Problem Set 11
Quantum Field Theory and Many Body Physics (SoSe2016)

Due: Monday, July 7, 2016 at the beginning of the lecture

In this problem set, we discuss response functions within the functional-integral approach to RPA, discuss the BCS gap equation at finite temperature, and work out some details of the microscopic derivation of Ginzburg-Landau theory.

Problem 1: Density-density response function in RPA (5 + 5 + 10 + 5 points)

In this problem, we discuss the density-density response function in the functional integral approach to RPA. Consider the action of the jellium model

$$S[J] = \int \mathrm{d}\tau \mathrm{d}\mathbf{r} \int \mathrm{d}\tau' \mathrm{d}\mathbf{r}' \left\{ \psi^*(\mathbf{r}\tau)\mathcal{G}_0(\mathbf{r}\tau;\mathbf{r}'\tau')\psi(\mathbf{r}'\tau') + \frac{1}{2}v(\mathbf{r}\tau;\mathbf{r}'\tau')n(\mathbf{r}\tau)n(\mathbf{r}'\tau') \right\} - \int \mathrm{d}\tau \mathrm{d}\mathbf{r} \mathrm{d}\tau n(\mathbf{r}\tau)J(\mathbf{r}\tau) \tag{1}$$

with $v(\mathbf{r}\tau;\mathbf{r}'\tau') = v(\mathbf{r}-\mathbf{r}')\delta(\tau-\tau')$. Here, $J(\mathbf{r}\tau)$ is a source field which allows us to derive correlators of the electron density (relative to the positive background) $n(\mathbf{r}\tau) = \psi^*(\mathbf{r}\tau)\psi(\mathbf{r}\tau) - n_0$ by means of functional derivatives of the grand potential $\Omega[J]$.

(a) Use the Kubo formula to identify the correct correlation function describing the (reducible) polarization operator

$$\rho(\mathbf{r}\tau) = -e^2 \int \mathrm{d}\tau' \mathrm{d}\mathbf{r}' \Pi(\mathbf{r}\tau;\mathbf{r}'\tau')\varphi(\mathbf{r}'\tau'), \tag{2}$$

where $\varphi(\mathbf{r}\tau)$ is an applied electric potential and $\rho(\mathbf{r}\tau) = en(\mathbf{r}\tau)$ is the induced charge density.

(b) Formulate the correct analytical continuation which yields the response function Π from a corresponding time-ordered correlation function. Give an expression for this correlation function as a functional derivative with respect to the source field J.

(c) Follow the functional integral approach to RPA, now including the additional source term in the action. You should decouple the interaction, integrate out the fermions, expand the resulting action to quadratic order in the Hubbard-Stratonovich field ϕ, and finally perform the integral over ϕ. Now use your result to obtain the response function Π in terms of the (irreducible) polarization operator Π_0 introduced in class,

$$\Pi = [\Pi_0{}^{-1} + v]^{-1}. \tag{3}$$

(d) Use the explicit expression for Π_0 in terms of Green functions, pass to momentum and frequency representation, and perform the Matsubara sum. Show that the resulting expression is the expression derived for the polarization operator Π_0 in an earlier problem set.

Problem 2: BCS gap equation at finite temperature (25 points)

In class, we derived the BCS gap equation

$$\Delta = g\frac{1}{V}\sum_{\mathbf{k}}\frac{1}{\beta}\sum_{\Omega}\frac{\Delta}{\Omega^2+\xi_k^2+\Delta^2}, \tag{4}$$

where Ω is a fermionic Matsubara frequency. Perform the sum over Ω and derive an (implicit) expression for the gap $\Delta(T)$ as function of temperature T as well as for the critical temperature T_c. Determine how does Δ vanishes at T_c. What is the order parameter exponent β?

Problem 3: Ginzburg-Landau theory (5+10+10 points)

As sketched in class, one can derive Ginzburg-Landau theory from the effective action which we obtained by decoupling the attractive interaction. In this problem, we fill in some of the details by computing the prefactor of the quadratic term of the Ginzburg-Landau functional. (You are welcome to extend this to include the derivative and quartic terms or even to include vector potentials if you like.)

The considerations start from the action which we obtained after integrating out the fermions,

$$S = \int \mathrm{d}\tau \mathrm{d}\mathbf{r} \frac{|\Delta|^2}{g} - \mathrm{tr} \ln\{1 + \mathcal{G}_0 \Delta \tau_+ + \mathcal{G}_0 \Delta^* \tau_-\} \tag{5}$$

for the order parameter field $\Delta(\mathbf{r}\tau)$.

(a) Explain the basic idea of deriving Ginzburg-Landau theory from this action, specifically why (and under which conditions) one can expand S in powers of Δ, why odd powers are absent, why one has to include the quartic term, why we can neglect the dependence of Δ on imaginary time, and why we can restrict attention to low-order derivatives (or equivalently slow spatial variations).

(b) Now expand the action to quadratic order in Δ and derive explicit expressions for the prefactor of both the $|\Delta|^2$ and the $|\nabla\Delta|^2$ terms, assuming that one can neglect the τ dependence of Δ.

(c) Use explicit expressions for $\mathcal{G}_0$ in frequency and momentum representation and perform the Matsubara and momentum sums to obtain an expression for the prefactor of the $|\Delta|^2$ term. You should find

$$\beta \nu_0 \frac{T - T_c}{T_c} |\Delta|^2. \tag{6}$$