**Problem Set 11**
**Quantum Field Theory and Many Body Physics (SoSe2015)**

Due: Monday, June 29, 2015 at the beginning of the lecture

This problem set

**Problem 1: Grassmann basics** (5+10+10 points)

Read your favorite book (e.g., Negele, Orland) to formulate and prove the following statements:

(a) Linear changes of variables in Grassmann integrals.

(b) Use (a) to prove the most general Gaussian integral for Grassmann variables as given in the class.

(c) Prove the resolution of the identity for fermionic coherent states.

**Problem 2: Resonant-level model** (10+5+5+5 points)

There are many physical situations in which a localized fermionic level is coupled to a (non-interacting) fermionic many-body system with a continuum of states. For instance, consider an atom placed on a metallic substrate. The atom has a large level spacing so that it can be a good approximation to consider only the atomic level which is closest to the Fermi energy of the substrate. We may then be interested in how the atomic level is influenced by the presence of the surface. Another situation where this model is relevant is a quantum dot coupled to two electronic electrodes. If the quantum dot is sufficiently small, its spectrum will also be discrete with large level spacing so that we can restrict attention to the level which is closest to the Fermi energy in the electrodes.

As it is non-interacting, this problem can of course be solved by elementary means. In this problem, we want to treat it by deriving an effective action for the localized level by integrating out the continuum field. The Hamiltonian of the system takes the form

$$H = \epsilon_d d^\dagger d + \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \psi^\dagger_{\mathbf{k}} \psi_{\mathbf{k}} + \frac{t}{\sqrt{V}} \sum_{\mathbf{k}} (\psi^\dagger_{\mathbf{k}} d + d^\dagger \psi_{\mathbf{k}}). \tag{1}$$

The first term accounts for the localized fermionic level with energy $\epsilon_d$, the second term of the fermionic continuum with dispersion $\epsilon_{\mathbf{k}}$ and volume $V$, and the last term allows the fermions to hop between the localized level and the continuum. Note that this hopping is local at the position of the localized level (taken to be at the origin) as reflected in the fact that the hopping amplitudes $t$ (taken as real) are assumed independent of momentum. We need not be very specific about the dispersion $\epsilon_{\mathbf{k}}$ of the continuum. We will simply assume that the continuum has a constant density of states $\nu_0$ and a with bandwidth $-D < \epsilon_{\mathbf{k}} < D$, where $D$ is some large energy. (This is sometimes refered to *wide-band limit.*)

(a) Write down the Grassmann functional integral for the partition function of this model. Integrate out the field $\psi_{\mathbf{k}}$ of the fermionic continuum and show that the effective action for the localized level becomes

$$S = d^*(\partial_\tau + \epsilon_d - \mu + \Sigma)d \tag{2}$$

in compact matrix notation or

$$S = \int \mathrm{d}\tau d^*(\tau)(\partial_\tau + \epsilon_d - \mu)d(\tau) + \int \mathrm{d}\tau \mathrm{d}\tau' d^*(\tau)\Sigma(\tau,\tau')d(\tau') \tag{3}$$

when keeping the explicit time integrals. The self energy in this action is found to be

$$\Sigma(\tau,\tau') = -\frac{1}{V}\sum_{\mathbf{k}} \langle\tau| \frac{t^2}{\partial_\tau + \epsilon_{\mathbf{k}} - \mu} |\tau'\rangle. \tag{4}$$

(b) Show that in Matsubara-frequency representation, the self-energy becomes

$$\Sigma(i\omega_n) = -\frac{1}{V}\sum_{\mathbf{k}} \mathcal{G}(\mathbf{k}, i\omega_n) \tag{5}$$

with the Green function

$$\mathcal{G}(\mathbf{k}, i\omega_n) = \frac{-1}{i\omega_n - \epsilon_{\mathbf{k}} + \mu} \tag{6}$$

of the fermionic continuum. To evaluate the self energy explicitly, replace the sum over momenta by an integral in the usual way,

$$\frac{1}{V}\sum_{\mathbf{k}} \to \nu_0 \int_{-D}^{D} d\epsilon_{\mathbf{k}} \tag{7}$$

and show that to leading order in $\mu/D$ and $\omega_n/D$ (i.e., assuming that the bandwidth $D$ is large)

$$\Sigma(i\omega_n) = -\nu_0 t^2 \ln(1 - 2\mu/D) - i\pi\nu_0 t^2 \mathrm{sgn}\omega_n. \tag{8}$$

The real part of the self energy can be interpreted as a small shift in the energy of the localized level. Note that it approaches zero as $D \to \infty$. Let us take this limit in the following and retain only the imaginary part of the self energy in the following.

(c) Now use the resulting effective action to obtain the thermal Green function $\mathcal{G}(i\omega_n)$ of the localized level and show that it becomes

$$\mathcal{G}(i\omega_n) = \frac{-1}{i\omega_n - \epsilon_d + \mu + i\pi\nu_0 t^2 \mathrm{sgn}\omega_n}. \tag{9}$$

Show that the corresponding spectral function is

$$\rho(\omega) = \frac{\Gamma/2\pi}{(\omega - \epsilon_d)^2 + (\Gamma/2)^2}, \tag{10}$$

where we introduced $\Gamma = 2\pi t^2 \nu_0$, i.e.,

$$\mathcal{G}(i\omega_n) = -\int d\omega' \frac{\rho(\omega')}{i\omega_n - \omega'}. \tag{11}$$

Thus, we see that the imaginary part of the self energy broadens the delta-like spectral function of the uncoupled localized level into a Lorentzian, with the broadening given by $\Gamma/2$. Remembering Fermi's golden rule, interpret the explicit expression for $\Gamma$.

(d) To further interpret the imaginary part of the self energy, perform the appropriate analytical continuation to obtain the retarded Green function and Fourier transform your result to real time. How does the broadening $\Gamma$ enter into the real-time retarded Green function?

**Problem 3: Effective action for interacting species of fermions** (10+5+10 points)

Consider the partition function of two species of fermions, labeled $a$ and $b$, that interact via a local, spatially-uniform, and repulsive density-density interaction. We assume that particles of type $a$ interact with particles of type $b$ and vice versa, but particles do not interact with other particles of their own kind. This is described by the action

$$S = \int_0^\beta d\tau \int d\mathbf{r} \left[ \sum_{j=a,b} \psi_j^*(\mathbf{r}) G_0^{-1} \psi_j(\mathbf{r}) + v n_a(\mathbf{r}) n_b(\mathbf{r}) \right] \tag{12}$$

where $G_0^{-1} = \partial_\tau + \mathbf{p}^2/2m - \mu$, $v > 0$ is the strength of the repulsive local interaction, $n_j(\mathbf{r}) = \psi_j^*(\mathbf{r})\psi_j(\mathbf{r}) - n_0$, and $n_0$ is a constant equal to the density of either species of particles.

(a) Express the partition function (normalized to the partition function of the non-interacting system) in terms of a functional integral and integrate out the $\psi_b$ field. Express your result explicitly in terms of determinants.

(b) Derive an effective action for the fermions of type $a$ by expansion of the result in (a) in powers of $v$. Show that the first order term in the expansion of the effective action is cancelled by the background term. *Hint:* Use the fact that $G_0(\mathbf{r},\tau;\mathbf{r},\tau) = -n_0$ (prove that!), due to charge neutrality.

(c) Show that to quadratic order in $v$, you obtain an effective interaction between particles of type $a$. Show that this interaction is governed by the polarization operator $\Pi$ of the particles of type $b$, i.e., that the effective action of the $a$ fermions to quadratic order in $v$ is given by

$$S = \int \mathrm{d}\tau \mathrm{d}\mathbf{r}\, \psi_a^*(\mathbf{r}) G_0^{-1} \psi_a(\mathbf{r}) + \int \mathrm{d}\tau \mathrm{d}\tau' \int \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{r}' n_a(\mathbf{r},\tau) v^2 \Pi(\mathbf{r},\tau;\mathbf{r}'\tau') n_a(\mathbf{r}',\tau'). \tag{13}$$

Thus, even though there is no bare interaction between particles of the same kind, the presence of the other kind of particles effectively mediates an interaction.