Problem Set 10
Quantum Field Theory and Many Body Physics (SoSe2015)

Due: Monday, June 22, 2015 at the beginning of the lecture

This problem set attempts to walk you through the field-theoretic (momentum-space) renormalization-group treatment of the Kosterlitz-Thouless transition. In the lecture, we saw that the low-temperature partition function of the 2-dimensional Bose liquid factorizes into a phonon and a vortex contribution, $Z = Z_0 Z_V$. We then mapped the vortex part Z_V to a Coulomb-gas Hamiltonian with partition function

$$Z_V = \sum_{n=0} \frac{1}{n!n!} \int \prod_{j=1}^{2n} \mathrm{d}\mathbf{r}_j \exp\left\{-\beta\left[2nE_c - 2\pi\eta \sum_{\alpha<\beta} k_\alpha k_\beta \ln\frac{|\mathbf{r}_\alpha - \mathbf{r}_\beta|}{a}\right]\right\} \tag{1}$$

and analyzed this partition function by a real-space renormalization procedure. The first problem finds yet another representation of this problem which is known as sine-Gordon model. This model lends itself much more readily to a momentum-space RG procedure than the Coulomb-gas model. This momentum space renormalization is then developed in the second problem, following the presentation in Wen's book. The third problem looks more closely at one integral that is used in Wen's treatment and shows by contour integration that this integral actually vanishes. This problem was discussed in a publication by Nozieres and Gallet, where one can also find a more accurate treatment which resolves the issue.

Problem 1: Coulomb gas and sine-Gordon model (5+10+10 points)

As a first step, we now show that the sine-Gordon model is equivalent to the phase model for the superfluid (also refered to as XY-model). We will show this by proving that the two models have the same partition function. The sine-Gordon model is defined by the action

$$S = \int d\mathbf{r} \left\{\frac{\kappa}{2}(\nabla\theta)^2 - g\cos\theta\right\}. \tag{2}$$

Note that for $g = 0$, this is just the action for the phase field of the superfluid. However, the field θ of the sine-Gordon model takes values in $\mathbf{R}$ and is *not* an angle degree of freedom! Thus, we do not need to consider vortex configurations of the θ-field. The $\cos\theta$-term explicitly breaks the $U(1)$ symmetry of the superfluid model. For large g, θ gets pinned near integer multiples of 2π.

(a) Consider the partition function of the sine-Gordon model in $d = 2$,

$$Z = \int [d\theta]\, e^{-S} = \int [d\theta]\, \exp\{-\int d\mathbf{r}\frac{\kappa}{2}(\nabla\theta)^2 + \int d\mathbf{r}\, g\cos\theta\} \tag{3}$$

and expand in powers of g to all orders. Show that the partition function can be rewritten as

$$Z = Z_0 \sum_{n=0}^{\infty} \frac{g^{2n}}{4^n n!n!} \int d\mathbf{r}_1\, d\mathbf{r}_2\, \ldots d\mathbf{r}_{2n}\, \langle e^{i\theta(r_1)} e^{i\theta(r_2)} \ldots e^{i\theta(r_n)} e^{-i\theta(r_{n+1})} \ldots e^{-i\theta(r_{2n})}\rangle_0, \tag{4}$$

where the average is taken with respect to the $g = 0$ action,

$$\langle \ldots \rangle_0 = \frac{1}{Z_0} \int [d\theta] \ldots e^{-\int d\mathbf{r}\frac{\kappa}{2}(\nabla\theta)^2}. \tag{5}$$

Note that this unperturbed action is just the action of the superfluid phase field (without vortex excitations), which we already studied in some detail.

Hint: Remember that $\langle e^{i\theta(\mathbf{r})}\rangle = 0$ (why?) and more generally

$$\int d\mathbf{r}_1\, d\mathbf{r}_2 \ldots d\mathbf{r}_{k+l}\, \langle e^{i\theta(r_1)} e^{i\theta(r_2)} \ldots e^{i\theta(r_k)} e^{-i\theta(r_{2k+1})} e^{i\theta(r_{2k+2})} \ldots e^{i\theta(r_{k+l})}\rangle_0 \tag{6}$$

is non-zero only when there are equal numbers of $e^{i\theta(r_i)}$ and $e^{-i\theta(r_i)}$ factors. This can be exploited when expanding in powers of g only after separating the cosine (and subsequently the exponent of the cosine) into $e^{i\theta}$ and $e^{-i\theta}$.

(b) Next, compute the averages on the right-hand side of Eq. (4). This can be conveniently done by introducing the function

$$f(r) = \sum_{j=1}^{2n} k_j \delta(r - r_j), \tag{7}$$

where $k_j = 1$ for $j = 1, \ldots, n$ and $k_j = -1$ for $j = n+1, \ldots 2n$. Show that the average becomes the Gaussian integral

$$\langle e^{i\theta(r_1)} e^{i\theta(r_2)} \ldots e^{i\theta(r_n)} e^{-i\theta(r_{n+1})} \ldots e^{-i\theta(r_{2n})}\rangle_0 = \frac{1}{Z_0} \int [d\theta]\, e^{-\int d\mathbf{r} \left\{\frac{\kappa}{2}(\nabla\theta)^2 - if(r)\theta(r)\right\}} \tag{8}$$

and can be readily performed with the result

$$Z = Z_0 \sum_{n=0}^{\infty} \frac{g^{2n}}{4^n n! n!} e^{-\frac{1}{2\kappa}\int d\mathbf{r} d\mathbf{r}' f(\mathbf{r}) \langle \mathbf{r}| \frac{1}{-\nabla^2} |\mathbf{r}'\rangle f(\mathbf{r}')}. \tag{9}$$

(c) Evaluate the exponent in Fourier space introducing short- and long-distance cutoffs a and R and obtain

$$Z = Z_0 \sum_{n=0}^{\infty} \frac{1}{n! n!} \int \prod_{j=1}^{2n} d\mathbf{r}_j e^{-2nS_c + \frac{1}{2\pi\kappa}\sum_{i<j} k_i k_j \ln\left(\frac{|\mathbf{r}_i - \mathbf{r}_j|}{a}\right)}. \tag{10}$$

Here, we defined $g/2 = e^{-S_c}$. Indeed, this is just the partition function (1) of the superfluid with vortices if we make the identifications $1/2\pi\kappa \leftrightarrow 2\pi\beta\eta$ and $g/2 \leftrightarrow e^{-\beta E_c}$. (The latter quantity is proportional to the fugacity y.)

Hint: You may find the integral

$$\int \frac{\mathrm{d}q}{q} e^{iq(r_i - r_j)} \simeq \ln(R/a) - \ln(|r_i - r_j|/a) \tag{11}$$

helpful.

Problem 2: Perturbative RG for the sine-Gordon model — 1st order (5+5+5+5+5 points)

We are now ready to consider the renormalization of the coupling constants κ and g of the sine-Gordon model. Rather than eliminating degrees of freedom in real space, we will follow a common momentum-shell renormalization scheme in which we integrate out modes in momentum space. We start from the sine-Gordon action

$$S = \int d\mathbf{r} \left\{\frac{\kappa}{2}(\nabla\theta)^2 - g\cos\theta\right\}. \tag{12}$$

As this is an effective low-energy theory, it has a short-distance (ultraviolet) cutoff l. The key steps of the RG procedure are:

- Integrating out the Fourier components of the field θ with wavelengths between l and λ. This step leaves us with an action in terms of the fast modes with wavelengths smaller than λ. This action may not necessarily be particularly simple, but we will assume that we need to keep only terms which have the same structure as in the original theory. (There are ways to decide which interactions one needs to retain and which interactions one can neglect. This separation of interactions into relevant, irrelevant, and marginal perturbations is beyond the scope of this problem.)

- Rescaling lengths by a factor l/λ such that the resulting theory returns to the original cutoff l.

As a result of these renormalization-group transformations,[1] we find that the coupling constants κ and g of the theory change. This procedure can be repeated many times, and the coupling constants flow as a function of the number of such RG transformations. We can finally analyze this coupling-constant flow to understand the physics of the model.

According to this program, we first integrate out the short-wavelength modes of the θ-field. Note that we effectively integrate out (spherical or circular) shells in momentum space with momenta in $[2\pi/\lambda, 2\pi/l]$. We assume that this is a thin (in fact, infinitesimal) shell so that it contains a small number of modes. We write the field as

$$\theta = \theta_\lambda + \delta\theta. \tag{13}$$

Here,

$$\theta_\lambda = \frac{1}{V} \sum_{|\mathbf{q}|>\mathbf{2}\pi/\lambda} \theta_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}} \tag{14}$$

describes all modes with wavelengths smaller than λ, while

$$\delta\theta = \frac{1}{V} \sum_{2\pi/l<|\mathbf{q}|<\mathbf{2}\pi/\lambda} \theta_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}} \tag{15}$$

collects the modes with wavelengths between l and λ. Since $\delta\theta$ contains only few modes, it can be treated as small and we can expand the action in powers of $\delta\theta$.

(a) Expand the action to second order in $\delta\theta$ to obtain

$$S = \int \mathrm{d}\mathbf{r} \left\{ \frac{\kappa}{2} (\nabla\theta_\lambda)^2 - g\cos\theta_\lambda \right\} + \int \mathrm{d}\mathbf{r} \left\{ \frac{\kappa}{2} (\nabla\delta\theta)^2 + g\sin\theta_\lambda \delta\theta + \frac{g}{2}\cos\theta_\lambda [\delta\theta]^2 \right\}. \tag{16}$$

Why is there a term linear in $\delta\theta$ originating from the cosine term but none from the $(\nabla\theta)^2$ term?

(b) Integrate out $\delta\theta$, i.e., the fast modes of the θ-field. Treat the $(\nabla\delta\theta)^2$ term as the unperturbed action and use the cumulant expansion to second order in g. (This step requires the coupling to be weak, so that we are dealing with perturbative RG. Still, this is much better than direct perturbation theory because the small changes in one iteration enter into later iterations.) Show that the action becomes

$$S = \int \mathrm{d}\mathbf{r} \left\{ \frac{\kappa}{2} (\nabla\theta_\lambda)^2 - g\cos\theta_\lambda \right\} + \int \mathrm{d}\mathbf{r}\, \mathrm{d}\mathbf{r}' \left[\frac{g}{2}\cos\theta_\lambda(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}') - \frac{g^2}{2}\sin\theta_\lambda(\mathbf{r})\sin\theta_\lambda(\mathbf{r}') \right] K(\mathbf{r}-\mathbf{r}'). \tag{17}$$

Here, $K(\mathbf{r}) = \langle \delta\theta(\mathbf{r})\delta\theta(\mathbf{0})\rangle_0$, where the average is over the free action $S_0 = (\kappa/2)\int d\mathbf{r}(\nabla\delta\theta)^2$.

(c) As a first approximation, consider only the leading order term in g in the action, neglecting terms $\propto g^2$ or higher. Show that the action has the same form as the original one, but with a new cut-off and with

$$g \to g - \frac{g}{2} K(\mathbf{0}), \tag{18}$$

while κ remains unchanged. Show that

$$K(\mathbf{0}) = \int_{2\pi/\lambda<k<2\pi/l} \frac{\mathrm{d}\mathbf{k}}{2\pi} \frac{1}{\kappa\mathbf{k}^2} = \frac{1}{2\pi\kappa} \ln(\lambda/l). \tag{19}$$

(d) We now turn to the second step of the RG procedure in which we reset the short-distance cutoff to its original value. It turns out that this is equivalent to introducing dimensionless coupling constants. Thus, we introduce dimensionless (but cutoff-dependent) coupling constants $\kappa_\lambda = \kappa$ (note that κ is already

[1] Do not bother too much with the name. In particular, do not try to make a connection to mathematical groups.

dimensionless) and $g_\lambda = \lambda^2 g$. Show that to linear order in g, the RG flow of these couplings with cutoff becomes

$$\frac{\mathrm{d}g_\lambda}{\mathrm{d}\ln\lambda} = \left(2 - \frac{1}{4\pi\kappa_\lambda}\right) g_\lambda \tag{20}$$

$$\frac{\mathrm{d}\kappa_\lambda}{\mathrm{d}\ln\lambda} = 0 \tag{21}$$

(e) Briefly discuss these RG equations and make the connection with the real-space RG presented in class explicit.

Problem 3: Perturbative RG for the sine-Gordon model — 2^{nd} order (10+5+5+5 points)

This problem deals with the quadratic order in g. It turns out that the calculation is subtle. We will follow a standard textbook presentation, although it is actually not fully accurate. Resolving these issue would take us too far, but you do show very explicitly in the last item of this problem that there is a problem with the treatment.

(a) Start from the result of the cumulant expansion in part (b) of the previous problem. Introduce the notation

$$K_0 = \int \mathrm{d}\mathbf{r}\, K(\mathbf{r}-\mathbf{r}') \qquad K_2 = \int \mathrm{d}\mathbf{r}(\mathbf{r}-\mathbf{r}')^2 K_0(\mathbf{r}-\mathbf{r}'). \tag{22}$$

Show that the g^2-term can be rewritten as

$$\int \mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{r}' \sin\theta_\lambda(\mathbf{r}) K(\mathbf{r}-\mathbf{r}') \sin\theta_\lambda(\mathbf{r}') = \int \mathrm{d}\mathbf{r} K_0 \sin^2\theta_\lambda(\mathbf{r}) - \frac{1}{2}\int \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{r}' \left[\sin\theta_\lambda(\mathbf{r}) - \sin\theta_\lambda(\mathbf{r}')\right]^2 K(\mathbf{r}-\mathbf{r}'). \tag{23}$$

Assuming that $K(\mathbf{r}-\mathbf{r}')$ falls off sufficiently fast with $|\mathbf{r}-\mathbf{r}'|$, expand $[\sin\theta_\lambda(\mathbf{r}) - \sin\theta_\lambda(\mathbf{r}')]^2$ to leading (second) order in $\mathbf{r}-\mathbf{r}'$ to obtain

$$\begin{aligned} S \;=\; & \int \mathrm{d}\mathbf{r} \left\{\frac{\kappa}{2}(\nabla\theta_\lambda)^2 - g\cos\theta_\lambda\right\} + \frac{g}{2}K(\mathbf{0}) \int \mathrm{d}\mathbf{r} \cos\theta_\lambda(\mathbf{r}) \\ - \; & \frac{g^2}{8}\int \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{r}' \cos^2\theta_\lambda(\mathbf{r})(\nabla\theta_\lambda)^2(\mathbf{r}-\mathbf{r}')^2 K(\mathbf{r}-\mathbf{r}') + \frac{g^2}{2}K_0 \int \mathrm{d}\mathbf{r} \sin^2\theta_\lambda(\mathbf{r}). \end{aligned}$$

(b) Rewrite the action using $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ and $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$, and neglect terms involving $\cos(n\theta_\lambda)$, with $n \geqslant 2$. These terms are examples for terms which are generated by the renormalization procedure but can be shown to be neglegible ("irrelevant in the renormalization-group sense"). With this approximation, the action retains the form of the original action with a new cutoff and modified coupling constants. Shat that

$$g \;\to\; g - \frac{1}{2}K(\mathbf{0}). \tag{24}$$

$$\kappa \;\to\; \kappa + \frac{1}{8}g^2 K_2. \tag{25}$$

Compute the coefficient K_2,

$$K_2 = \int \mathrm{d}\mathbf{r} \int_{2\pi/\lambda<k<2\pi/l} \frac{d^2\mathbf{k}}{2\pi}\frac{\mathbf{r}^2 e^{i\mathbf{k}\cdot\mathbf{r}-\epsilon|\mathbf{r}|}}{\kappa\mathbf{k}^2}. \tag{26}$$

Here, we introduce a large-distance cut-off, $1/\epsilon$, with ϵ a positive infinitesimal quantity. Perform the integrals over $\mathbf{k}$ and $\mathbf{r}$ in polar coordinates, approximating

$$\int_0^{2\pi} d\theta \frac{1}{(\cos\theta + i\epsilon)^4} \simeq 8\pi. \tag{27}$$

This "follows" by ignoring the divergences at $\theta = \pm\pi/2$ due to the regularization and replacing $\cos^2\theta$ by its average 1/2. (See below for a more accurate evaluation.) Show that this yields

$$K_2 = \frac{3(\lambda - l)l^4}{2\pi^4 l\kappa}. \tag{28}$$

(c) Using this expression for K_2, show that the flow equations for the dimensionless coupling constants introduced in the previous problem becomes

$$\frac{dg_\lambda}{d\ln\lambda} = \left(2 - \frac{1}{4\pi\kappa_\lambda}\right) g_\lambda, \tag{29}$$

$$\frac{d\kappa_\lambda}{d\ln\lambda} = \frac{3g_\lambda^2}{16\pi^4\kappa_\lambda}. \tag{30}$$

Compare your result to the real-space RG equations discussed in the lecture.

(d) In (e) you approximated

$$A = \int_0^{2\pi} d\theta \frac{1}{(\cos\theta + i\epsilon)^4} \simeq 8\pi. \tag{31}$$

Now compute this integral explicitly by substituting $z = e^{i\theta}$. This yields $A = \int_C dz\, F(z)$, where $\mathcal{C}$ is the unit circle in the complex plane. Give the explicit expression of $F(z)$, and show that its only pole inside the unit circle is at $z = i(1 - \epsilon)$. Compute the integral and show that $A = 0$. Clearly, something goes wrong with the procedure sketched above. Nevertheless, it turns out that the procedure is a classic case of compensating errors and actually produces the correct RG equations (up to numerical constants which are not controlled within this order anyway). If you want to understand these issues in more detail, you may consult: Nozieres and Gallet, J. Phys. (Paris) **48**, 353 (1987).