Problem Set 2
Quantum Field Theory and Many Body Physics (SoSe2015)

Due: Monday, April 27, 2015 before the beginning of the class

In this problem set, we continue our discussion of Gaussian integrals. We compute an integral explicitly which we encountered and used in the lecture. Here, the main challenge is to compute the inverse of a large matrix. We also extend our considerations to Gaussian integrals over complex variables. This is important because many field theories involve complex-valued fields. Finally, we introduce the concept of generating functionals of moments and cumulants which are widely used in quantum field theory.

Problem 1: A Gaussian integral (10+10 points)

In the course of discussing the thermodynamics of the harmonic chain, we encountered the Gaussian integral

$$\langle \phi_i \phi_j \rangle = \frac{\int [\mathrm{d}\boldsymbol{\phi}] \phi_i \phi_j \exp\left\{-\frac{1}{2}\beta D \sum_{j=1}^{N} (\phi_{j+1} - \phi_j)^2\right\}}{\int [\mathrm{d}\boldsymbol{\phi}] \exp\left\{-\frac{1}{2}\beta D \sum_{j=1}^{N} (\phi_{j+1} - \phi_j)^2\right\}} \tag{1}$$

where ϕ_j denotes the (real) displacement of the jth atom of the chain. In this problem, we want to perform the explicit calculation which yields the results quoted in the lecture, also in order to learn how to deal with Gaussian integrals in practice.

(a) Rewrite the exponent in the integrands in matrix notation and show that

$$\beta D \sum_{j=1}^{N} (\phi_{j+1} - \phi_j)^2 = \boldsymbol{\phi}^T \mathrm{M} \boldsymbol{\phi} \tag{2}$$

where

$$\mathrm{M} = \beta D \begin{bmatrix} 2 & -1 & 0 & 0 & \dots & -1 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & & -1 & 2 & -1 \\ -1 & 0 & \dots & 0 & -1 & 2 \end{bmatrix} \tag{3}$$

(b) Now we know from the general expression for Gaussian integrals that

$$\langle \phi_i \phi_j \rangle = [M^{-1}]_{ij}, \tag{4}$$

so that we need to compute the inverse of the matrix M. This can be done by noting that M describes a system on a ring which is translationally invariant. Thus, it is natural to suspect that the eigenvectors $\boldsymbol{\phi}_k$ are plane-wave like, with

$$[\phi_k]_j = \frac{1}{\sqrt{N}} e^{ikja}. \tag{5}$$

These eigenvectors are labeled by momentum k. Show that the periodic boundary conditions imply that k takes on the values $k = (2\pi/a)(n/N)$ with $n \in \mathbf{Z}$ and n restricted to the first Brillouin zone. Show that this corresponds to normalized vectors $\boldsymbol{\phi}$ and that these are indeed eigenvectors of M with

$$\mathrm{M}\boldsymbol{\phi} = 4\beta D \sin^2 \frac{ka}{2} \boldsymbol{\phi}. \tag{6}$$

Thus, we can write the matrix M as

$$\mathrm{M} = 4\beta D \sum_k |k\rangle \sin^2 \frac{ka}{2} \langle k|, \tag{7}$$

where we use bra-ket notation for the eigenvectors for notational simplicity. Finally, explain why this yields

$$\mathrm{M}^{-1} = \sum_k |k\rangle \frac{1}{4\beta D \sin^2 \frac{ka}{2}} \langle k| \tag{8}$$

and

$$[\mathrm{M}^{-1}]_{jl} = \frac{1}{N} \sum_k \frac{e^{ika(j-l)}}{4\beta D \sin^2 \frac{ka}{2}}. \tag{9}$$

This gives the result

$$\langle \phi_j \phi_l \rangle = \frac{1}{N} \sum_k \frac{e^{ika(j-l)}}{4\beta D \sin^2 \frac{ka}{2}} \tag{10}$$

which we used in the lecture.

Problem 2: Complex Gaussian integrals (10+5+5+5 points)

In problem set 1, we derived the Gaussian integral

$$\int \prod_{n=1}^{N} \mathrm{d}\phi_n \exp\left\{-\frac{1}{2}\boldsymbol{\phi}^T \mathrm{M} \boldsymbol{\phi} + \mathbf{j}^T \boldsymbol{\phi}\right\} = \frac{(2\pi)^{N/2}}{(\det \mathrm{M})^{1/2}} \exp\left\{\frac{1}{2}\mathbf{j}^T \mathrm{M}^{-1} \mathbf{j}\right\} \tag{11}$$

for a positive definite, real and symmetric $N \times N$ matrix M. In this problem, we want to consider integrals over complex variables $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_N)$. Here, you should not think of countour integrals in the complex plane! Instead, these integrals are simply independent integrals over the real and imaginary part of the ϕ_j. One usually writes the integration measure in terms of ϕ_j and its complex conjugate ϕ_j^*. This should be interpreted as

$$\mathrm{d}\phi_j \mathrm{d}\phi_j^* = 2\, \mathrm{dRe}\phi_j \, \mathrm{dIm}\phi_j. \tag{12}$$

This definition is motivated by formally computing the Jacobian for passing from ϕ_j and ϕ_j^* to $\mathrm{Re}\phi_j$ and $\mathrm{Im}\phi_j^*$ as integration variables. Using $\phi_j = \mathrm{Re}\phi_j + i\mathrm{Im}\phi_j$ and $\phi_j^* = \mathrm{Re}\phi_j - i\mathrm{Im}\phi_j$, this Jacobian is

$$\left| \frac{\partial(\phi_n, \phi_n^*)}{\partial(\mathrm{Re}\phi_n, \mathrm{Im}\phi_n)} \right| = 2. \tag{13}$$

We will also use the shorthand notation $[\mathrm{d}\boldsymbol{\phi}][\mathrm{d}\boldsymbol{\phi}^*] = \mathrm{d}\phi_1 \mathrm{d}\phi_1^* \ldots \mathrm{d}\phi_N \mathrm{d}\phi_N^*$ for the integration measure.

(a) Now show that the complex Gaussian integral gives

$$\int [\mathrm{d}\boldsymbol{\phi}][\mathrm{d}\boldsymbol{\phi}^*] \exp\left\{-\boldsymbol{\phi}^\dagger \mathbf{M} \boldsymbol{\phi} + \mathbf{J}^\dagger \boldsymbol{\phi} + \boldsymbol{\phi}^\dagger \mathbf{J}\right\} = \frac{(2\pi)^N}{\det \mathbf{M}} \exp\left\{\mathbf{J}^\dagger \mathbf{M}^{-1} \mathbf{J}\right\} \tag{14}$$

This is very similar to the result of the Gaussian integral over real variables except that the prefactor on the right hand side does not involve a square root. This is just a consequence of the fact that we are integrating over twice as many variables, namely N real and N imaginary parts.

Finally, we need to comment on the matrix M. Since M enters the integrand as a quadratic form and the integrand should be real, the matrix M must be Hermitian. Then, M can be diagonalized by a unitary transformation with all eigenvalues being real. Clearly, these eigenvalues need to be strictly positive for the integral to be convergent.

(b) We can again define averages over the complex fields $\boldsymbol{\phi}$ through

$$\langle \ldots \rangle = \frac{\int [\mathrm{d}\boldsymbol{\phi}][\mathrm{d}\boldsymbol{\phi}^*] \ldots \exp\left\{-\boldsymbol{\phi}^\dagger \mathrm{M} \boldsymbol{\phi}\right\}}{\int [\mathrm{d}\boldsymbol{\phi}][\mathrm{d}\boldsymbol{\phi}^*] \exp\left\{-\boldsymbol{\phi}^\dagger \mathrm{M} \boldsymbol{\phi}\right\}}. \tag{15}$$

Use the result of (a) to show that

$$\langle \phi_i^* \phi_j \rangle = [M^{-1}]_{ij}. \tag{16}$$

What would you get if you were to compute $\langle \phi_i \phi_j \rangle$ or $\langle \phi_i^* \phi_j^* \rangle$?

(c) Now use the results of (a) and (b) to formulate Wick's theorem for complex Gaussian integrals, i.e., explain how to express any average of the form

$$\left\langle \phi_{i_1}^* \ldots \phi_{i_n}^* \phi_{j_1} \ldots \phi_{j_n} \right\rangle \tag{17}$$

in terms of the $\langle \phi_i^* \phi_j \rangle$. Also explain what happens for averages with different numbers of ϕ_j and ϕ_j^*.

Problem 3: Cumulant expansion and generating functionals (10+10+5+5 points)

In this problem, we want to discuss some basics of generating functions for probability distributions and generalize this concept to field theories. Generating functions are a standard tool in probability theory. Consider a random variable x with probability distribution $P(x)$ and denote the corresponding averages by $\langle \ldots \rangle$. Then, the moment generating function

$$\mathcal{G}(J) = \langle e^{Jf(x)} \rangle \tag{18}$$

succinctly summarizes all moments $\langle [f(x)]^n \rangle$ $(n = 0, 1, 2, \ldots)$ of some function $f(x)$. Indeed,

$$\langle [f(x)]^n \rangle = \left. \frac{\mathrm{d}^n}{\mathrm{d}J^n} \mathcal{G}(J) \right|_{J=0} \tag{19}$$

or

$$\mathcal{G}(J) = \sum_{n=0}^{\infty} \frac{J^n}{n!} \langle [f(x)]^n \rangle. \tag{20}$$

For the special case of $f(x) = x$, the moments are just the averages $\langle x^n \rangle$.

Instead of the moments, it is often useful to characterize the probability distribution through its cumulants. Examples are the average $C_1 = \langle f(x) \rangle$, which is the first cumulant, or the variance $C_2 = \langle [f(x)]^2 \rangle - \langle f(x) \rangle^2$, which is the second cumulant. It turns out that the entire series of cumulants C_n is generated by the cumulant generating functional

$$\mathcal{W}(J) = \ln \mathcal{G}(J) = \ln \langle e^{Jf(x)} \rangle \tag{21}$$

through

$$C_n = \left. \frac{\mathrm{d}^n}{\mathrm{d}J^n} \mathcal{W}(J) \right|_{J=0} \tag{22}$$

and

$$\mathcal{W}(J) = \sum_{n=1}^{\infty} \frac{J^n}{n!} C_n. \tag{23}$$

(a) Give explicit expressions for the first four cumulants C_1, C_2, C_3, C_4 in terms of the moments of $f(x)$. The third cumulant is known as skewness, the fourth as kurtosis.

(b) Compute all cumulants of x for a Gauss distribution

$$P(x) = \frac{1}{\sqrt{2\pi\sigma}}\,\mathrm{e}^{-\frac{x^2}{2\sigma}}. \tag{24}$$

and all cumulants of n ($n = 1, 2, 3, \ldots$) for the Poisson distribution

$$P(n) = \tfrac{1}{n!}\,\mathrm{e}^{-\lambda}\,\lambda^n, \tag{25}$$

where $n = 0, 1, 2, \ldots$. You should do this by explicitly computing the cumulant generating functions for these distributions.

(c) Now consider a multivariate complex Gaussian distribution as introduced above in problem 2,

$$P[\boldsymbol{\phi}] = \frac{\exp\left\{-\boldsymbol{\phi}^\dagger \mathrm{M} \boldsymbol{\phi}\right\}}{\int [\mathrm{d}\boldsymbol{\phi}][\mathrm{d}\boldsymbol{\phi}^*] \exp\left\{-\boldsymbol{\phi}^\dagger \mathrm{M} \boldsymbol{\phi}\right\}} \tag{26}$$

and introduce the moment generating functional

$$\mathcal{G}[\mathbf{J}] = \langle \exp\{\mathbf{J}^\dagger \boldsymbol{\phi} + \boldsymbol{\phi}^\dagger \mathbf{J}\}\rangle \tag{27}$$

as well as the cumulant generating functional

$$\mathcal{W}[\mathbf{J}] = \ln\langle \exp\{\mathbf{J}^\dagger \boldsymbol{\phi} + \boldsymbol{\phi}^\dagger \mathbf{J}\}\rangle. \tag{28}$$

Compute both of these generating functions explicitly (i.e., perform the average). Use this to compute the second moment $\langle \phi_i^* \phi_j \rangle$ and the second cumulant $\langle \phi_i^* \phi_j \rangle - \langle \phi_i^* \rangle \langle \phi_j \rangle$.

(d) Now consider the slightly modified multivariate complex Gaussian distribution,

$$P[\boldsymbol{\phi}] = \frac{\exp\left\{-(\boldsymbol{\phi} - \boldsymbol{\phi}_0)^\dagger \mathrm{M} (\boldsymbol{\phi} - \boldsymbol{\phi}_{\mathbf{0}})\right\}}{\int [\mathrm{d}\boldsymbol{\phi}][\mathrm{d}\boldsymbol{\phi}^*] \exp\left\{-(\boldsymbol{\phi} - \boldsymbol{\phi}_0)^\dagger \mathrm{M} (\boldsymbol{\phi} - \boldsymbol{\phi}_0)\right\}} \tag{29}$$

with some fixed $\boldsymbol{\phi}_0$. Compute the generating functions as defined in (c). Use your result to obtain the first and second moments and cumulants. (The first average and cumulant are just the average $\langle \phi_i \rangle$).