Problem Set 3
Quantum Field Theory and Many Body Physics (SoSe2015)

Due: Monday, May 4, 2015 before the beginning of the class

In this problem set, we fill in some of the missing steps in the quantum mechanics of the harmonic chain. We also learn how to compute path integrals explicitly. For the most part, the only path integrals that can be computed exactly are those for a free particle and for the harmonic oscillator, and it is these path integrals which we want to do. To do this, we exploit that the semiclassical calculation is exact in these cases.

Problem 1: Quantum mechanics of the harmonic chain (5+5+5+10 points)

Consider the quantized harmonic chain with Hamiltonian

$$H = \int \mathrm{d}r \left\{ \frac{1}{2\rho}\hat{\pi}^2(r) + \frac{1}{2}\rho c^2 (\partial_x \hat{\phi}(r))^2 \right\} \tag{1}$$

and fundamental commutation relations $[\hat{\pi}(x), \hat{\phi}(x')] = -i\hbar\delta(x-x')$ and $[\hat{\phi}(x), \hat{\phi}(x')] = [\hat{\pi}(x), \hat{\pi}(x')] = 0$. In this problem, we will use the operator approach as sketched in the lecture to compute the *quantum* expectation value

$$\langle \hat{\phi}_q \hat{\phi}_{-q} \rangle \tag{2}$$

at *finite temperature*. Remember that the quantum expectation value of an operator $\hat{O}$ at finite temperature T is given by

$$\langle \hat{O} \rangle = \frac{\mathrm{tr}\rho\hat{O}}{\mathrm{tr}\hat{\rho}}, \tag{3}$$

where the density operator is $\rho = \exp(-\beta H)$ and $\beta = 1/k_B T$.

We will also use this problem to perform some of the steps explicitly which were only sketched in the lecture.

(a) Compute the fundamental commutation relations of the Fourier components of the fields $\hat{\phi}(x)$ and $\hat{\pi}(x)$,

$$\begin{aligned}\hat{\phi}(x) &= \frac{1}{L}\sum_q \hat{\phi}_q e^{iqx} \\ \hat{\pi}(x) &= \frac{1}{L}\sum_q \hat{\pi}_q e^{iqx},\end{aligned}$$

namely show that

$$[\hat{\pi}_q, \hat{\phi}_{q'}] = -i\hbar L \delta_{q,-q'} \qquad [\hat{\phi}_q, \hat{\phi}_{q'}] = [\hat{\pi}_q, \hat{\pi}_{q'}] = 0. \tag{4}$$

Express the Hamiltonian in terms of the Fourier components $\hat{\phi}_q$ and $\hat{\pi}_q$, i.e., show that

$$H = \frac{1}{L}\sum_q \left[\frac{1}{2\rho}\hat{\pi}_q\hat{\pi}_{-q} + \frac{1}{2}\rho c^2 q^2 \hat{\phi}_q \hat{\phi}_{-q} \right]. \tag{5}$$

(b) Introduce creation and annihilation operators through

$$\hat{a}_q = \sqrt{\frac{\rho c|q|}{2L\hbar}}(\hat{\phi}_q + \frac{i}{\rho c|q|}\hat{\pi}_q), \tag{6}$$

compute their commutators $[\hat{a}_q, \hat{a}^\dagger_{q'}]$, $[\hat{a}_q, \hat{a}_{q'}]$, and $[\hat{a}^\dagger_q, \hat{a}^\dagger_{q'}]$, and express the Hamiltonian in terms of these operators.

(c) Explain how to derive the spectrum and the eigenstates using the creation and annihilation operators.

(d) Now use these results to find the result

$$\left\langle \hat{\phi}_q \hat{\phi}_{-q} \right\rangle = \frac{\hbar L}{2\rho c\,|q|} \coth\left(\frac{\beta \hbar c\,|q|}{2}\right) \tag{7}$$

and check that this reduces to the results derived in the lecture for $T \ll \hbar c|q|$ (quantum limit) and $T \gg \hbar c\,|q|$ (thermal limit).

Problem 2: Path integral for a free particle (10+10+5 points)

Calculate the path integral for a free particle in one dimension, i.e., compute the propagator $i\,G(xt, x't')$ for $H = p^2/2m$. When written in its discrete form, the path integral for this problem becomes a Gaussian integral which we can compute explicitly.

(a) We start from the configuration-space path integral and write the path as

$$x(\tau) = x_{\rm cl}(\tau) + \delta x(\tau). \tag{8}$$

Here,

$$x_{\rm cl}(\tau) = x' + \frac{x - x'}{t - t'}(\tau - t') \tag{9}$$

is the classical path. By Hamilton's principle, the classical path extremizes the action. Explain why this implies that

$$S = \int_{t'}^{t} d\tau \frac{1}{2} m \dot{x}^2 = S_{\rm cl} + \int_{t'}^{t} d\tau \frac{1}{2} m \delta\dot{x}^2 \tag{10}$$

with the classical action

$$S_{\rm cl} = \frac{1}{2} m \frac{(x - x')^2}{t - t'}. \tag{11}$$

Now, write the path integral in its discrete form and show that

$$iG(xt, x't') = \left(\frac{m}{2\pi i\,\Delta t}\right)^{N/2} \frac{(2\pi)^{(N-1)/2}}{\sqrt{\det[-iM]}} e^{iS_{\rm cl}}, \tag{12}$$

with $\Delta t = \frac{t-t'}{N}$, $S_{\rm cl} = \frac{m}{2}\frac{(x-x')^2}{t-t'}$, and M the $(N-1) \times (N-1)$ matrix

$$M = \frac{m}{\Delta t} \begin{pmatrix} 2 & -1 & 0 & \dots & & & \\ -1 & 2 & -1 & 0 & \dots & & \\ 0 & -1 & 2 & -1 & & & \\ \vdots & 0 & \ddots & \ddots & \ddots & & \\ & \vdots & & \ddots & & -1 \\ & & & & -1 & 2 \end{pmatrix}. \tag{13}$$

(b) We now need to compute $\det M$. Let us actually consider the slightly more general $N \times N$ matrix with matrix elements

$$(M_N)_{ij} = \begin{cases} 2\cosh u & i = j \\ -1 & i = j \pm 1\,. \\ 0 & \text{else} \end{cases} \tag{14}$$

First show that $\det M_N$ satisfies the recursion relation

$$\begin{aligned}
\det M_N &= 2\cosh u \, \det M_{N-1} - \det M_{N-2} && (15)\\
\det M_1 &= 2\cosh u && (16)\\
\det M_2 &= 4\cosh^2 - 1. && (17)
\end{aligned}$$

Solve this recursion relation with the ansatz $\det M_N = a\, e^{Nn} + b\, e^{-Nn}$, to show that

$$\det M_N = \frac{\sinh(N+1)u}{\sinh u}. \tag{18}$$

(c) Now use the result of (b) to show that

$$iG(xt, x't') = \left(\frac{m}{2\pi i\,\Delta t}\right)^{1/2} e^{iS_{\text{cl}}(xt,x't')}, \tag{19}$$

which is the final result.

Note that this way of solving the path integral relies on a semiclassical approximation, which turns out to be exact in this simple problem. In this approximation, one expands the path about the classical path, $x = x_{\text{cl}} + \delta x$. Since the action is stationary for the classical path, the expansion of the action in δx has no linear term. Moreover, we already argued in the lecture that the vicinity of the classical path gives the dominant contribution. Expanding the action up to the quadratic term in δx, we recover a Gaussian integral which can be performed. Mathematically, this is known as a stationary phase approximation since the exponent becomes stationary for the classical path.

Problem 3: Path integral for the harmonic oscillator (25 points)

Use the same approach as in the previous problem to derive the path integral for the harmonic oscillator,

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2. \tag{20}$$

You should find the result $iG(xt, x'0) = Ae^{iS_{\text{cl}}}$, with

$$A = \left(\frac{m\omega}{2\pi i \sin\omega t}\right)^{1/2}, \text{ and } S_{\text{cl}} = \frac{m\omega}{2\sin\omega t}\left[\left(x^2 + x'^2\right)\cos\omega t - 2xx'\right]. \tag{21}$$