta\, d\phi$ denotes the solid angle corresponding to the unit vector $\hat{\mathbf{n}}$. You can show this by evaluating the integrals explicitly for the individual matrix elements in the usual spin basis. The set of states $\hat{\mathbf{n}}$ is *over*complete because is contains more than the minimum number of required basis states (which would be two for a spin-$\frac{1}{2}$ problem).

(c) Consider the propagator for $H = 0$,

$$i\, G(\hat{\mathbf{n}}t, \hat{\mathbf{n}}'t') = \langle\hat{\mathbf{n}}|\mathcal{U}(t,t')|\hat{\mathbf{n}}'\rangle, \tag{19}$$

with $\mathcal{U}(t,t') = \mathbf{1}$. Following the derivation of the path integral, insert resolutions of the identity in terms of the overcomplete sets $|\hat{\mathbf{n}}(t_j)\rangle$ at the $N-1$ equally spaced intermediate times $t_1, \ldots, t_{N-1}$ (also define $t_0 = t'$ and $t_N = t$). Show that

$$i\, G(\hat{\mathbf{n}}t, \hat{\mathbf{n}}'t') = \int \left[\frac{d^2\hat{\mathbf{n}}(\tau)}{2\pi}\right] e^{i\int_{t'}^{t} d\tau\, i\langle\hat{\mathbf{n}}(\tau)|\frac{d}{dt}|\hat{\mathbf{n}}(\tau)\rangle}. \tag{20}$$

Give explicit discrete versions of the action and the integration measure of this path integral. Note that remarkably, the action

$$S = i\int_{t'}^{t} d\tau\, i\langle\hat{\mathbf{n}}(\tau)|\frac{d}{d\tau}|\hat{\mathbf{n}}(\tau)\rangle = i\int_{t'}^{t} d\tau\, \frac{1}{2}\dot{\phi}(1+\cos\theta) \tag{21}$$

is non-zero even for a vanishing Hamiltonian! To arrive at this result, you may find the following little calculation for the overlap of the spin states at two neighboring times useful,

$$\begin{aligned} \langle\hat{\mathbf{n}}(t_j)|\hat{\mathbf{n}}(t_{j-1})\rangle &= 1 - \langle\hat{\mathbf{n}}(t_j)(|\hat{\mathbf{n}}(t_j)\rangle - |\hat{\mathbf{n}}(t_{j-1})\rangle) \\ &= \exp\{-\langle\hat{\mathbf{n}}(t_j)(|\hat{\mathbf{n}}(t_j)\rangle - |\hat{\mathbf{n}}(t_{j-1})\rangle)\}. \end{aligned} \tag{22}$$

Explain in which sense the second equality is valid.

(d) Show that the action generalizes to

$$S = \int_{t'}^{t} d\tau \left(i\langle\hat{\mathbf{n}}(\tau)|\frac{d}{d\tau}|\hat{\mathbf{n}}(\tau)\rangle + g\hat{\mathbf{n}}(\tau)\cdot\mathbf{B}(\tau) \right) \tag{23}$$

for $H = -g\boldsymbol{\sigma}\cdot\mathbf{B}(\tau)$. To arrive at this result, write the time evolution operator for this Hamiltonian as a time-ordered exponential (as usual for time-dependent Hamiltonians – see the derivation of linear-response theory in the lecture). You may want to use the identity (prove!)

$$\langle\hat{\mathbf{n}}|\boldsymbol{\sigma}|\hat{\mathbf{n}}\rangle = \hat{\mathbf{n}}. \tag{24}$$

(e) Show that Hamilton's principle for the action (23) correctly reproduces the classical equation of motion of a spin

$$\dot{\hat{\mathbf{n}}} = 2g\hat{\mathbf{n}}\times\mathbf{B}(t). \tag{25}$$

(You may find it useful to parametrize the unit vector $\hat{\mathbf{n}}$ in terms of polar and azimuthal angles as these can be varied independently.) Note that this equation of motion is quite different from the usual equations of motion in classical mechanics. It is a first-order differential equation in time and the force involves a vector product. (The latter also appears for particles in a magnetic field whose path-integral description actually has some similarities with the present problem.)

Remark: To understand the physics of the action for spin more deeply, you may want to learn (or remember) the concept of a Berry phase in quantum mechanics and recognize that the first term of the action (23) is just such a Berry phase.