Thermal rectification in nonlinear quantum circuits

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(Dated: December 18, 2008)

We present a theoretical study of radiative heat transport in nonlinear solid-state quantum circuits. We give a detailed account of heat rectification effects, i.e. the asymmetry of heat current with respect to a reversal of the thermal gradient, in a system consisting of two reservoirs at finite temperatures coupled through a nonlinear resonator. We suggest an experimentally feasible superconducting circuit employing the Josephson nonlinearity to realize a *controllable* low temperature heat rectifier with a maximal asymmetry of the order of 10%. Strikingly, we discover that rectification can change sign as a function of temperature.

Heat transport in nanoscale structures has become an active and rapidly growing research area. Progress in experimental methods has enabled the study of fundamental issues, and lately the field has seen major breakthroughs, such as the measurement of quantized heat transport [1], and manipulation of thermal currents using external control fields [2, 3]. In solid-state systems electron–electron and electron–phonon scattering are the most important channels for small systems to exchange energy with the environment. However, recently it was understood that at low temperatures one needs to take into account the radiative channel which becomes the dominant relaxation method in mesoscopic samples below the phonon–photon crossover [2, 4, 5].

In this paper we study rectification effects in thermal transport mediated by electromagnetic fluctuations in solid-state nanostructures. In a two-terminal geometry a finite rectification means that heat current is not simply reversed when the thermal gradient changes sign, but also the absolute magnitude of the current changes. We define the rectification $\mathcal{R}$ as

$$\mathcal{R} = (J_+ - J_-)/\max\{J_+,\, J_-\}, \tag{1}$$

where J_+ and J_- are the magnitudes of the heat currents in forward and reverse bias configurations, respectively (see Fig. 1). Previously rectification has been shown to take place in systems where a classical [6, 7, 8] or quantized [9, 10] nonlinear chain is coupled asymmetrically to linear reservoirs, or when nonlinear reservoirs are coupled through a harmonic oscillator [11]. Here we demonstrate rectification in a fully quantum-mechanical and experimentally realizable model where photon-mediated heat current flows between two linear reservoirs coupled asymmetrically to a nonlinear resonator.

Our analysis is based on a nonequilibrium Green's function method developed in Ref. [14], and the nonlinear transport problem is solved with a self-consistent mean-field approximation. Rectification is studied as a function of the operating temperatures, reservoir coupling strengths and admittances, and the strength of the nonlinearity. We also propose a concrete setup based on a Superconducting QUantum Interference Device (SQUID) where the rectification effects can be realized within current experimental technology at sub-Kelvin temperatures. A similar circuit, operated in the linear regime, was employed in the pioneering experiment demonstrating photonic heat transport [2]. By adjusting the external magnetic flux through the circuit it is possible to tune the rectification continuously between zero and the maximum value. Using realistic parameters we find a rectification of over 10%, and identify a regime where $\mathcal{R}$ changes sign as a function of temperature. Experimentally rectification has been observed in phonon transport through a nanotube at room temperature with $\mathcal{R} = 7\%$ [12] and in electron transport through a quantum dot at 80 mK with $\mathcal{R}$ up to 10% [13].

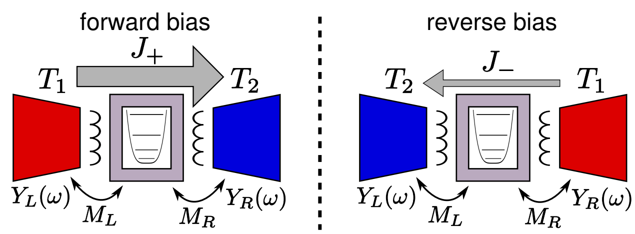

FIG. 1: In a heat transport experiment, thermal energy flows from a hot reservoir (temperature T_1) to a cold reservoir (temperature T_2). To obtain the rectification $\mathcal{R}$ one must measure the current for both thermal bias directions. In our model the heat is transported by inductive magnetic coupling between the reservoirs and the central nonlinear resonator.

The thermal transport setup is depicted in Fig. 1. It consists of two linear reservoir circuits with admittances $Y_L(\omega)$ and $Y_R(\omega)$. Temperatures of the left and right reservoirs are T_1 and $T_2 < T_1$ in the forward bias setting, and vice versa for reverse bias. We assume that heat can flow between the reservoirs only through a mediating nonlinear resonator circuit. The couplings between the reservoirs and the resonator are taken to be inductive with mutual inductances M_L and M_R. Using the

Caldeira–Leggett mapping between linear admittances and bosonic reservoir modes the total Hamiltonian takes the form $H = H_L + H_R + H_M + H_C$, where the middle circuit and reservoir terms are

$$H_M = \hbar\omega_0(\hat{b}^\dagger\hat{b} + \frac{1}{2}) + \frac{\hbar\epsilon}{2}(\hat{b} + \hat{b}^\dagger)^4, \tag{2}$$

$$H_{L/R} = \sum_{j\in L/R} \hbar\omega_j(\hat{a}_j^\dagger\hat{a}_j + \frac{1}{2}), \tag{3}$$

and the inductive coupling term is

$$H_C = \hat{I}\left(M_L\hat{i}_L + M_R\hat{i}_R\right), \tag{4}$$

which involves the current operators for the central device $\hat{I}$ and for the reservoirs $\hat{i}_{L/R} = \sum_{j\in L/R} g_j(\hat{a}_j + \hat{a}_j^\dagger)$, respectively. The electric current operator for the central device can be expressed as $\hat{I} = I_0(\hat{b} + \hat{b}^\dagger)$ with $I_0 = \sqrt{\hbar\omega_0/2L}$ and $\omega_0 = 1/\sqrt{LC}$ where L and C are the linear inductance and capacitance of the resonator; $\hat{b}$, $\hat{b}^\dagger$ and reservoir operators are bosonic creation and annihilation operators, $[\hat{b}, \hat{b}^\dagger] = 1$. The nonlinearity of the central circuit is characterized by the the second term in Eq. (2), corresponding to a quartic potential whose strength is controlled by the parameter ϵ. It must be emphasized that Eq. (4) has a generic bilinear form and therefore the results are relevant for other types of systems beyond the studied realization.

The basis of our analysis is provided by the Meir–Wingreen formula for the heat current [14, 15]

$$J = \int_0^\infty \frac{d\omega\omega^2 M_L^2}{2\pi}\left\{2\left[S_I(-\omega) - S_I(\omega)\right]\mathrm{Re}[Y_L(\omega)]n_L(\omega) + S_I(-\omega)2\mathrm{Re}[Y_L(\omega)]\right\}. \tag{5}$$

Here $n_L(\omega)$ is the Bose function of the left reservoir and $S_I(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega(t-t')}\langle\hat{I}(t)\hat{I}(t')\rangle$ is the current noise power of the central circuit. The admittances $Y_{L/R}(\omega)$ are related to the current correlation functions of the free reservoirs [14]. In the absence of the nonlinear term ($\epsilon = 0$) the transport problem can be solved exactly for arbitrary couplings and reservoir admittances [14]. No rectification takes place in this regime. In the following we solve the nonlinear transport problem in a self-consistent mean-field approximation, which is expected to be accurate for small values of the nonlinearity. This approach does not fully account for the correlation effects due to the interplay of nonlinearity and tunneling which are potentially important in the ultra-low temperature regime T_1, $T_2 \ll \hbar\omega_0/k_B$. However, analogously to interacting electron transport problems, the mean-field approach is accurate in the sequential tunneling regime when the temperatures are of the order of $\hbar\omega_0/k_B$.

As a first step we approximate the resonator Hamiltonian as

$$H_M \approx \hbar\omega_0(\hat{b}^\dagger\hat{b} + \frac{1}{2}) + \hbar\epsilon\Phi(\hat{b}^\dagger + \hat{b})^2, \tag{6}$$

where we have defined the mean field $\Phi = \langle(\hat{b}^\dagger + \hat{b})^2\rangle$. Because Eq. (6) is again quadratic in bosonic operators, it is possible to bring it to a diagonal form by a canonical transformation. However, now we have the added complication of an a priori unknown mean field, which has to be evaluated self-consistently in a nonequilibrium state. The transformed Hamiltonian and current operators are

$$H_M = \hbar\tilde{\omega}_0(\tilde{b}^\dagger\tilde{b} + \frac{1}{2}), \qquad \hat{I} = \tilde{I}_0(\tilde{b} + \tilde{b}^\dagger), \tag{7}$$

where $\tilde{\omega}_0 = \omega_0\sqrt{1 + \frac{4\epsilon\Phi}{\omega_0}}$ and $\tilde{I}_0 = \sqrt{\frac{\omega_0}{\tilde{\omega}_0}}I_0$. Thus the effect of the nonlinear term is incorporated by a mean-field dependent renormalization of the resonance frequency of the oscillator and its current operator. For further development it is convenient to introduce the correlation functions $\langle\hat{I}(t)\hat{I}(t')\rangle^r = -i\theta(t-t')\langle[\hat{I}(t), \hat{I}(t')]\rangle$ and $\langle\hat{I}(t)\hat{I}(t')\rangle^< = -i\langle\hat{I}(t')\hat{I}(t)\rangle$. A nonequilibrium equation-of-motion analysis [17], similar to the one presented in Ref. [14], reveals that the current correlators are given by

$$\begin{aligned}\langle\hat{I}\hat{I}\rangle^r(\omega) &= \frac{1}{\left(\langle\hat{I}\hat{I}\rangle_0^r(\omega)\right)^{-1} - \tilde{I}_0^{-2}\Sigma^r(\omega)}, \\ \langle\hat{I}\hat{I}\rangle^<(\omega) &= \tilde{I}_0^{-2}|\langle\hat{I}\hat{I}\rangle^r(\omega)|^2\Sigma^<(\omega)\end{aligned} \tag{8}$$

where $\langle\hat{I}\hat{I}\rangle_0^r(\omega) = 2\tilde{I}_0^2\tilde{\omega}_0/(\omega^2 - \tilde{\omega}_0^2)$ arises from the uncoupled oscillator. The self-energies

$$\begin{aligned}\Sigma^r(\omega) &= -\frac{i\tilde{I}_0^2\omega}{\hbar}\left[M_L^2Y_L(\omega) + M_R^2Y_R(\omega)\right] \\ \Sigma^<(\omega) &= -\frac{i\tilde{I}_0^2\omega}{\hbar}\left[M_L^2\mathrm{Re}(Y_L(\omega))n_L(\omega) + M_R^2\mathrm{Re}(Y_R(\omega))n_R(\omega)\right],\end{aligned} \tag{9}$$

take into account the presence of reservoirs. Furthermore, the mean field Φ is related to the lesser correlator via

$$\Phi = \langle(\hat{b}^\dagger + \hat{b})^2\rangle = -I_0^{-2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi i}\langle\hat{I}\hat{I}\rangle^<(\omega). \tag{10}$$

Equations (8)–(10) form a closed set of equations which needs to be solved to find the current correlation functions. The self-consistent solution proceeds by making an initial guess for the mean field, calculating the correlation functions (8) corresponding to the initial value and calculating the updated value of the mean field by evaluating the integral in Eq. (10). The procedure is repeated until convergence is achieved. The current noise then follows immediately from the lesser function $S_I(\omega) = -\mathrm{Im}\langle\hat{I}\hat{I}\rangle^<(-\omega)$ which yields the heat current after evaluating Eq. (5). In the case of a vanishing nonlinearity ($\epsilon = 0$) this procedure recovers the exact solution of the linear problem.

For numerical calculations explicit expressions for the admittances $Y_{L/R}(\omega)$ are needed. Here we assume that the reservoir circuits effectively consist of a resistor, a capacitor, and an inductor in series, resulting in $Y_{L/R}(\omega) = R_{L/R}^{-1}[1 - iQ_{L/R}(\frac{\omega}{\omega_{L/R}} - \frac{\omega_{L/R}}{\omega})]^{-1}$, where $R_{L/R}$, $Q_{L/R}$, and $\omega_{L/R}$ are the resistance, quality factor and resonance frequency of the left and right reservoir, respectively. The behavior of the system is now uniquely determined by nine dimensionless parameters: ϵ/ω_0, $k_BT_{1/2}/\hbar\omega_0$, $M_{L/R}^2 I_0^2/\hbar R_{L/R}$, $Q_{L/R}$, and $\omega_{L/R}/\omega_0$. Rectification can then be calculated from Eq. (1), by computing the forward and reverse bias currents, $J_{+/-}$, with the above prescription.

Let us illustrate some generic features of the model with the simple setup of two purely dissipative reservoirs, $Q_L = Q_R = 0$, in which case the frequencies ω_L and ω_R are irrelevant. In Fig. 2 we plot the rectification against three different variables. First, from Fig. 2(a) we see that already at moderate values of the nonlinearity, $\epsilon \sim 0.2\,\omega_0$, the rectification has essentially reached its maximum value. Such values for ϵ are well within the regime of validity of our approximations and should also be easily achieved in the experimental setup proposed below. Next, Fig. 2(b) exemplifies a very generic feature: having $M_L^2/R_L < M_R^2/R_R$ tends to produce $J_+ > J_-$, and vice versa. Finally Fig. 2(c) shows that the rectification increases logarithmically with the temperature ratio T_1/T_2. Therefore, to see an appreciable effect, the temperature difference $T_1 - T_2$ should be of the same order of magnitude as the temperatures themselves. For

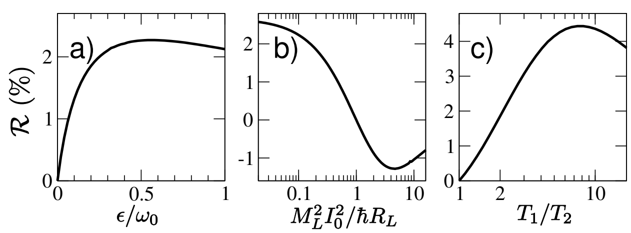

FIG. 2: Rectification with purely resistive reservoirs, $Q_L = Q_R = 0$, as a function of (a) nonlinearity ϵ, (b) coupling M_L, and (c) temperature ratio T_1/T_2. In all panels we have $\epsilon/\omega_0 = 0.2$, $k_BT_1/\hbar\omega_0 = 0.2$, $k_BT_2/\hbar\omega_0 = 0.1$, $M_L^2 I_0^2/\hbar R_L = 0.2$, and $M_R^2 I_0^2/\hbar R_R = 1$, except for the variable on the horizontal axis. In panel (c) T_1 is varied.

purely resistive reservoirs maximal value for the rectification is about 2% (Fig. 2(a)). Larger values can be obtained by adding a reactive part to one of the reservoir circuits. Then, as Fig. 3 shows, $\mathcal{R}$ can be made an order of magnitude higher. The inset shows the current J_+, normalized with respect to the universal single-channel maximum heat current $J_{\max} = \frac{\pi k_B^2}{3\hbar}(T_1^2 - T_2^2)$ [16]. According to Fig. 3, the highest values for $\mathcal{R}$ are obtained for high temperatures, where J_+ tends to zero. High rectification and large current are thus competing effects, and the optimal operating point depends on the experimental constraints. In any case, it is possible to obtain a rectification of $\sim 5\%$ with $J \sim 0.1\,J_{\max}$ and up to $\sim 15\%$ with $J \sim 0.01\,J_{\max}$.

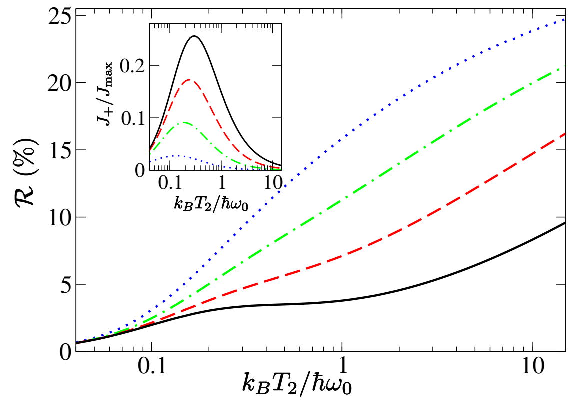

FIG. 3: Rectification with one reactive reservoir ($Q_L = 0.1$). Here $T_1/T_2 = 2$ and the different curves correspond to $\omega_L/\omega_0 = 0.2$ (solid), 0.1 (dashed), 0.05 (dash-dotted), 0.02 (dotted). Other parameters as in Fig. 2.

From Fig. 3 we also see that decreasing ω_L increases the rectification, so both small ω_L and the condition $M_L^2/R_L < M_R^2/R_R$ favor the direction $J_+ > J_-$. We can also combine these two trends in an opposing manner by making ω_L large. This way one can produce a system where the *direction* of rectification changes as a function of temperature (see Fig. 4). It is especially noteworthy that in this case the extrema of rectification coincide with reasonably high current levels, $J \sim 0.1\,J_{\max}$, i.e., at current levels which should be detectable in experiments.

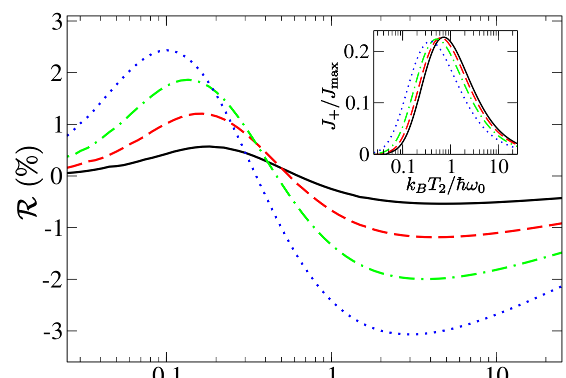

FIG. 4: Rectification as a function of temperature. Here $Q_L = 0.1$, $\omega_L = 10\,\omega_0$ and the curves correspond to $T_1/T_2 = 1.2$ (solid), 1.5 (dashed), 2 (dash-dotted), 3 (dotted). Other parameters as in Fig. 2.

For low operating temperatures, with T_1, T_2 approximately in the range 100 mK–1 K, the studied model can be realized by the setup shown in Fig. 5. The system consists of a superconducting loop containing a Josephson junction characterized by its Josephson energy E_J

and shunt capacitance C. The loop itself is assumed to have a finite inductance dominating the potential landscape. The Hamiltonian of the system is [18]

$$H_M = E_C\hat{q}^2 + E_L(\hat{\phi} - \phi_x)^2 - E_J \cos\hat{\phi}, \qquad (11)$$

where the charging and inductive energies are $E_C = e^2/2C$, $E_L = (\hbar/2e)^2/2L$, and ϕ_x denotes the external magnetic flux through the loop (in units of $\hbar/2e$). The superconducting phase across the junction $\hat{\phi}$ and the charge at the capacitor $\hat{q}$ (in units of electron charge) are treated as conjugate observables $[\hat{\phi}, \hat{q}] = 2i$. The charging term can be thought of as the kinetic energy and the ϕ-dependent terms as an effective potential energy of a fictitious particle. In the following we assume that $\phi_x \approx \pi$ and $E_J < 2E_L$ so that the potential has a single minimum at $\hat{\phi} = \phi_0$, with $\phi_0 \approx \pi$. With these assumptions the phase is bound close to the minimum so that we can approximate the potential accurately by expanding the cosine term to the 4th order:

$$\begin{aligned} H_M &= E_C\hat{q}^2 + (E_L + \frac{1}{2}E_J\cos\phi_0)\hat{\phi}^2 - \frac{1}{24}E_J\cos\phi_0\,\hat{\phi}^4 \\ &\equiv E_C\hat{q}^2 + E_2\hat{\phi}^2 + E_4\hat{\phi}^4, \end{aligned} \qquad (12)$$

the second line defining the quantities E_2 and E_4. In general there should also be a $\hat{\phi}^3$ term, but with $\phi_0 \approx \pi$ this is small. Further, within the mean-field approximation one has $\hat{\phi}^3 \sim \hat{\phi}\langle\hat{\phi}^2\rangle$, producing just a shift in the origin. Writing the charge and phase in terms of bosonic creation and annihilation operators we recover exactly Eq. (2) with parameters $\hbar\omega_0 = 4\sqrt{E_C E_2}$ and $\hbar\epsilon = 2\frac{E_C E_4}{E_2}$. The current operator of the circuit is given by $\hat{I} = I_0(\hat{b} + \hat{b}^\dagger)$, where $I_0 = 4e(E_C E_2^3)^{1/4}/\hbar$. Thus, in the parameter regime $E_2 \gg E_4$ we have effectively realized the previously studied weakly nonlinear resonator model.

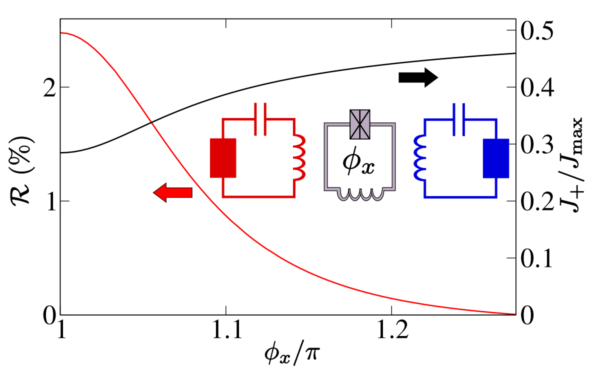

FIG. 5: Rectification and heat current through the SQUID system as a function of the control flux ϕ_x. Parameter values at $\phi_x = \pi$ as in Fig. 2, except $T_1 = 2T_2 = 0.4\hbar\omega_0/k_B$. Inset: Schematic of the SQUID setup with two linear reservoir circuits inductively coupled to a superconducting loop containing one Josephson junction.

As the above considerations show, varying the externally applied field ϕ_x about π moves the potential minimun ϕ_0 which in turn changes the values of the parameters ω_0 and ϵ. In particular, ϵ is maximized at $\phi_x = \phi_0 = \pi$ and vanishes when $\phi_0 \to \pi \pm \pi/2$. Figure 5 demonstrates the resulting continuous tuning of the rectification performance.

In conclusion, we have analyzed heat rectification effects in radiative heat transport through a nonlinear quantum resonator. This system is particularly interesting because it can be realized by an experimentally feasible superconducting circuit. The proposed system is operated in a low-temperature regime and is, as far as we know, the first suggestion for controllable heat rectification not based on electron transport. Despite its simplicity, the system is capable of producing a rectification of over 10%. We have also discovered an unexpected effect where the direction of rectification changes as a function of temperature.

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