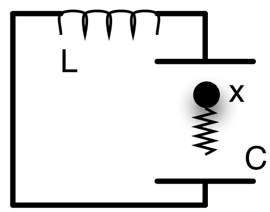
Problem Set 7
Quantum Field Theory and Many Body Physics (SoSe2015)

Due: Monday, June 1, 2015 with Yang Peng

In this problem set, we study examples of effective field theories.

Problem 1: Effective action of an LC circuit (25 points)

To illustrate the concept of an effective "field" theory, consider an LC circuit coupled to a harmonically bound charge e:



The charge with coordinate x has a Hamiltonian

$$H_d = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0 x^2 - eEx, \tag{1}$$

where E is the electric field of the capacitor. As familiar from elementary physics, the LC circuit is also a harmonic oscillator, as reflected in its energy

$$H_{\rm LC} = \frac{1}{2}I\dot{Q}^2 + \frac{Q^2}{2C}. \tag{2}$$

This can also be written in terms of the electric field $E = Q/Cd$ in the capacitor (d is the distance between the capacitor plates),

$$H_{\rm LC} = \frac{1}{2g}\left(\dot{E}^2 + \omega_{\rm LC}^2 E^2\right). \tag{3}$$

Here $\omega_{\rm LC} = 1/LC$ is the resonance frequency of the LC circuit and $g = 1/C^2Ld^2$. Thus, we can express the partition function of this system as

$$Z = \int [dE][dx] \, \exp\left[-\int_0^\beta d\tau \left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0 x^2 - eEx + \frac{1}{2g}\left(\dot{E}^2 + \omega_{\rm LC}^2 E^2\right)\right)\right]. \tag{4}$$

Now consider the limit $\omega_{\rm LC} \ll \omega_0$ and derive an effective action for the LC circuit by tracing out the charge coordinate x (i.e. perform the integral over $[dx]$). Show that to leading order in this limit, the effective action is again a harmonic-oscillator action with renormalized parameters. You should find that to leading order, only the frequency becomes renormalized and the "mass" prefactor of $\dot{E}^2$ remains unchanged.

Problem 2: Friction in quantum mechanics (5+5+5+5+5 points)

Friction is an important phenomenon in everyday life but cannot be described within a Hamiltonian language. This makes it difficult to describe at the quantum level. In this problem, we show that friction can be captured in quantum mechanics within the language of effective "field" theories.

First consider a classical particle subject to a frictional force $-\gamma\dot{x}$ and driving force $F(t)$. The equation of motion is

$$m\ddot{x} = -\gamma\dot{x} + F. \tag{5}$$

In frequency space we can define the response function χ through

$$x(\omega) = \chi(\omega) F(\omega), \tag{6}$$

and read off from the equation of motion that

$$\chi(\omega) = \frac{1}{-m\omega^2 - i\omega\gamma}. \tag{7}$$

Let us now reproduce this form a quantum perspective.
Consider a quantum particle subject to a uniform force F, coupled to an environment consisting of (many) harmonic oscillators,

$$H = \frac{1}{2} m\dot{x}^2 - Fx + \sum_i \left(\frac{1}{2}\mu_i \dot{x}_i^2 + \frac{1}{2}\mu_i \left(\omega_i x_i - g_i x\right)^2 \right). \tag{8}$$

The harmonic oscillators are meant to model the environmental degrees of freedom which dissipate the energy of the particle. Define the spectral density of the environmental oscillators

$$n(\omega) = \sum_i \delta(\omega - \omega_i), \tag{9}$$

and the coupling function

$$g^2(\omega) = \frac{1}{n(\omega)} \sum_i \mu_i g_i^2 \delta(\omega - \omega_i). \tag{10}$$

The system can also be described by the imaginary-time action

$$S = \int \mathrm{d}\tau \left(\frac{1}{2} m\dot{x}^2 - Fx + \sum_i \left(\frac{1}{2}\mu_i \dot{x}_i^2 + \frac{1}{2}\mu_i \left(\omega_i x_i - g_i x\right)^2 \right) \right). \tag{11}$$

(a) Integrate out the harmonic-oscillator environment to obtain the effective action for x (setting $F = 0$),

$$S_{\text{eff}} = \int \mathrm{d}\tau \, \frac{1}{2} m\dot{x}^2 + \int \mathrm{d}\tau \mathrm{d}\tau' \frac{1}{4} \mathcal{G}(\tau - \tau') \left(x(\tau) - x(\tau')\right)^2, \tag{12}$$

where $\mathcal{G}(\tau - \tau')$ is the (Matsubara) Fourier transform of

$$\mathcal{G}(i\omega) = \sum_i \frac{\mu_i \omega_i^2 g_i^2}{\omega^2 + \omega_i^2}. \tag{13}$$

(b) Use this result to show that the response function defined by

$$x(\tau) = \int d\tau' \, \chi(\tau - \tau') F(\tau') \tag{14}$$

is given by

$$\chi(\omega + i\eta) = \frac{1}{-m(\omega + i\eta)^2 - \left(\mathcal{G}(\omega + i\eta) - \mathcal{G}(0 + i\eta)\right)}. \tag{15}$$

By comparison with the classical response function, we can thus identify the friction coefficient

$$\gamma = \mathrm{Im} \frac{\mathcal{G}(\omega + i\eta) - \mathcal{G}(0 + i\eta)}{\omega}, \tag{16}$$

and the mass renormalization

$$m^* = m + \mathrm{Re} \frac{\mathcal{G}(\omega + i\eta) - \mathcal{G}(0 + i\eta)}{\omega^2}. \tag{17}$$

(c) Show that

$$\gamma = \frac{\pi}{2} n(|\omega|) g^2(|\omega|). \tag{18}$$

You might find the identity $\frac{1}{x-i\eta} = \mathcal{P}\left(\frac{1}{x}\right) + i\pi\delta(x)$ useful. Show also that

$$m^* = m + \mathcal{P}\sum_i \mu_i g_i^2 \frac{1}{\omega_i^2 - \omega^2}, \tag{19}$$

where $\mathcal{P}$ denotes the principal value.

(d) Choose $n(\omega)g^2(\omega) = n_0 g_0 \theta(\Omega - \omega)$, where Ω is an upper frequency (ultraviolet) cutoff of the phonon spectrum, and the step function is given by $\theta(x) = 1$ ($\theta(x) = 0$) for $x > 0$ ($x < 0$). Then

$$\gamma = \frac{\pi}{2} n_0 g_0^2. \tag{20}$$

Show that the associated mass renormalization is given by

$$m^* = m - \frac{n_0 g_0^2}{\Omega}. \tag{21}$$

(e) Now, assume $n(\omega)g^2(\omega) = n_0 g_0^2 e^{-\frac{\omega^2}{\Omega^2}}$, i.e., a phonon spectrum with a smooth cutoff. Show that

$$\mathcal{G}(\tau - \tau') = \int d\omega' \frac{\omega}{2} n(\omega) g^2(\omega) e^{-\omega|\tau|}, \tag{22}$$

and perform the frequency integral to obtain the popular effective action

$$S_{\text{eff}} = \int d\tau \, \frac{1}{2m}\dot{x}^2 + \int d\tau \, d\tau' \, \frac{\gamma \left(x(\tau) - x(\tau')\right)^2}{4\pi(\tau - \tau')^2}. \tag{23}$$

Problem 3: Landau theory of phase transitions (25 points)

For understanding the material of the lecture, it is useful to have at least a nodding acquaintance with the Landau theory of phase transitions. For full benefit, you should be fully conversant in this circle of ideas. This problem asks you to brush up your knowledge of Landau theory as a reading assignment which you should document by writing a concise summary. Your text should include the terms order parameter, free energy, first and second-order phase transitions, discrete and continuous symmetry, spontaneous symmetry breaking, critical exponents, correlation functions, and correlation length.

You find appropriate reading material in P. Chaikin & T. Lubensky, *Principles of Condensed Matter Physics*, Sec. 4.2. and 4.3. or Chapter 5 in N. Goldenfeld, *Lectures on phase transitions and critical phenomena.*