PHASE TRANSITIONS AND RENORMALIZATION GROUP: FROM THEORY TO NUMBERS

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Introduction

A fundamental very difficult problem, which the renormalization group allowed solving at the beginning of the 1970th, was the determination of the long distance behaviour and the singularities of correlation functions in systems with short range interactions, at a continuous phase transition.

The early standard approach was based on mean field theory (MFT). MFT predicts that the large distance physics has some very universal properties, that is, properties independent of the specific form of the microscopic interactions, symmetries and dimension of space. However, MFT, while it gives a qualitative description of phase transitions, is inaccurate because it relies on the wrong assumption of scale decoupling.

In the first part, we consider phase transitions corresponds to a breaking of a $\mathbb{Z}_2$ reflection symmetry (the Ising model universality class).

Moreover, we consider only models invariant under space translations and with rotation or hypercubic symmetry.

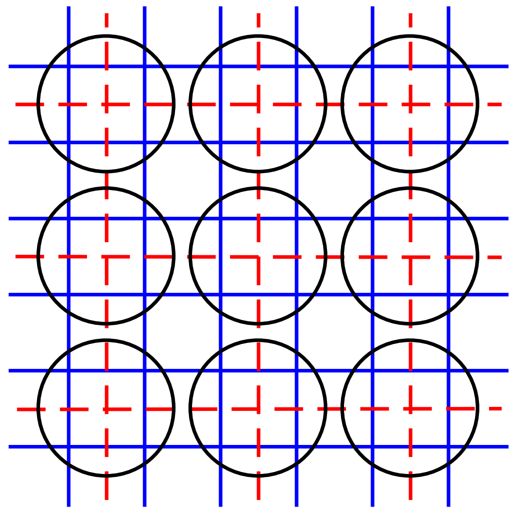

Fig. 1 Decimation: initial (blue) lattice with lattice size a and (red) lattice with size $2a$.

Decimation and Landau–Ginzburg–Wilson Hamiltonian

To understand why MFT is incorrect and to investigate whether some universality nevertheless survives, Kadanoff has proposed a form of renormalization group (RG) based on a decimation idea, which consists in integrating step by step the shortest distance degrees of freedom.

Wilson realized that, even if the initial model is defined in terms of a space lattice and the order parameter takes discrete values (like in the Ising model), after many iterations one can replace the initial model by an effective model in continuum space with, as order parameter, a field $\phi(x)$ that takes continuous values. The partition function is then given by the field integral

$$\mathcal{Z} = \int [\mathrm{d}\phi(x)] \exp\left[-\mathcal{H}(\phi)\right], \tag{1}$$

where the Landau–Ginzburg–Wilson (LGW) Hamiltonian is a linear combination of an infinite number of local monomials in ϕ (as a consequence of short range interactions), that is, products of powers of the field $\phi(x)$ and its derivatives at the same point x, for example,

$$\phi^2(x),\ \phi(x)\left(\nabla_x\right)^{2n}\phi(x),\ \phi^4(x), \ldots .$$

We write the sum symbolically as

$$\mathcal{H}(\phi) = \int \mathrm{d}^d x \sum_\alpha \mathcal{H}_\alpha(\phi, x),$$

where $\mathcal{H}_\alpha(\phi, x)$ is a local monomial. The coefficients multiplying the monomials are regular functions of the thermodynamic parameters like the temperature because only non-critical modes have been eliminated, and are assumed to be generically of order 1.

One coefficient has to be adjusted for the model to be critical, that is, for the correlation length to diverge. For reasons that will become clearer later, one chooses in general the coefficient of $\phi^2(x)$, which thus plays the role of the temperature near T_c.

Finally, and this is a consequence of non-decoupling of scales, a momentum cut-off, providing an artificial short distance structure, is required to ensure that all local monomials have a finite expectation value.

From the viewpoint of the field integral (1) such a cut-off has the effect that only smooth (infinitely differentiable) fields contribute.

Renormalization group arguments are then needed to prove that replacing the initial short distance structure by a simpler artificial structure does not affect the large scale universal properties.

Space rescaling and renormalization group

The next step is to rescale distances, taking the large physical scale as a reference rather than the microscopic scale. Thus we introduce a large momentum Λ or a microscopic length $1/\Lambda$ scale and set

$$x = \Lambda x' .$$

In the Hamiltonian, the contribution of a monomial $\int \mathrm{d}^d x \, \mathcal{H}_\alpha(\phi, x)$ with $2k$ derivatives is then multiplied by Λ^{d-2k}.

Kadanoff's decimation idea then suggests integrating, in Fourier space, over field Fourier modes corresponding approximately to a shell $[\Lambda, \Lambda + \mathrm{d}\Lambda]$. The partial integration over high momentum modes yields Wegner–Wilson's type Renormalization Group (RG) equations, functional flow equations for the effective Hamiltonian.

They have the general form

$$\Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} \mathcal{H}(\phi, \Lambda) = \tfrac{1}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \tilde{D}_\Lambda(k) \left[\frac{\delta^2 \mathcal{H}}{\delta \tilde{\phi}(k) \delta \tilde{\phi}(-k)} - \frac{\delta \mathcal{H}}{\delta \tilde{\phi}(k)} \frac{\delta \mathcal{H}}{\delta \tilde{\phi}(-k)} \right] + \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left(\tilde{D}_\Lambda(k) / \tilde{\Delta}_\Lambda(k) \right) \frac{\delta \mathcal{H}}{\delta \tilde{\phi}(k)} \tilde{\phi}(k). \tag{2}$$

The function $\tilde{D}_\Lambda(k)$ has a support concentrated around $|k| = \Lambda$, implementing the notion of momentum shell integration.

Together, with this partial integration it is necessary to renormalize the field (*cf.*, the factor $\sqrt{n}$ in the central limit theorem of probabilities) to ensure that correlation functions remain finite. One thus sets

$$\phi(x) = \sqrt{Z(\Lambda)} \phi'(x).$$

Thus, a monomial with $2n$ fields and $2k$ derivatives is multiplied by

$$\int \mathrm{d}^d x \, \mathcal{H}_\alpha(\phi, x) \mapsto Z^n(\Lambda) \Lambda^{d-2k} \int \mathrm{d}^d x \, \mathcal{H}_\alpha(\phi, x). \tag{3}$$

Gaussian fixed point and perturbation theory

We consider the quadratic Hamiltonian (for $d>2$)

$$\mathcal{H}(\phi)=\tfrac{1}{2}\int \mathrm{d}^d x\,\left(\nabla_x\phi(x)\right)^2 .$$

It is critical, because the two-point correlation function decreases algebraically like $\langle\phi(x)\phi(y)\rangle\propto|x-y|^{2-d}$, and reproduces the result of mean field theory (MFT) at criticality.

It corresponds to an RG fixed point, the Gaussian fixed point, because a partial integration in a Gaussian integral reproduces a Gaussian integral.

The correlation function $\langle\phi(x)\phi(y)\rangle\propto|x-y|^{2-d}$ is finite after the renormalization (3) if

$$Z(\Lambda)\Lambda^{d-2}=1\ \Rightarrow\ Z(\Lambda)=\Lambda^{2-d}.$$

The field renormalization amounts to give to the field, which initially was dimensionless, a momentum dimension $\frac{1}{2}(d-2)$.

With this choice, adapted to discuss perturbations of the Gaussian fixed point, a monomial with $2n$ fields and $2k$ derivatives is multiplied by

$$\int \mathrm{d}^d x\, \mathcal{H}_\alpha(\phi, x) \mapsto \Lambda^{\delta_{n,k}} \int \mathrm{d}^d x\, \mathcal{H}_\alpha(\phi, x) \text{ with } \delta_{n,k} = d - 2k - n(d-2). \quad (4)$$

Critical domain and mean field theory

To be able to describe physics in the critical domain, in the neighbourhood of the critical temperature T_c, above T_c when the correlation is large but not infinite or below T_c in the several phase region, it is necessary to perturb the quadratic critical Hamiltonian adding local terms like $\int \mathrm{d}^d x\, \phi^2(x)$ and $\int \mathrm{d}^d x\, \phi^4(x)$.

The fundamental assumption justifying MFT is that these unavoidable perturbations can indeed be considered as small perturbations.

Stability of the Gaussian fixed point

The stability of the Gaussian fixed point with respect to local perturbations reduces to the study of the behaviour of the renormalized monomials as given by equation (4) when $\Lambda \to \infty$.

The largest values of

$$\delta_{n,k} = d - 2k - n(d-2), \tag{5}$$

correspond to the largest perturbations.

For $d > 2$ (otherwise the Gaussian fixed point does not exist), the largest values of the dimension of $\delta_{n,k}$ correspond to the smallest values of n and k. We now classify local monomials ordering them according the values of $\delta_{n,k}$.

Quadratic perturbations. In d dimensions, the coefficient of the monomial

$$\frac{1}{2}\int \mathrm{d}^d x\, \phi(x)(\nabla_x)^{2k}\phi(x)$$

is $\Lambda^{d-(d-2)-2k} = \Lambda^{2-2k}$. The term $\phi^2(x)$ ($k=0$) corresponds to a direction of instability. It is called a relevant operator. When it is the only perturbation, its coefficient, which we denote here by r, has to be positive and it induces a finite correlation length ξ proportional to $1/\Lambda\sqrt{r}$. The condition $\xi \gg 1/\Lambda$ then implies a fine-tuning $r \ll 1$, that is, in the initial parametrization the temperature has to be sufficiently close to T_c.

The term with $k=1$ generates a small renormalization of the field and has no physical effect. It is called redundant.

All terms with $k>1$ correspond to directions of stability and are called irrelevant.

The ϕ^4 operator and the role of dimension four. The next most important monomial is

$$\int \mathrm{d}^d x \, \phi^4(x).$$

It is multiplied by Λ^{4-d} and the stability depends on the dimension of space.

For $d > 4$, the monomial is irrelevant and since increasing the power of the field or the number of derivatives makes the power $\delta_{n,k}$ (equation(5)) even more negative, no other monomial can be relevant. Therefore, on the critical surface ($T = T_c$) the Gaussian fixed point is stable. At leading order in the large distance limit, MFT is correct.

For $d = 4$, the coefficient is Λ-independent and the operator is called marginal. The stability of the Gaussian fixed point cannot be determined by this leading order analysis. Higher order perturbative calculations are required to determine whether ϕ^4 is marginally stable or unstable.

All other operators are irrelevant.

For $d < 4$, the ϕ^4 perturbation is relevant and the Gaussian fixed point is unstable. This is reflected by the property that a perturbative expansion in the coefficient of ϕ^4 at criticality does not exist. Universality then depends on the possible existence of another fixed point.

The ϕ^6 operator. The ϕ^6 is multiplied by Λ^{6-3d}. If the contribution of ϕ^4 is tuned to zero (this corresponds to a multicritical point), then the operator is irrelevant for $d > 3$ and marginal to $d = 3$. However, note that if the contribution of ϕ^4 is generic the question of the dimension of the operator ϕ^6 becomes non-perturbative.

The dimension $d = 2$ is special and requires a specific analysis. The Gaussian fixed point does not exist and its stability is no longer an issue. An infinite number of other multicritical fixed points are found for models with an Ising-like $\mathbb{Z}_2$ symmetry.

Moreover, Mermin–Wagner–Coleman's theorem forbids spontaneous symmetry breaking of continuous symmetries in two dimensions.

Dimensional continuation and ϕ^4 field theory

There exists methods to define Feynman diagrams (and thus a field theory but only in a perturbative sense) for generic real or complex values of the dimension d. Integrals for generic d can be manipulated to a large extent like integrals in integer dimensions. One method to define dimensional continuation is to express the propagator (the Gaussian two-point function), in the Fourier representation, as a Laplace transform. An example is

$$\tilde{\Delta}(p) = \int_{1/\Lambda^2}^{\infty} \mathrm{d}t \; \mathrm{e}^{-t(p^2+m^2)} = \frac{\mathrm{e}^{-(p^2+m^2)/\Lambda^2}}{p^2+m^2}\,, \tag{6}$$

where Λ is the cut-off.

The introduction of this representation in Feynman diagrams reduces all momentum integrations to Gaussian integrations that can be explicitly performed. The dependence in the dimension d is then explicit and can be continued to generic real or complex values.

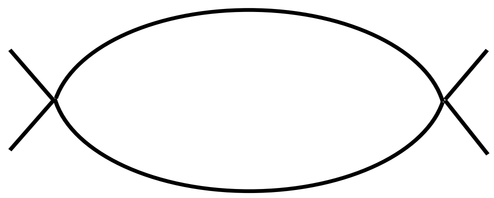

Fig. 2 One-loop contribution to the four-point function.

For example, the Feynman diagram of Fig. 2 in dimension $d < 4$ is given by

$$B_d(p) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d q \, \tilde{\Delta}(q) \tilde{\Delta}(p-q) \underset{\substack{m=0,\\ \Lambda\to\infty}}{\sim} -\frac{2^{3-2d}\pi^{(3-d)/2}}{\sin(\frac{1}{2}\pi d)\Gamma\left(\frac{1}{2}(d-1)\right)} |\mathbf{p}|^{d-4}. \quad (7)$$

Dimensional regularization. For d small enough (possibly negative), all integrals have a finite $\Lambda \to \infty$ limit. The outcome is called dimensional regularization.

Dimensional regularization provides a very effective framework for leading order calculation of universal quantities, based on the introduction of renormalized correlation functions.

The ϕ^4 effective field theory

After dimensional continuation, one can explore the neighbourhood of the Gaussian fixed point for dimensions $d = 4 - \varepsilon$ with $\varepsilon > 0$ small. It is then expected that, on the critical surface, ϕ^4 is still the only relevant perturbation. Therefore, one expects to be able to determine critical properties near $d = 4$, at leading order, from the study of the simple effective Hamiltonian

$$\mathcal{H}(\boldsymbol{\phi}) = \int \mathrm{d}^d x \left\{ \tfrac{1}{2} \left[\nabla_x \boldsymbol{\phi}(x) \right]^2 + \tfrac{1}{2} \Lambda^2 r_c(g) \boldsymbol{\phi}^2(x) + \frac{g \Lambda^{4-d}}{4!} \left[\boldsymbol{\phi}^2(x) \right]^2 \right\}, \quad (8)$$

where the coefficient $r_c(g)$ is determined by the criticality condition, for example, the divergence of the correlation length.

Due to scale non-decoupling, it is necessary to introduce an explicit large-momentum cut-off (a procedure called regularization). Here, to define perturbation theory we need only to restrict the field integration to continuous fields. This can be achieved by adding quadratic irrelevant terms,

$$\left[\nabla_x \boldsymbol{\phi}(x) \right]^2 \mapsto \nabla_x \boldsymbol{\phi}(x) \big(1 - \alpha_1 \nabla_x^2 / \Lambda^2 + \alpha_2 (\nabla_x^2)^2 / \Lambda^4 \big) \nabla_x \boldsymbol{\phi}(x), \ \alpha_1, \alpha_2 > 0 \, .$$

However, for $d < 4$ fixed, the perturbative expansion of the critical theory is still not defined as the factor Λ^{4-d} already suggests, but due to zero momentum divergences, another indication that the Gaussian fixed point is unstable.

For example, the contributions to the two-point function proportional to

$$\int \frac{\mathrm{d}^d q}{(p-q)^2} B_d^n(q) \propto \int \frac{\mathrm{d}^d q}{(p-q)^2 q^{n(4-d)}} \quad \forall n \ ,$$

where B_d is the diagram (7), diverge at $q = 0$ for $n > d/(4-d)$.

The ε-expansion. Apart from its applications in quantum field theory as relevant to particle physics, dimensional continuation is at the basis of Wilson–Fisher's ε-expansion.

While the critical perturbation theory does not exist for $d < 4$ fixed, it can be defined as a double series expansion in powers of the interaction strength g and $\varepsilon = 4 - d$. The ε-expansion has allowed discovering a new fixed point relevant for the large distance behaviour of correlation functions.

The N-vector model

A number of interesting phase transitions are described by the N-vector model, an $O(N)$ symmetric model with an N-component field $\boldsymbol{\phi}(x)$. The partition function reads

$$\mathcal{Z} = \int [\mathrm{d}\boldsymbol{\phi}(x)] \exp\left[-\mathcal{H}(\boldsymbol{\phi})\right],$$

where the Hamiltonian at criticality generalizes expression (8),

$$\mathcal{H}(\boldsymbol{\phi}) = \int \mathrm{d}^d x \left\{ \tfrac{1}{2} \left[\nabla_x \boldsymbol{\phi}(x)\right]^2 + \tfrac{1}{2}\Lambda^2 r_c(g) \boldsymbol{\phi}^2(x) + \frac{g\Lambda^{4-d}}{4!} \left[\boldsymbol{\phi}^2(x)\right]^2 \right\}.$$

The first values of N correspond to the transitions:

$N = 1$: liquid–vapour, binary mixtures, Ising-like magnetic systems

$N = 2$: Helium superfluidity

$N = 3$: isotropic ferromagnetic systems

and the limit $N = 0$ to statistical properties of long polymers.

Perturbative renormalization group

Once the effective Hamiltonian is reduced to the simple form (8), the functional RG equations can be reduced, for Λ large, to partial differential equations. Applied to critical connected correlation functions in Fourier representation, they take the form (Zinn-Justin 1973)

$$\left(\Lambda\frac{\partial}{\partial\Lambda}+\beta(g)\frac{\partial}{\partial g}+\frac{n}{2}\eta(g)\right)\tilde{W}^{(n)}(p_i;g,\Lambda)=0\,, \tag{9}$$

where contributions that are related to other irrelevant operators (from the viewpoint of the Gaussian fixed point) and are subleading by factors $(\ln\Lambda)^{\ell}/\Lambda^2$, ℓ increasing with the order, have been neglected.

These equations can be proved using the results of renormalization theory in quantum field theory. They are formally similar to the more standard RG equations satisfied by renormalized correlation functions.

Fixed points correspond to zeros of the β-function. Zeros with a positive slope are attractive while zeros with a negative slope are repulsive.

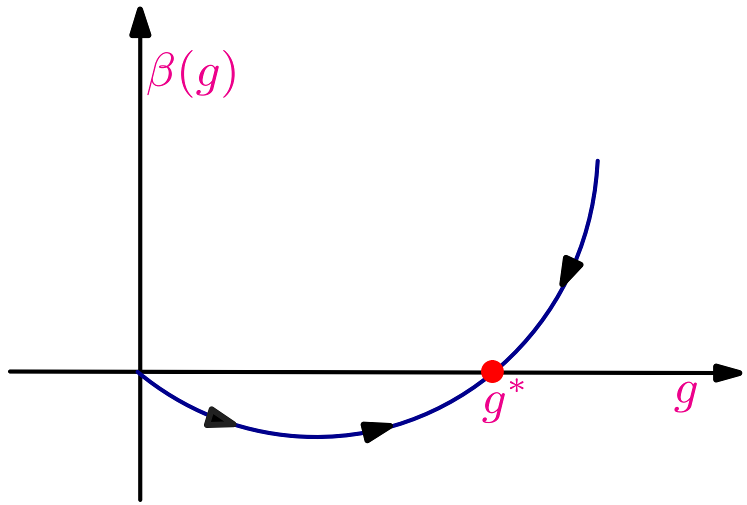

Fig. 3 The RG β-function for $d = 4 - \varepsilon$.

The RG functions $\beta(g)$ and $\eta(g)$ can be calculated in a double series expansion in g and $\varepsilon = 4 - d$. At leading non-trivial order, for the $O(N)$ symmetric model, one finds

$$\beta(g) = -\varepsilon g + \frac{(N+8)g^2}{48\pi^2} + O(g^3, g^2\varepsilon), \;\; \eta(g) = \frac{N+2}{18(4\pi)^4} g^2 + O(g^3, g^2\varepsilon).$$

At order ε, $\beta(g)$ has two zeros $g = 0$ and $g = g^* = 48\pi^2\varepsilon/(N+8)$.

The first corresponds to the Gaussian fixed point, has a slope $-\varepsilon$ and thus is repulsive while the second one g^* has a positive slope $+\varepsilon$ and thus governs the large distance behaviour.

At the fixed point, the RG equations reduce to

$$\left(\Lambda \frac{\partial}{\partial \Lambda} + \frac{n}{2}\eta\right) \tilde{W}^{(n)}(p_i; g^*, \Lambda) = 0 \text{ with } \eta \equiv \eta(g^*) \,.$$

Combining the solution with the dimensional relation

$$\tilde{W}^{(n)}(\lambda p_i, g, \lambda\Lambda) = \lambda^{d-n(d+2)/2} \tilde{W}^{(n)}(p_i, g, \Lambda),$$

one obtains the scaling form

$$\tilde{W}^{(n)}(\lambda p_i, g, \Lambda) \underset{\lambda \to 0}{\propto} \lambda^{d-n(d+2-\eta)/2} \;\Rightarrow\; \tilde{W}^{(2)}(p, \Lambda) \underset{p \to 0}{\propto} \frac{\Lambda^{-\eta}}{p^{2-\eta}} \,.$$

The critical domain

The critical domain is defined by the property that the correlation length is large but not infinite. This situation is realized by modifying the coefficient of $\boldsymbol{\phi}^2$ term in the critical Hamiltonian (8),

$$\mathcal{H}(\boldsymbol{\phi}) = \int \mathrm{d}^d x \left\{ \tfrac{1}{2} \left[\nabla_x \boldsymbol{\phi}(x)\right]^2 + \tfrac{1}{2}\left(\Lambda^2 r_c(g) + t\right)\boldsymbol{\phi}^2(x) + \frac{g\Lambda^{4-d}}{4!}\left[\boldsymbol{\phi}^2(x)\right]^2 \right\},$$

where the constant $t \propto T - T_c$ characterizes the deviation from the critical temperature.

Correlation functions satisfy RG equations of the form

$$\left(\Lambda \frac{\partial}{\partial \Lambda} + \beta(g)\frac{\partial}{\partial g} + \tfrac{1}{2} n\eta(g) - \eta_2(g) t \frac{\partial}{\partial t}\right) \tilde{W}^{(n)} = 0\,,$$

where the new RG function $\eta_2(g)$ appears.

At the fixed point $g = g^*$, the equation reduces to $(\eta_2 \equiv \eta_2(g^*))$

$$\left(\Lambda \frac{\partial}{\partial \Lambda} + \tfrac{1}{2} n\eta - \eta_2 t \frac{\partial}{\partial t}\right) \tilde{W}^{(n)} = 0\,.$$

Using the dimensional relation

$$\tilde{W}^{(2)})(p,t,\Lambda) = \Lambda^2 \tilde{W}^{(2)}(p/\Lambda, t/\Lambda^2, 1)$$

and applying the corresponding RG equation, one finds the scaling relations

$$\tilde{W}^{(2)})(p,t,\Lambda) \propto |t|^{-\gamma} F(p/|t|^{\nu})$$

with

$$\nu = 1/(2+\eta_2), \quad \gamma = \nu(2-\eta).$$

Thus, the exponent ν characterizes the divergence of the correlation length at T_c.

Conclusion. Within the framework of the ε-expansion and using the perturbative renormalization group, one can prove scaling properties to all orders in ε and calculate universal quantities in the form of ε-expansions.

Callan–Symanzik (CS) equations

Another scheme involves working directly in the massive field theory (the critical domain) where the mass $m = 1/\xi$. Within this massive scheme, the perturbative expansion exists in any dimension. However, at fixed dimension $d < 4$, one faces the problem that the coupling constant $g\Lambda^{4-d}$ diverges with the cut-off. One proceeds then in two steps. First, one fixes $g_0 = g\Lambda^{4-d}$ and introduces renormalized correlation functions defined by

$$W_{\mathrm{r}}^{(n)}(p_i, m, g_{\mathrm{r}}) = \lim_{\substack{\Lambda \to \infty \\ m, g_{\mathrm{r}} \text{ fixed}}} Z^{-n/2} W^{(n)}(p_i, t, \Lambda, g_0),$$

where the parameters m, g_{r} and the field renormalization Z are determined by the renormalization conditions

$$\left[\tilde{W}_{\mathrm{r}}^{(2)}(p; m, g_{\mathrm{r}})\right]^{-1} = m^2 + p^2 + O(p^4)$$
$$\tilde{W}_{\mathrm{r}}^{(4)}(0,0,0,0) = 1/g_{\mathrm{r}} m^{4+d}\,.$$

Connected correlations functions then satisfy the CS equations

$$\left[m\frac{\partial}{\partial m}+\beta_{\mathrm{r}}(g_r)\frac{\partial}{\partial g_r}+\frac{n}{2}\eta_{\mathrm{r}}(g_{\mathrm{r}})\right]\tilde{W}_{\mathrm{r}}^{(n)}(p_i;m,g_{\mathrm{r}})=m^2(2-\eta)\tilde{W}_{\mathrm{r},\phi^2}^{(n)}(p_i;m,g_{\mathrm{r}}), \tag{10}$$

where $W_{\mathrm{r},\phi^2}^{(n)}$ correspond to correlation functions with one insertion of the operator $\frac{1}{2}\int \mathrm{d}^d x\,\phi^2(x)$.

In a second step, one takes the infinite $g_0 = g\Lambda^{4-d}$ limit since g is fixed and $\Lambda \to \infty$. One then verifies that when $g_0 \to \infty$, g_{r} converges toward a zero of $\beta_{\mathrm{r}}(g_r)$ with a positive slope.

Within the ε-expansion, such a zero $g^* = O(\varepsilon)$ of the β-function is found and one is back to the scenario of Fig. 3. A universal scaling behaviour is recovered since correlation functions, up to the renormalization constant, depend only on p_i/m.

In the same way, the right hand side of equation (10) is negligible for $|p_i| \gg m$ (but still $|p_i| \ll \Lambda$) and the critical scaling can be proved.

However, for $d < 4$ fixed, there is no small parameter and the zeros of the β-function have to be determined numerically. Moreover, within the perturbative expansion, the right hand side is no longer negligible and the property becomes non-perturbative.

Finally, the validity of the scheme relies on a property that is not rigorously established: that it is possible to take the large Λ limit in two steps.

However, Nickel managed to generate longer series (6 and partially 7 loops) in three dimensions because he noticed that it is easier to calculate Feynman diagrams in three dimensions than in generic dimension d. The results, obtained in this way after summation, are more precise than those coming from the summed ε-expansion.

Table 1

Sum of the successive terms of the ε-expansion of γ and η for $\varepsilon = 1$ and $N = 1$.

k	0	1	2	3	4	5
γ	1.000	1.1667	1.2438	1.1948	1.3384	0.8918
η	0.0 . . .	0.0 . . .	0.0185	0.0372	0.0289	0.0545

Practical calculations: summation of divergent series

The ε-expansion

After a calculation of physical quantities as a double series expansion in powers of g and ε, one first solves the equation $\beta(g) = 0$ in the form of a ε-expansion. One then inserts the fixed point value of g in other physical quantities. Table 1 shows immediately that for $\varepsilon = 1$, the successive partial sums for two exponents up to the available order ε^5 do not converge.

The fixed dimension scheme

Following Parisi's suggestion, one can also evaluate the β-function directly in dimension 3 but, since there is no longer a 'small' expansion parameter, a summation method is required. Nickel managed to calculate, in dimension 3, all diagrams contributing to η, η_2 up to seven loops (in the terminology of Feynman diagrams), and the diagrams contributing to the β-function, which are more difficult, only up to six loops.

For example, to six loop order, for $N = 1$, Nickel has obtained

$$\begin{aligned}\beta(\tilde{g}) = -\tilde{g} + \tilde{g}^2 - \tfrac{308}{729}\tilde{g}^3 + 0.3510695978\tilde{g}^4 \\ - 0.3765268283\tilde{g}^5 + 0.49554751\tilde{g}^6 - 0.749689\tilde{g}^7 + O\left(\tilde{g}^8\right),\end{aligned}$$

where $\tilde{g} = 3g/(16\pi))$.

Like for the ε-expansion, the series are also divergent. The difference is that one must then first determine numerically the zero of the β-function, which is a number of order 1.

But in both cases a summation of the series is required.

Large order behaviour and instanton calculus

The apparent divergence of the perturbative expansion has a simple explanation: for any $g < 0$ the Hamiltonian is not bounded from below, thus $g = 0$ corresponds to a singularity. It is expected that physical observables are functions analytic in a cut-plane with a cut on the whole negative axis.

The ϕ^4 Hamiltonian (8) has also the interpretation of the action of a quantum field theory in imaginary time. The imaginary part of observables on the cut when $g < 0$ is then related to quantum barrier penetration.

In the semi-classical limit (here g plays the formal role of $\hbar$), quantum barrier penetration is associated with propagation in imaginary time. The imaginary part is related the solutions of classical field equations in imaginary time with finite action, called instantons.

The instanton solution with the smallest action gives the leading contribution.

If A is the smallest instanton action, for any observable $F(g)$ in the ϕ^4 field theory, one finds

$$\operatorname{Im} F(g) \underset{g \to 0_-}{\propto} \mathrm{e}^{A/g} \,.$$

The function $F(g)$ is a real function, analytic in a cut-plane. It has the Cauchy representation

$$F(g) = \frac{1}{\pi} \int_{-\infty}^{0} \frac{\operatorname{Im} F(g')}{g' - g} \mathrm{d}g' .$$

The function $F(g)$ can be expanded in a power series for $g \to 0_+$. Then,

$$F(g) = \sum_k F_k g^k \,, \;\; F_k = \frac{1}{\pi} \int_{-\infty}^{0} \mathrm{d}g g^{-k-1} \operatorname{Im} F(g).$$

The behaviour of the integral for $k \to \infty$, is governed by the behaviour of $\operatorname{Im} F(g)$ for $g \to 0_-$. Thus,

$$F_k \underset{k \to \infty}{\propto} \int^{0_-} \frac{\mathrm{e}^{A/g}}{g^k} \mathrm{d}g \propto (-1)^k A^{-k} k! \,.$$

Instanton solutions

In the tree approximation, in the CS framework, for $N = 1$, the Hamiltonian reduces to (here g_r is denoted g)

$$\mathcal{H}(\phi) = \int \mathrm{d}^d x \left[\tfrac{1}{2}\bigl(\nabla_x \phi(x)\bigr)^2 + \tfrac{1}{2} m^2 \phi^2(x) + \tfrac{1}{4!} g m^{4-d} \phi^4(x)\right].$$

The field equation is

$$\frac{\delta \mathcal{H}}{\delta \phi(x)} = (-\nabla_x^2 + m^2)\phi(x) + g m^{4-d} \phi^3(x)/6 = 0\,.$$

The smallest action comes from solutions of the form

$$\phi(x) = \frac{1}{\sqrt{-g}} m^{(d-2)/2} f(mr), \quad r = |x|.$$

The function f for $d < 4$ satisfies the non-linear differential equation

$$-\ddot{f} + (d-1)\dot{f}/r + f - f^3/6 = 0\,.$$

The equation can be solved numerically for $d = 3$ and $d = 2$. A more precise calculation of the instanton contributions leads to, for example, for the β-function in three dimensions:

$$\beta_k \underset{k\to\infty}{\propto} (-a)^k k^b k!$$

with $a = 0.147774232\ldots$.

Borel summation

To deal with the divergence problem, when the coupling constant g is not small, it is necessary to introduce summation techniques. In three dimensions, the perturbative expansion is proved to be Borel summable. It is thus natural to introduce the Borel–Laplace transformation (here, Borel–Leroy):

$$B_\sigma(g) = \sum_k \frac{\beta_k}{\Gamma(k+\sigma+1)} g^k .$$

Then, formally in the sense of power series

$$\beta(g) = \int_0^{+\infty} t^\sigma \, \mathrm{e}^{-t} \, B_\sigma(gt) \mathrm{d}t \, .$$

From the large order behaviour, we infer that the function $B_\sigma(g)$ is analytic in a circle of radius $1/a$ with a singularity at $-1/a$. The series is said Borel summable if, in addition, $B_\sigma(g)$ is analytic in a neighbourhood of the real positive semi-axis and the integral converges.

Table 2

Series summed by a method based on Borel transformation and mapping for the zero $\tilde{g}^*$ of the $\beta(g)$ function and the exponents γ and ν in the ϕ^4_3 field theory.

k	2	3	4	5	6	7
$\tilde{g}^*$	1.8774	1.5135	1.4149	1.4107	1.4103	1.4105
ν	0.6338	0.6328	0.62966	0.6302	0.6302	0.6302
γ	1.2257	1.2370	1.2386	1.2398	1.2398	1.2398

Since the series defines $B_\sigma(g)$ only in a circle, an analytic continuation is required. In practice, with a small number of terms, the continuation requires a large domain of analyticity. Le Guillou and Zinn-Justin (1977–1980) have assumed maximal analyticity, that is, analyticity in a cut-plane.

The continuation has then be obtained by a conformal mapping of the cut-plane onto a circle.

Slight modification of summation techniques and the additional seven-loop contributions have lead later to improved estimates of critical exponents (Guida, Zinn-Justin 1998).

A systematic comparison between these field theory and renormalization group based calculations and available experimental results, as well as lattice calculations, shows excellent agreement. Nevertheless, in the case of superfluid Helium transition, low gravity experiments have given

$$\nu = 0.6705 \pm 0.0006\,, \quad \nu = 0.6708 \pm 0.0004$$
$$\alpha = -0.01285 \pm 0.00038\,,$$

a precision that is now a challenge to field theory, which yields

$$\nu = 0.6703 \pm 0.0015\,, \quad \alpha = -0.011 \pm 0.004\,.$$

A noticeable improvement could be expected from a seven-loop calculation of the β-function, since the value of g^* enters in the calculation of all other universal quantities.

However, to give an idea of the problem one faces, at seven-loop about 3500 diagrams have to be evaluated, which are integrals of rather singular functions over 21 variables.

A set of technical tricks, some already used Nickel, and a complete automatisation of the calculation (Guida–Ribeca), which, in particular, allows finding many sub-integrations that can be performed analytically, reduces somewhat the difficulty. However, the problem of automatic numerical integration of the irreducible diagrams remains unsolved.

Reference: R. Guida and J. Zinn-Justin, *J. Phys. A* 31 (1998) 8103, cond-mat/9803240, an improvement over the results published in

J.C. Le Guillou and J. Zinn-Justin, *Phys. Rev. Lett.* 39 (1977) 95; *Phys. Rev.* B21 (1980) 3976.

Critical exponents from the $O(N)$ symmetric $(\phi^2)^2_3$ field theory

N	0	1	2	3
$\tilde{g}^*$	1.413 ± 0.006	1.411 ± 0.004	1.403 ± 0.003	1.390 ± 0.004
g^*	26.63 ± 0.11	23.64 ± 0.07	21.16 ± 0.05	19.06 ± 0.05
γ	1.1596 ± 0.0020	1.2396 ± 0.0013	1.3169 ± 0.0020	1.3895 ± 0.0050
ν	0.5882 ± 0.0011	0.6304 ± 0.0013	0.6703 ± 0.0015	0.7073 ± 0.0035
η	0.0284 ± 0.0025	0.0335 ± 0.0025	0.0354 ± 0.0025	0.0355 ± 0.0025
β	0.3024 ± 0.0008	0.3258 ± 0.0014	0.3470 ± 0.0016	0.3662 ± 0.0025
α	0.235 ± 0.003	0.109 ± 0.004	-0.011 ± 0.004	-0.122 ± 0.010
ω	0.812 ± 0.016	0.799 ± 0.011	0.789 ± 0.011	0.782 ± 0.0013
$\omega\nu$	0.478 ± 0.010	0.504 ± 0.008	0.529 ± 0.009	0.553 ± 0.012

Critical exponents from the $O(N)$ symmetric $(\phi^2)^2_3$ field theory

N	0	1	2	3
γ	1.1596 ± 0.0020	1.2396 ± 0.0013	1.3169 ± 0.0020	1.3895 ± 0.0050
ν	0.5882 ± 0.0011	0.6304 ± 0.0013	0.6703 ± 0.0015	0.7073 ± 0.0035
α	0.235 ± 0.003	0.109 ± 0.004	-0.011 ± 0.004	-0.122 ± 0.010
β	0.3024 ± 0.0008	0.3258 ± 0.0014	0.3470 ± 0.0016	0.3662 ± 0.0025
$\omega\nu$	0.478 ± 0.010	0.504 ± 0.008	0.529 ± 0.009	0.553 ± 0.012

Critical exponents from $O(N)$ symmetric lattice models

N	0	1	2	3
γ	1.1575 ± 0.0006	1.2385 ± 0.0025	1.322 ± 0.005	1.400 ± 0.006
ν	0.5877 ± 0.0006	0.631 ± 0.002	0.674 ± 0.003	0.710 ± 0.006
α	0.237 ± 0.002	0.103 ± 0.005	-0.022 ± 0.009	-0.133 ± 0.018
β	0.3028 ± 0.0012	0.329 ± 0.009	0.350 ± 0.007	0.365 ± 0.012
$\omega\nu$	0.56 ± 0.03	0.53 ± 0.04	0.60 ± 0.08	0.54 ± 0.10

Table 6

Critical exponents in the $(\phi^2)^2_3$ *field theory from the* ε*-expansion.*

N	0	1	2	3
γ (free)	1.1575 ± 0.0060	1.2355 ± 0.0050	1.3110 ± 0.0070	1.3820 ± 0.0090
γ (bc)	1.1571 ± 0.0030	1.2380 ± 0.0050	1.317	1.392
ν (free)	0.5875 ± 0.0025	0.6290 ± 0.0025	0.6680 ± 0.0035	0.7045 ± 0.0055
ν (bc)	0.5878 ± 0.0011	0.6305 ± 0.0025	0.671	0.708
η (free)	0.0300 ± 0.0050	0.0360 ± 0.0050	0.0380 ± 0.0050	0.0375 ± 0.0045
η (bc)	0.0315 ± 0.0035	0.0365 ± 0.0050	0.0370	0.0355
β (free)	0.3025 ± 0.0025	0.3257 ± 0.0025	0.3465 ± 0.0035	0.3655 ± 0.0035
β (bc)	0.3032 ± 0.0014	0.3265 ± 0.0015		
ω	0.828 ± 0.023	0.814 ± 0.018	0.802 ± 0.018	0.794 ± 0.018
θ	0.486 ± 0.016	0.512 ± 0.013	0.536 ± 0.015	0.559 ± 0.017

Equation of state

Using the series provided by Nickel, combined with a few new technical tricks, it has been possible to obtain a precise representation of the equation of state for models in the $N = 1$ Ising class. In particular, from the equation of state, a number of universal combinations of amplitudes of the singularities at T_c can be derived (see table 10). For example, the magnetic susceptibility, diverges at T_c with a susceptibility exponent γ and

$$\chi_+ \sim C_+(T - T_c)^{-\gamma}, \quad \chi_- \sim C_-(T_c - T)^{-\gamma}.$$

The ratio C_+/C_- is universal.

The singular part of the specific heat behaves like

$$\mathcal{C}_+ \sim A_+(T - T_c)^{-\alpha}, \quad \mathcal{C}_- \sim A_-(T_c - T)^{-\alpha},$$

and the ratio A_+/A_- is also universal.

Reference: R. Guida and J. Zinn-Justin, *Nucl. Phys.* B489 [FS] (1997) 626.

Table 10

Amplitude ratios: models and binary critical fluids.

	ε-expansion	Fixed dim. $d = 3$	Lattice models	Experiment
A^+/A^-	0.527 ± 0.037	0.537 ± 0.019	$\begin{cases} 0.523 \pm 0.009 \\ 0.560 \pm 0.010 \end{cases}$	0.56 ± 0.02
C^+/C^-	4.73 ± 0.16	4.79 ± 0.10	$\begin{cases} 4.75 \pm 0.03 \\ 4.95 \pm 0.15 \end{cases}$	4.3 ± 0.3
f_1^+/f_1^-	1.91	2.04 ± 0.04	1.96 ± 0.01	1.9 ± 0.2
R_ξ^+	0.28	0.270 ± 0.001	0.266 ± 0.001	0.25–0.32
R_c	0.0569 ± 0.0035	0.0574 ± 0.0020	0.0581 ± 0.0010	0.050 ± 0.015
$R_\xi^+ R_c^{-1/3}$	0.73	0.700 ± 0.014	0.650	0.60–0.80
R_χ	1.648 ± 0.036	1.669 ± 0.018	1.75	1.75 ± 0.30
Q_2	1.13		1.21 ± 0.04	1.1 ± 0.3
Q_3	0.96		0.896 ± 0.005	