Ginzburg-Landau Theory for Bosonic Gases in Optical Lattices

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Outline

1. **Bosons in Optical Lattices**
   - Optical Lattices
   - Bose-Hubbard Model
   - Superfluid-Mott Insulator Transition

2. **Ginzburg-Landau Theory**
   - Generating Functionals
   - Diagrammatic Expansion
   - Diagrammatic Rules

3. **Results**
   - Quantum Phase Diagram
   - Excitation Spectra
   - Collapse and Revival of Matter Waves
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Optical Lattices

Counter-propagating laser beams create periodic potential
Optical Lattices

- Counter-propagating laser beams create periodic potential
- Hopping and interactions are highly controllable

\[ a = \lambda/2 \]
\[ V_0 = \text{Re} [\alpha(\omega)] I/2 \]
- Counter-propagating laser beams create periodic potential
- Hopping and interactions are highly controllable
- Different possible topologies at 1D, 2D, and 3D
Counter-propagating laser beams create periodic potential
Hopping and interactions are highly controllable
Different possible topologies at 1D, 2D, and 3D
Model for condensate matter systems
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Bose-Hubbard Model

- Bose-Hubbard Hamiltonian

\[ \hat{H}_{\text{BH}} = \hat{H}_0 + \hat{H}_J \]

\[ \hat{H}_0 = \sum_i \frac{U}{2} (\hat{n}_i^2 - \hat{n}_i) - \mu \hat{n}_i \]

\[ \hat{H}_J = - \sum_{ij} J_{ij} \hat{a}_i^\dagger \hat{a}_j \]

\[ \hat{n}_i = \hat{a}_i^\dagger \hat{a}_i \]

\[ J_{ij} = \begin{cases} J , & \text{if } i, j \text{ nearest neighbors} \\ 0 , & \text{otherwise.} \end{cases} \]
Bose-Hubbard Model

Bose-Hubbard Hamiltonian

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\[ J_{ij} = \begin{cases} J, & \text{if } i, j \text{ nearest neighbors} \\ 0, & \text{otherwise.} \end{cases} \]

\[ \hat{H}_0 \left| n \right\rangle = N_S E_n \left| n \right\rangle \]

\[ E_n = \frac{U}{2} n(n-1) - \mu n \]

\[ \hat{H}_0 \text{ is diagonal} \]

Expansion in series of \( J \)
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Superfluid-Mott Insulator Transition

- Increasing the laser intensity localizes atoms
Superfluid-Mott Insulator Transition

- Increasing the laser intensity localizes atoms
- Detectable in time-of-flight pictures
Superfluid-Mott Insulator Transition

- Increasing the laser intensity localizes atoms
- Detectable in time-of-flight pictures
- Inaccurate analytical methods prior to this work
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Generating Functionals

Symmetry breaking source:

\[ \hat{H}_{BH}(\tau) = \hat{H}_{BH} + \sum_i \left[ j_i^*(\tau)\hat{a}_i + j_i(\tau)\hat{a}^\dagger_i \right] \]
Generating Functionals

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  \[ \hat{H}_{BH}(\tau) = \hat{H}_{BH} + \sum_i \left[ j_i^*(\tau) \hat{a}_i + j_i(\tau) \hat{a}_i^\dagger \right] \]

- Evolution operator: \[ \hat{U}(\tau, \tau_0) = \hat{T} e^{-\int_{\tau_0}^{\tau} d\tau' \hat{H}_{BH}(\tau')} \]
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Generalized partition function: \( Z = \text{Tr} \left[ \hat{U}(0, \beta) \right] \)
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- Generator of connected functions:
  \[ W [j_i^*(\tau), j_i(\tau)] = \log Z = -\beta F [j_i^*(\tau), j_i(\tau)] \]
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  \[ \mathcal{W}[j_i^*(\tau), j_i(\tau)] = \log Z = -\beta F[j_i^*(\tau), j_i(\tau)] \]

- Order parameter field:
  \[ \psi_i(\tau) = \beta \frac{\delta F}{\delta j_i^*(\tau)} \]
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Effective action:
\[ \Gamma[\psi_i^*(\tau), \psi_i(\tau)] = F - \frac{1}{\beta} \sum_i \int_0^\infty d\tau [\psi_i^*(\tau) j_i(\tau) + \psi_i(\tau) j_i^*(\tau)] \]
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Equation of motion:

\[ \frac{\delta \Gamma}{\delta \psi_i(\tau)} = 0 \]
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Diagrammatic Expansion

- General formula: \[ Z \langle j^*, j \rangle = e^{\sum_{ii'} J_{ii'} \int_0^\beta d\tau \frac{\delta}{\delta j_i^*(\tau) \delta j_{ii'}(\tau)} Z_0 \langle j^*, j \rangle} \]
Diagmmatic Expansion

- General formula: \( Z[j^*, j] = e^{\sum_{ii'} J_{ii'} \int_0^\beta d\tau \frac{\delta}{\delta j_i^*(\tau) \delta j_{i'}^{\prime}(\tau)} Z_0[j^*, j]} \)

- Diagrammatic representation:

\[ W_0[j^*, j] = \bullet + \quad + \quad + \quad + \cdots \]
Diagrammatic Expansion

- General formula: \( Z [j^*, j] = e^{\sum_{ii'} J_{ii'} \int_0^\beta d\tau \frac{\delta}{\delta j^*_i(\tau)} \frac{\delta}{\delta j_{i'}(\tau)} Z_0 [j^*, j]} \)

- Diagrammatic representation:

\[
W_0 [j^*, j] = \mathcal{O} + \frac{1}{2!^2} \times \times + \frac{1}{3!^2} \times \times + \cdots
\]

- Perturbative expansion:

\[
W [j^*, j] = \mathcal{O} + \frac{1}{2!^2} \times \times + \frac{1}{3!^2} \times \times + \cdots
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Diagrammatic Expansion

- General formula: \( Z [j^*, j] = e^{\sum_{ii'} J_{ii'} \int_0^\beta d\tau \frac{\delta}{\delta j_{ii'}^{*}(\tau)} \frac{\delta}{\delta f_{ii'}^{*}(\tau)} Z_0 [j^*, j]} \)
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- Perturbative expansion:

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W [j^*, j] = \bullet + \frac{1}{2!} + \frac{1}{2!} + \cdots
\]

- Effective action has only 1PI diagrams

\[
- \beta \Gamma [\psi^*, \psi] = \Gamma^{(0)} + \frac{1}{2!} + \cdots
\]
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GL Theory for Bosonic Gases in Optical Lattices
Diagrammatic Rules

- Vertices represent connected functions $W^{(n)}(\tau', \tau)$
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- Each vertex means a sum over all lattice sites
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- Each vertex means a sum over all lattice sites
- Internal lines multiply by the hopping matrix

$V_n(\tau')$ multiplies by the hopping matrix

Each line means an integral from 0 to $\beta$

Symmetry factor is number of ways of joining vertices and lines
Diagrammatic Rules

- Vertices represent connected functions $W^{(n)}(\tau', \tau)$
- Each vertex means a sum over all lattice sites
- Internal lines multiply by the hopping matrix
- External lines inward (outward) lines multiply $j^*(j)$
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Quantum Phase Diagram

- System enters the MI phase when $\psi_{eq}$ vanishes
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Phase diagram at second hopping order

F.E.A. Santos, and A. Pelster PRA 79:013614, 2009
Quantum Phase Diagram

- System enters the MI phase when $\psi_{eq}$ vanishes
- Phase diagram at second hopping order
- Error smaller than 3% in 3D

F.E.A. Santos, and A. Pelster PRA 79:013614, 2009
Quantum Phase Diagram

- System enters the MI phase when $\psi_{eq}$ vanishes
- Phase diagram at second hopping order
- Error smaller than 3% in 3D
- Fast convergence: N. Teichmann et. al. PRB 79: 195131

![Graphs showing Quantum Phase Diagram for d=2 and d=3](image)

F.E.A. Santos, and A. Pelster PRA 79:013614, 2009
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Excitation Spectra

- MI phase:
  Particle and hole excitations
Excitation Spectra

- **MI phase:**
  Particle and hole excitations

- **SF phase:**
  Density and phase excitations

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**Bosons in Optical Lattices**

**Ginzburg-Landau Theory**

**Results**

**Summary and Outlook**

**Quantum phase diagram**

**Excitation spectra**

**Collapse and revival of matter waves**
Excitation Spectra

- MI phase: Particle and hole excitations
- SF phase: Density and phase excitations
- Different universality class at the tip
Excitation Spectra

- MI phase: Particle and hole excitations
- SF phase: Density and phase excitations
- Different universality class at the tip

B. Bradlyn, F.E.A. Santos, and A. Pelster PRA 79:013615, 2009
T.D. Grass, F.E.A. Santos, and A. Pelster PRA 84:013613, 2011
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Sample of $2 \times 10^5 \ ^{87}$Rb atoms:


Periodic potential depth suddenly changed from $8E_R$ to $22E_R$
Collapse and Revival of Matter Waves

- Sample of $2 \times 10^5 \ ^{87}\text{Rb}$ atoms: 
- Periodic potential depth suddenly changed from $8 E_R$ to $22 E_R$
- Inhomogeneous chemical potential: $\mu \rightarrow \mu - \frac{m}{2} \omega^2 r_i$
- Wick rotation $\tau \rightarrow it$

F.E.A. dos Santos, and A. Pelster in preparation 84:013613, 2011
A Ginzburg-Landau theory was developed for bosons in optical lattices.

Analytical calculations are performed using diagrammatic methods.

High accuracy to equilibrium and out-of-equilibrium systems.

Outlook:
- Different geometries
- Bose-Fermi mixtures
- Optical QED lattices