Problem Set 13
Quantum Field Theory and Many Body Physics (SoSe2015)

Due: Problem set as preparation for class exam

This problem set discusses some aspects of R. Shankar, *Renormalization group approach to interacting fermions*, Rev. Mod. Phys. **66**, 129 (1994). We will focus in particular on Sec. IV. All problems should be solved for zero temperature.

Problem 1: Polarization operator of (non-interacting) 1d electron systems

Consider a one-dimensional system of electrons with Hamiltonian

$$H = \int dx \psi^\dagger(x) \left(-\frac{\nabla^2}{2m} \right) \psi(x). \tag{1}$$

We want to study the density response of the system to external perturbations.

(a) Use linear-response theory and exploit the relations between the various correlation functions to compute its polarization operator defined through

$$\delta n(x,t) = -\int \mathrm{d}x' \mathrm{d}t' \Pi(x,t;x',t') e\phi(x',t') \tag{2}$$

in momentum and frequency representation. Here, $\delta n(x,t)$ is the change in density relative to the ground state and $\phi(x,t)$ is an applied scalar potential. You should find

$$\Pi(q,\omega) = \frac{m}{2\pi q} \left\{ \ln \frac{\omega + i\eta + q^2/2m + qk_F/m}{\omega + i\eta - q^2/2m + qk_F/m} - \ln \frac{\omega + i\eta + q^2/2m - qk_F/m}{\omega + i\eta - q^2/2m - qk_F/m} \right\}. \tag{3}$$

(b) Specify to the static limit and consider what happens to it in the limits of $q \to 0$ and $q \to 2k_F$. Show that the polarization operator becomes equal to the density of states in the former case and diverges logarithmically at $2k_F$.

The latter divergence suggests that a one-dimensional interacting system might want to spontaneously form density modulations with wave vector $2k_F$. Such a state with spontaneous density modulations is known as a charge density wave.

Problem 2: Mean-field approach to 1d charge density waves

We have already seen in a previous problem set that the Jordan-Wigner transformation maps the XXZ chain to a spinless fermion system with nearest-neighbor interactions at half filling. We will now study this interacting fermion problem in more detail. Specifically, we will be interested in whether this system exhibits charge density wave order. The Hamiltonian is

$$H = \sum_j \left\{ -t[\psi^\dagger_{j+1}\psi_j + \psi^\dagger_j \psi_{j+1}] + U_0 (n_j - \frac{1}{2})(n_{j+1} - \frac{1}{2}) \right\}. \tag{4}$$

Here, $n_j = \psi^\dagger_j \psi_j$ is the occupation number of site j and the interaction $U_0 > 0$ is assumed to be repulsive. We will also assume a vanishing chemical potential, $\mu = 0$, which puts the system at half filling.

(a) Briefly describe the ground states in the limits of vanishing and infinitely strong repulsive interaction.

(b) Consider the mean-field theory for a charge-density wave. Make the ansatz

$$n_j = \langle n_j \rangle + (n_j - \langle n_j \rangle) \tag{5}$$

with the average

$$\langle n_j \rangle = \frac{1}{2} + \frac{1}{2}(-1)^j \Delta. \tag{6}$$

and neglect terms in the Hamiltonian which are quadratic in the fluctuations about the average. Show explicitly that this leads to the gap equation

$$\Delta = \int_{-\pi/2}^{\pi/2} \frac{\mathrm{d}k}{2\pi} \frac{2U_0\Delta}{\sqrt{4t^2\cos^2 k + \Delta^2 U_0^2}}. \tag{7}$$

(b) Obtain the nontrivial solution of the gap equation explicitly by linearizing the spectrum in the vicinity of the Fermi points.

(c) What kind of order would the charge density wave correspond to in the XXZ model?

Problem 3: Renormalization group treatment

In the previous problem, we found that in mean-field theory the interacting fermion chain exhibits charge density wave order at arbitrarily weak interactions. We will now address this problem by means of a simple RG treatment at weak interactions. We will make several approximations to simplify the problem. Justifications and further discussions for all these approximations can be found in the paper mentioned above.

(a) Rewrite the nearest neighbor interaction

$$H_I = U_0 \sum_j \psi_{j+1}^\dagger \psi_j^\dagger \psi_j \psi_{j+1} \tag{8}$$

in momentum space and show that after antisymmetrization, this yields

$$H_I = -U_0 \int \frac{\mathrm{d}k_1}{2\pi}\frac{\mathrm{d}k_2}{2\pi}\frac{\mathrm{d}k_3}{2\pi}\frac{\mathrm{d}k_4}{2\pi} 2\pi\delta(k_1+k_2-k_3-k_4) \sin\frac{k_1-k_2}{2}\sin\frac{k_3-k_4}{2} c_{k_4}^\dagger c_{k_3}^\dagger c_{k_2} c_{k_1}. \tag{9}$$

Here, we already used that umklapp processes in which momentum is conserved only modulo reciprocal lattice vectors can be neglected.

(b) Now explain why at low energies, we can decompose the Fermi field into right- and left movers, defined through

$$L_k = c_{k-\pi/2} \qquad R_k = c_{k+\pi/2}, \tag{10}$$

where k is measured from the Fermi points and is assumed small compared to π. Use this decomposition, exploit momentum conservation, and drop all terms which involve derivatives of the fields (i.e., factors of the k's measured from the Fermi points) to obtain the simplified interaction

$$\begin{aligned} H_I &= U_0 \int \frac{\mathrm{d}k_1}{2\pi}\frac{\mathrm{d}k_2}{2\pi}\frac{\mathrm{d}k_3}{2\pi}\frac{\mathrm{d}k_4}{2\pi} 2\pi\delta(k_1+k_2-k_3-k_4) \\ &\quad \times \left\{ R_{k_4}^\dagger L_{k_3}^\dagger L_{k_2} R_{k_1} + L_{k_4}^\dagger R_{k_3}^\dagger R_{k_2} L_{k_1} - R_{k_4}^\dagger L_{k_3}^\dagger R_{k_2} L_{k_1} - L_{k_4}^\dagger R_{k_3}^\dagger L_{k_2} R_{k_1} \right\}. \end{aligned} \tag{11}$$

Each term can be represented by an interaction vertex as discussed in the lecture, see Fig. 1(a) for the second term as an example. Explain how this representatin is related to that used in Shankar's paper.

(c) Now consider an momentum-space RG procedure in which we integrate out a slice of momentum space and track the change in the coupling constant. (We will not consider the rescaling step of the RG procedure here.) Renrmalizations of the interaction constant begin in second order in the interaction. Using the representation given above, draw all diagrams to this order which renormalize the second term in Eq. (11). An example term is shown in Fig. 1(b). Show explicitly that the sum of these diagrams vanishes. Note that in evaluating these diagrams, you are allowed to put all external lines right at the relevant Fermi

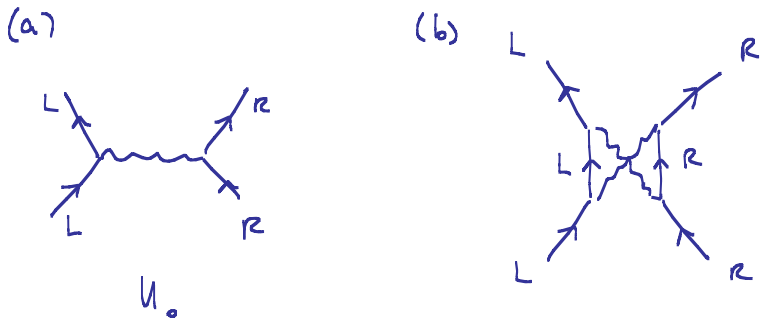

Figure 1: Diagrammatic representation of (a) the effective interaction and (b) a representative contribution to the interaction renormalization to second order in the coupling.

point, i.e., there are no external momenta or energies to consider as we neglect the momentum (as well as frequency) dependence of the interaction. Note also that you neither need the true prefactor of the diagram nor a complete evaluation of the integrals.

The last result shows that the free Hamiltonian is actually not unstable against charge-density wave formation. The cancellation is actually between diagams describing charge-density wave and superconducting instabilities. Mean-field theory retains only one of them and thus predicts an instability.