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Ultracold quantum gases in optical lattices, continued

Superfluid-Mott insulator transition

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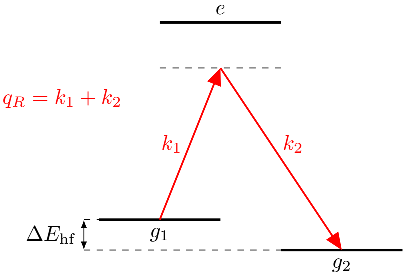
Measurement of the fine structure constant α :

$$\alpha^2 = \frac{4\pi R_\infty}{c} \times \frac{M}{m_e} \times \frac{\hbar}{M}$$

R_∞: Rydberg constant
m_e: electron mass
M: atomic mass

- possible window on physics beyond QED : interactions with hadrons and muons, constraints on theories postulating an internal structure of the electron, ...

Measurement of $\frac{\hbar}{M}$: Experiment in the group of F. Biraben (LKB, Paris)

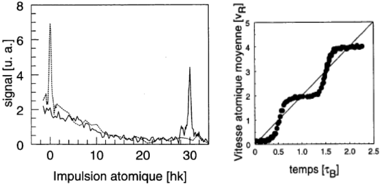


Doppler-sensitive Raman spectroscopy :

$$\hbar\omega_{\rm res} = \Delta E + \frac{\hbar^2}{2M}\left(p_i + \Delta k + q_R\right)^2$$

$$\Longrightarrow \frac{\hbar}{M} = \frac{\omega_{\rm res}(p_i + \Delta k) - \omega_{\rm res}(p_i)}{q_R \Delta k}$$

Large momentum beamsplitter using Bloch oscillations :

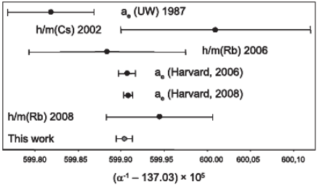


After N Bloch oscillations, momentum transfer of $\Delta k = 2N\hbar k_L$ to the atoms in the lab frame.

This transfer is perfectly coherent and enables beamsplitters where part of the wavepacket remains at rest while the other part is accelerated.

Measurement of $\frac{\hbar}{M}$:

- $N \sim 10^3$: Comparable uncertainty as current best measurement (anomalous magnetic moment of the electron – Gabrielse group, Harvard).
- Independent of QED calculations



[Bouchendira *et al.*, PRL 2011]

- Other applications in precision measurements: measurement of weak forces, *e.g.* Casimir-Polder [Beaufils *et al.*, PRL 2011].

1. Bose-Hubbard model
2. Ground state : Superfluid -Mott insulator transition
3. Phase coherence
4. Dynamics and transport
5. Shell structure

Basic Hamiltonian for bosons interacting *via* short-range forces :

$$\hat{H} = \hat{H}_0 + \hat{H}_{\rm int},$$
$$\hat{H}_0 = \int d\mathbf{r} \;\; \hat{\Psi}^\dagger(\mathbf{r})\Big[-\frac{\hbar^2}{2M}\Delta + V_{\rm lat}(\mathbf{r})\Big]\hat{\Psi}(\mathbf{r}),$$
$$\hat{H}_{\rm int} = \frac{g}{2}\int d^{(3)}\mathbf{r} \;\hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}(\mathbf{r})\hat{\Psi}(\mathbf{r}).$$

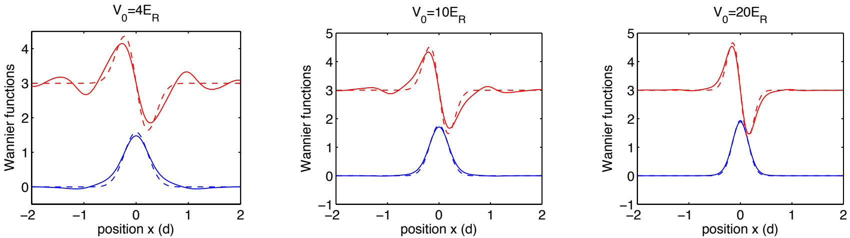
- $\hat{\Psi}(\mathbf{r})$: field operator annihilating a boson a position $\mathbf{r}$
- $V_{\rm lat}(\mathbf{r})$: lattice potential
- $g = 4\pi\hbar^2 a/m$: coupling constant (scattering length a)

Not simpler in the Bloch basis.

We are going to introduce a new basis of so-called Wannier functions that permit to simplify drastically the problem.

Wannier functions : $w_n(x - x_n) = \frac{1}{\sqrt{N_s}} \sum_{q \in BZ1} e^{-iqx_n} \phi_{n,q}(x)$
discrete Fourier transforms with respect to the site locations of the Bloch wave functions,

- All Wannier functions deduced from $w_n(x)$ by translations.
- Exactly N_s such functions per band (as many as Bloch functions).
- Basis of Hilbert space (but not an eigenbasis of $\hat{H}$).

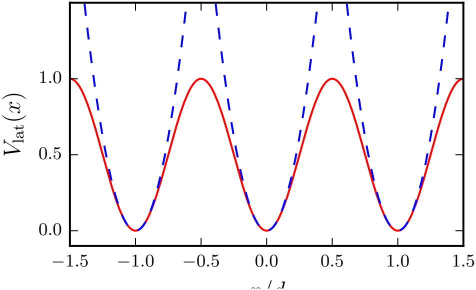


Cautionary note: Bloch functions are defined up to a $q-$dependent phase which needs to be fixed to obtain localized Wannier functions [W. Kohn, Phys. Rev. (1959)].

Harmonic approximation for each well :

$$V_{\text{lat}}(x \approx x_i) \approx \frac{1}{2} M\omega_{\text{lat}}^2 (x - x_i)^2,$$
$$\hbar\omega_{\text{lat}} = 2\sqrt{V_0 E_R}.$$

The bands are centered around the energy $\overline{E}_n \approx (n + 1/2)\hbar\omega_{\text{lat}}$.



First correction : quantum tunneling across the potential barriers

Bloch basis :

$$H = \sum_{n,k \in BZ1} \varepsilon_n(k) \hat{b}^\dagger_{n,k} \hat{b}_{n,k}.$$

$\hat{b}_{n,k}$: annihilation operator for Bloch state (n, k).

Wannier basis :

$$H = -\sum_{n,i,j} J_n(i-j) \hat{a}^\dagger_{n,i} \hat{a}_{n,j},$$

$\hat{a}_{n,i}$: annihilation operator for Wannier state $w_n(x - x_i)$.

Tunneling matrix elements :

$$J_n(i-j) = \int dx \; w_n^*(x - x_j) \left(\frac{\hbar^2}{2m} \Delta - V_{\rm lat}(x) \right) w_n(x - x_i).$$

(also called hopping parameters) depend only on the relative distance $x_i - x_j$ between the two sites.

Useful form :

$$J_n(i-j) = -\frac{1}{N_s} \sum_{q \in BZ1} \varepsilon_n(q) e^{iq\cdot(x_i - x_j)}.$$

Tight-binding limit

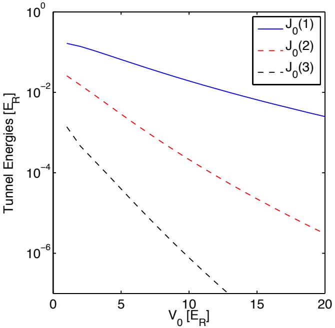
For deep lattices (roughly $V_0 \gg 5E_R$), the tunneling energies fall off exponentially quickly with distance.

- On-site term $(i = j)$: Mean energy of band n

$$J_n(0) = -\frac{1}{N_s}\sum_{q\in BZ1} \varepsilon_n(q) = -\overline{E}_n$$

- Nearest-neighbor tunneling $(i = j \pm 1)$:

$$J_n(1) = -\frac{1}{N_s}\sum_{q\in BZ1} \varepsilon_n(q)e^{iqx} = -J_n$$



Two useful approximations :

- Tight-binding approximation : keep only the lowest terms
- Single-band approximation : keep only the lowest band–drop band index and let $J_0(1) \equiv J$

$$\hat{H}_{TB} = \sum_i \overline{E}_0 \hat{a}_i^\dagger \hat{a}_i - \sum_{\langle i,j\rangle} J \hat{a}_i^\dagger \hat{a}_j,$$

Basis of Wannier functions $W_\nu(\boldsymbol{r}-\boldsymbol{r}_i)$:

$$\hat{\Psi}(\mathbf{r}) = \sum_{\nu,i} W_\nu(\boldsymbol{r}-\boldsymbol{r}_i)\hat{a}_{\nu,i}.$$

$\boldsymbol{r}_i$: position of site i,
ν : band index
$\hat{a}_{\nu,i}$: annihilation operator

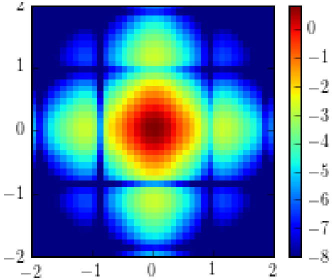
$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}},$$

$$\hat{H}_0 \to \hat{H}_{TB} = -\sum_{\langle i,j\rangle} J\hat{a}_i^\dagger \hat{a}_j,$$

$$\hat{H}_{\text{int}} \to \frac{1}{2}\sum_{i,j,k,l} U_{ijkl}\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_k\hat{a}_l,$$

$$U_{ijkl} = g\int d^3\boldsymbol{r}\, W^*(\boldsymbol{r}-\boldsymbol{r}_i)W^*(\boldsymbol{r}-\boldsymbol{r}_j) \times W(\boldsymbol{r}-\boldsymbol{r}_k)W(\boldsymbol{r}-\boldsymbol{r}_l)$$

$\log(|W_(x,y,0)|^2)$ for $V_0 = 5E_R$:



In the tight binding regime, strong localization of Wannier function $W(\boldsymbol{r}-\boldsymbol{r}_i)$ around $\boldsymbol{r}_i$. In the interaction energy, on-site interactions ($i=j=k=l$) are strongly dominant.

1. Single band approximation
2. Tight-binding approximation
3. On-site interactions

Bose-Hubbard model :

$$H_{\rm BH} = -J \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j + \frac{U}{2} \sum_i \hat{n}_i \left(\hat{n}_i - 1 \right).$$

$\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$: operator counting the number of particles at site i.

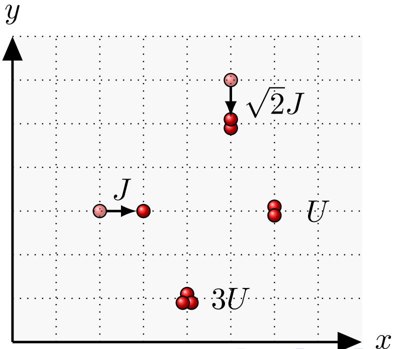
- Tunneling energy :

$$J = \frac{\max \varepsilon(\boldsymbol{q}) - \min \varepsilon(\boldsymbol{q})}{2z}$$

$z = 6$: number of nearest neighbors

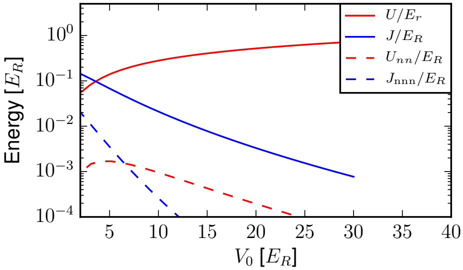
- On-site interaction energy :

$$U = g \int d\mathbf{r} \; w(\mathbf{r})^4.$$



Parameters of the Bose Hubbard model

Calculation for ^{87}Rb atoms [a=5.5 nm] in a lattice at $\lambda_L = 820$ nm:



Harmonic oscillator approximation :

$$\frac{\Delta_{\rm band}}{E_R} \approx \frac{\hbar\omega_{\rm lat}}{E_R} = \sqrt{\frac{2V_0}{E_R}},\ \ \frac{U}{E_R} \approx \sqrt{\frac{8}{\pi}} k_L a \left(\frac{V_0}{E_R}\right)^{3/4}.$$

1. Single band approximation :
   - $V_0 \gg E_R$
   - $U \ll \Delta_{\rm band}$: $k_L a \ll \left(\frac{E_R}{V_0}\right)^{1/4}$
2. Tight-binding approximation : $V_0 \gg 5E_R$
3. On-site interactions : $V_0 \gg E_R$

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BEC in the lowest energy Bloch state $\boldsymbol{q} = 0$:

$$|\Psi\rangle_N = \frac{1}{\sqrt{N!}} \left(\hat{b}^\dagger_{\mathbf{q}=0}\right)^N |\varnothing\rangle = \frac{1}{\sqrt{N!}} \left(\frac{1}{\sqrt{N_s}} \sum_i \hat{a}_i^\dagger \right)^N |\varnothing\rangle$$

- Fixed number of particles N: canonical ensemble

Probability to find n_i atoms at one given site i :

$$p(n_i) \approx e^{-\overline{n}} \frac{\overline{n}^{n_i}}{n_i!} + \mathcal{O}\left(\frac{1}{N}, \frac{1}{N_s}\right)$$

Poisson statistics, mean $\overline{n}$, standard deviation $\sim \sqrt{\overline{n}}$ In the thermodynamic limit

$N \to \infty, N_s \to \infty$, one finds the same result as for a coherent state with the same average number of particles N:

$$|\Psi\rangle_{\rm coh} = \mathcal{N} e^{\sqrt{N} \hat{b}^\dagger_{\mathbf{q}=0}} |\varnothing\rangle = \prod_i \left(\mathcal{N}_i e^{\sqrt{\overline{n}} \hat{a}_i^\dagger} |\varnothing\rangle \right)$$

- Fluctuating number of particles N: grand canonical ensemble
  $H_{\rm BH} \to G = H_{\rm BH} - \mu N$

BEC in the lowest energy Bloch state $\boldsymbol{q} = 0$, grand canonical ensemble :

$$|\Psi\rangle_{\text{coh}} = \prod_i |\alpha_i\rangle, \qquad |\alpha_i\rangle = \mathcal{N}_i \sum_{n_i=0}^{\infty} \frac{\alpha_i^{n_i}}{\sqrt{n_i!}} |n_i\rangle_i$$

One can relate the presence of the condensate to a non-zero expectation value of the matter wave field $\alpha_i = \langle \hat{a}_i \rangle$, playing the role of an order parameter :

- Condensate wavefunction : $\alpha_i = \langle \hat{a}_i \rangle = \sqrt{\frac{N}{N_s}} e^{i\phi}$
- Mean density : $\overline{n} = |\alpha_i|^2 =$ condensate density

Spontaneous symmetry breaking point of view.

Starting point to formulate a Gross-Pitaevskii (weakly interacting) theory :

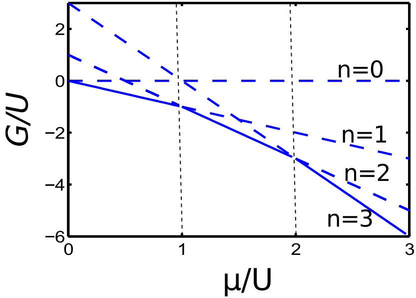
variational ansatz with self-consistent α_i determined by the total (single-particle + interaction)Hamiltonian.

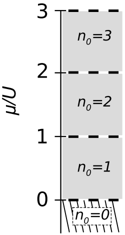
"Adiabatic continuation" from the ideal Bose gas.

Lattice $\equiv$ many independent trapping wells

Many-body wavefunction : product state running over all lattice sites $|\Psi\rangle = \prod_i |\overline{n}\rangle$

Free energy for one well: $G_{J=0} = \frac{U}{2}\hat{n}_i(\hat{n}_i - 1) - \mu\hat{n}_i$
with $\overline{n} = \mathrm{int}(\mu/U) + 1$: (integer) filling that minimizes $\mathcal{H}_{\mathrm{int}}$.





$\mu/U = p$ integer : p and $p+1$ degenerate, on-site wave function = any superposition of the two.

Gutzwiller ansatz for the ground state

Variational wavefunction :

$$|\Psi\rangle_{\text{Gutzwiller}} = \prod_i |\phi_i\rangle,$$

$$|\phi_i\rangle = \sum_{n_i=0}^{\infty} c(n_i)|n_i\rangle_i.$$

- Correct in both limits $J \to 0$ and $U \to 0$
- Minimize $\langle\Psi|H_{\text{BH}} - \mu\hat{N}|\Psi\rangle$ with respect to $\{c(n_i)\}$ with the constraint $\overline{n} = \sum_{n=0}^{\infty} |c(n)|^2 n$

Equivalent to mean-field theory : $\alpha_i = \langle\hat{a}_i\rangle = \langle\phi_i|\hat{a}_i|\phi_i\rangle \neq 0$

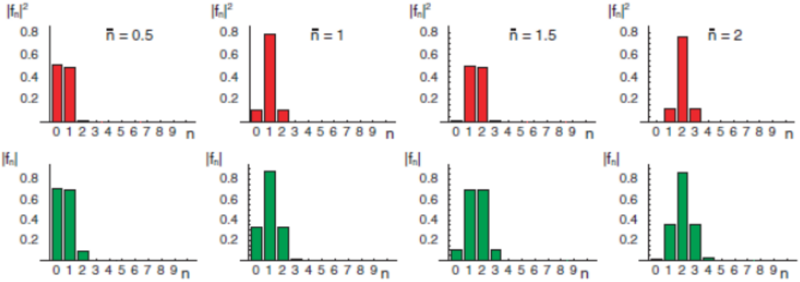
$$\begin{aligned}\langle\Psi|H_{\text{BH}} - \mu\hat{N}|\Psi\rangle &= -J\sum_{\langle i,j\rangle}\alpha_i^*\alpha_j + \sum_i\langle\phi_i|\left(\frac{U}{2}\left(\hat{n}_i^2 - \hat{n}_i\right) - \mu\hat{n}_i\right)|\phi_i\rangle \\ &= \sum_i\langle\phi_i|\hat{h}_i|\phi_i\rangle\end{aligned}$$

Effective single-site Hamiltonian : $\hat{h}_i = -J\left(\sum_{j\text{nn}}\alpha_j^*\right)\hat{a}_i + \frac{U}{2}\left(\hat{n}_i^2 - \hat{n}_i\right) - \mu\hat{n}_i.$

Uniform system : $|\phi\rangle_i$ identical for all sites i

Strong interactions $U \geq J$: on-site number fluctuations become costly.

The on-site statistics $p(n_i)$ evolves from a broad Poisson distribution to a peaked one around some integer n_0 closest to the average filling : number squeezing.



Numerical calculation close to Mott transition [from M. Greiner's PhD thesis (2003)]

Analytical results [Altman & Auerbach, PRL 2001] :

- Truncate local Hilbert space to three states $|n_0-1\rangle, |n_0\rangle, |n_0+1\rangle$ (n_0 closest integer to the (fixed) filling fraction $\overline{n}$),
- For Commensurate filling : $\overline{n}=n_0$, parametrize the amplitudes as : $c(n_0\pm 1)=\frac{1}{\sqrt{2}}\sin(\theta)$, $c(n_0)=\cos(\theta)$ $(\theta\in[0,\pi/2])$.

Variational free energy :

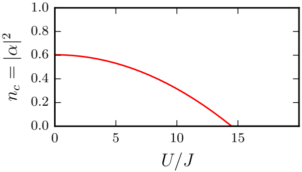
$$\langle \mathcal{G}_{\rm BH}\rangle_{\rm Gutzwiller}=\mathcal{G}_{J=0}+\frac{U}{2}\sin^2(\theta)-\frac{zJA(n_0)}{2}\sin^2(2\theta)$$

with $A(n_0)=(\sqrt{n_0}+\sqrt{n_0+1})^2/4$ and with $z=6$ the number of nearest neighbors.

$$\text{minimum}=\begin{cases}\sin(2\theta)=0 & \text{if } U\geq U_c \quad \text{(Mott insulator)}\\ \cos(2\theta)=\frac{U}{U_c} & \text{if } U\leq U_c \quad \text{(superfluid)}\end{cases}$$

critical interaction : $U_c=4zJA(n_0)$

OP for $\overline{n}=1$



Order parameter for $U\leq U_c$:

$$\alpha=\langle\hat{a}_i\rangle=\sqrt{\frac{A(n_0)}{2}}\sin(2\theta)=\sqrt{\frac{A(n_0)}{2}}\sqrt{1-\left(\frac{U}{U_c}\right)^2}$$

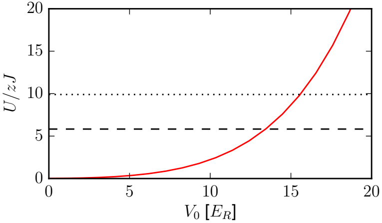
Transition from a delocalized superfluid state to a localized Mott insulator state above a critical interaction strength U_c

Superfluid:

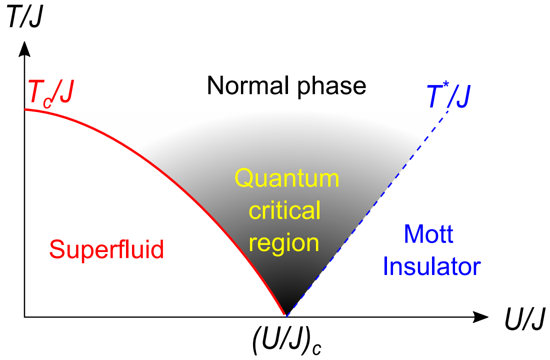
- non-zero condensed fraction $|\alpha|^2$
- on-site number fluctuations
- Gapless spectrum
- Long wavelength superfluid flow can carry mass across the lattice

Mott insulator :

- zero condensed fraction
- on-site occupation numbers pinned to the *same* integer value
- Energy gap $\sim U$ (far from transition)
- No flow possible unless one pays an extensive energy cost $\sim U$



Mott transition : prototype of a quantum phase transition driven by two competing terms in the Hamiltonian



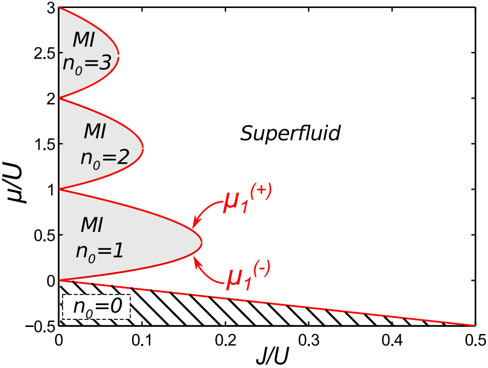
- Different from standard phase transition (competition between energy and entropy)
- Thermal crossover in the Mott insulator regime
- Strongly fluctuating quantum critical region

Generalization to incommensurate fillings:

Superfluid stable when

$$\mu_{n_0}^{(+)} \leq \mu \leq \mu_{n_0+1}^{(-)}.$$

$\mu_{n_0}^{(\pm)}$: upper/lower boundaries of the Mott region with occupation number n_0



$$\mu_{n_0}^{(\pm)} = U(n_0 - \frac{1}{2}) - \frac{zJ}{2} \pm \sqrt{U^2 - 2UzJ(2n_0+1) + (zJ)^2}$$

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Time-of-flight interferences

Evolution of field operator after suddenly switching off the lattice :

$$\hat{\psi}(\mathbf{r}, t=0) = \sum_i W(\mathbf{r}-\mathbf{r}_i)\hat{b}_i \rightarrow \hat{\psi}(\mathbf{k}) \propto \tilde{W}(\mathbf{K}) \sum_i e^{i\mathbf{K}\cdot\mathbf{r}_i}\hat{b}_i$$

Time of flight signal, far-field regime ($\mathbf{K} = \frac{M\mathbf{r}}{\hbar t}$) :

$$n_{\mathrm{tof}}(\mathbf{K}) = \langle \hat{\psi}^\dagger(\mathbf{K})\hat{\psi}(\mathbf{K})\rangle \approx \mathcal{G}(\mathbf{K})\,\mathcal{S}(\mathbf{K})$$

- $\mathcal{G}(\mathbf{K}) = \left(\frac{M}{\hbar t}\right)^3 |\tilde{W}(\mathbf{K})|^2$

  smooth enveloppe function

- $\mathcal{S}(\mathbf{K}) = \sum_{i,j} e^{i\mathbf{K}\cdot(\mathbf{r}_j - \cdot\mathbf{r}_i)}\langle \hat{b}_i^\dagger b_j\rangle$

  structure factor.

Key quantity : single-particle correlation function (also called $g^{(1)}(\boldsymbol{r}, \boldsymbol{r}'))$

$$\mathcal{C}(i,j) = \langle \hat{b}_i^\dagger b_j\rangle$$

Determines the structure factor and the interference pattern (or lack thereof)

Time-of-flight interferences across the Mott transition

$$\mathcal{C}(i,j) = \langle \hat{b}_i^\dagger b_j \rangle \qquad \mathcal{S}(\mathbf{K}) = \sum_{i,j} e^{i\mathbf{K}\cdot(\mathbf{r}_j - \cdot \mathbf{r}_i)} \langle \hat{b}_i^\dagger b_j \rangle$$

Superfluid/BEC :

$$\mathcal{C}_{\mathrm{BEC}}(i,j) = \sqrt{\overline{n}_i \overline{n}_j}$$

$$\mathcal{S}_{\mathrm{SF}}(\mathbf{K}) \approx |\sum_i e^{i\mathbf{K}\cdot\mathbf{r}_i} \sqrt{\overline{n}_i}|^2,$$

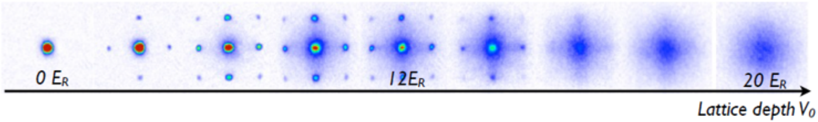
Bragg spots (height $\sim N_s^3$, width $\sim 1/N_s$)

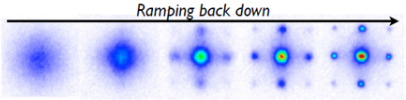
Mott insulator :

$$\mathcal{C}_{\mathrm{Mott}}(i,j) = n_0 \delta_{i,j}$$

$$\mathcal{S}_{\mathrm{Mott}}(\mathbf{K}) \approx N_s$$

Featureless





M. Greiner *et al.*, Nature 2002

Atomic limit $J = 0$:

- ground state : $|\Psi_0\rangle = \prod_i |n_0\rangle_i$, Energy : $E_0 = N_s \left(\frac{U}{2} n_0(n_0 - 1) - \mu n_0\right)$.
- Lowest excited states :

$$\text{Particle states: } \begin{cases} |p : \boldsymbol{r}_i\rangle & = \frac{1}{\sqrt{n_0+1}} \hat{a}_i^\dagger |\Psi_0\rangle, \\ \text{Energy} & E_p = E_0 + U n_0 - \mu \end{cases}$$

$$\text{Hole states: } \begin{cases} |h : \boldsymbol{r}_i\rangle & = \frac{1}{\sqrt{n_0}} \hat{a}_i |\Psi_0\rangle, \\ \text{Energy} & E_h = E_0 + \mu - U(n_0 - 1) \end{cases}$$

- another interpretation of the superfluid-Mott boundaries : instability towards proliferation of quasi-particle or holes

Ground state for $J \ll U$, small but finite:

- The true ground state is not a regular array of Fock state (as found from mean-field theory), but mixes "bound" particle-hole excitations.

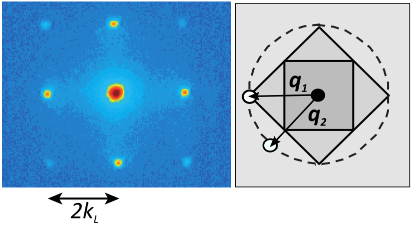
$$|\Psi_g\rangle \approx \Psi_0 + \sum_{\nu \neq 0} \frac{J}{E_\nu - E_0} \hat{T} |\Psi_0\rangle = \Psi_0 + \frac{J\sqrt{n_0(n_0+1)}}{U} \sum_{\langle i,j \rangle} |p : \boldsymbol{r}_i ; h : \boldsymbol{r}_j\rangle$$

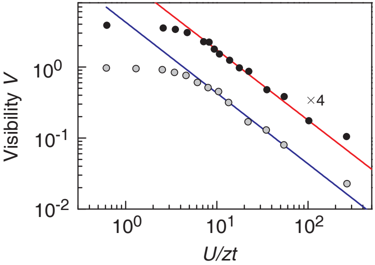
- Short-range phase coherence : $\mathcal{C}(i,j) \sim \left(\frac{J}{U}\right)$ for i, j nearest neighbours.

Structure factor:

$$\mathcal{S}(\boldsymbol{q}) = N_s \left(1 + \frac{2zJn_0(n_0+1)}{U}\gamma_{\boldsymbol{q}}\right)$$

- $\delta = \pm \boldsymbol{e}_x, \pm \boldsymbol{e}_y, \pm \boldsymbol{e}_z$: vector joining nearest neighbors
- $\gamma_{\boldsymbol{q}} = \frac{2}{z}\sum_{\alpha=x,y,z}\cos(q_\alpha d)$: dimensionless form of the single-particle dispersion relation.





Gerbier *et al.*, PRL 2005

Visibility :

$$\mathcal{V} = N_s \frac{\mathcal{S}(\boldsymbol{q}_1) - \mathcal{S}(\boldsymbol{q}_2)}{\mathcal{S}(\boldsymbol{q}_1) + \mathcal{S}(\boldsymbol{q}_2)}.$$

MI transition : $V_0 \approx 13\,E_R$ for $n_0 = 1$, $V_0 \approx 15\,E_R$ for $n_0 = 2$

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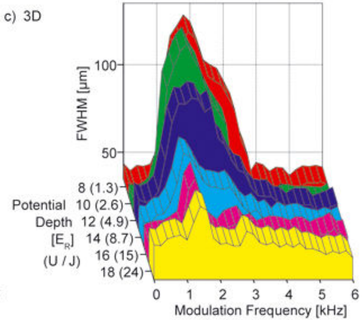
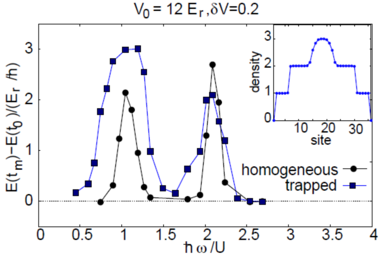
5. Shell structure

- Modulation of the lattice height : $V_0(t) = V_0 + \delta V_0 \cos(\omega_{\mathrm{mod}} t)$
- Main effect for deep lattices : $\delta \hat{V} = -\delta J \cos(\omega_{\mathrm{mod}} t) \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j$
- Superfluid regime : broad response at all frequencies
- Mott insulator regime : Coupling to particle-hole excitations $\Longrightarrow$ peaks at $\omega_{\mathrm{mod}} \approx \frac{U}{\hbar}$



Exp: Schori *et al.*, PRL 2004

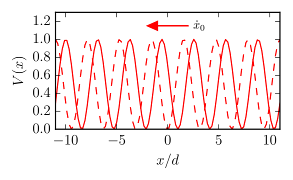


Th.: Kollath *et al.*, PRL 2006

Uniformly accelerated lattice : $V_{\rm lat}[x - x_0(t)]$ with $x_0 = -\frac{Ft^2}{2m}$

Lab frame:

$$H_{\rm lab} = \frac{p^2}{2m} + V_{\rm lat}[x - x_0(t)]$$

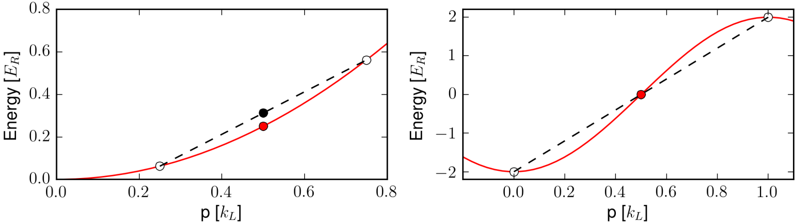


Non-interacting atoms undergo Bloch oscillations.

What happens with interactions ?

Collision of two atoms with momentum $\boldsymbol{p}_0$:

$$2\boldsymbol{p}_0 = \boldsymbol{p}_1 + \boldsymbol{p}_2$$
$$2\varepsilon(\boldsymbol{p}_0) = \varepsilon(\boldsymbol{p}_1) + \varepsilon(\boldsymbol{p}_2)$$

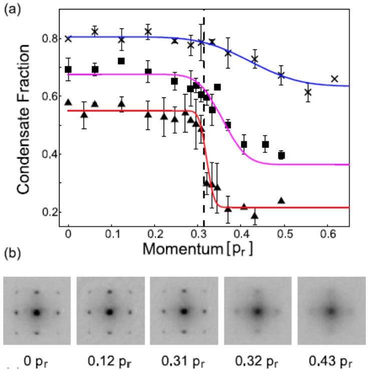


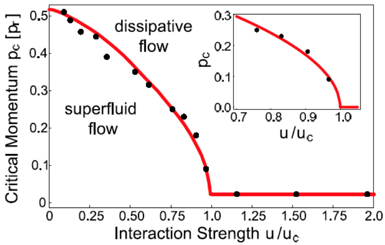
Because of the band structure, collisions redistributing quasi-momentum in the Brillouin zone are kinematically allowed in a lattice (for $|q| > \frac{\pi}{2d}$ in 1D).

This leads to a *dynamical instability* of wavepackets exceeding a certain critical velocity ($v_c = \frac{\hbar k_L}{M}$ in 1D).

MIT experiment [Mun *et al.*, PRL 2007]:

- moving lattice dragging the cloud along
- $p_r = \hbar k_L$: momentum unit
- cycle the lattice back and forth through the cloud (period 10 ms)





Critical point near $U/J \approx 34.2(2)$

Mean field theory predicts 34.8 , quantum Monte-Carlo 29.3 : ?

1. Bose-Hubbard model

2. Ground state : Superfluid -Mott insulator transition

3. Phase coherence

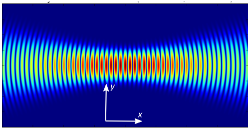
4. Dynamics and transport

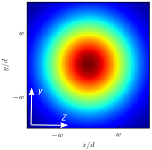
5. Shell structure

Non-uniform lattices : Harmonic trapping

Actual laser beams have Gausssian profile :
Lattice potential of the form

$$V_{\rm 1D} = -V_0 \cos^2\left(k_L x\right) e^{-2\frac{y^2+z^2}{w^2}}.$$





In Wannier basis, additional potential energy term :
$\delta V_{\rm x} \approx \sum_i \frac{1}{2} M\Omega^2 \left(y_i^2 + z_i^2\right) \hat{a}_i^\dagger \hat{a}_i$, with $\Omega^2 \approx \frac{8V_0}{Mw_x^2}\left(1 - \frac{k_L \sigma_w}{2}\right)$.

For a 3D lattice : $\delta V \approx \sum_i V_h(\boldsymbol{r}_i)\hat{a}_i^\dagger \hat{a}_i$, with V_h a harmonic potential.

Local density approximation for a smooth potential :

$$\mu_{\rm loc}(\mathbf{r}) = \mu - V_{\rm h}(\mathbf{r}),$$

μ : global chemical potential fixed by constraining the total atom number to N

The density profile is given by the equation of state $n[\mu]$ for the uniform system, evaluated at $\mu = \mu_{\rm loc}(\mathbf{r})$.

Shell structure in a trap

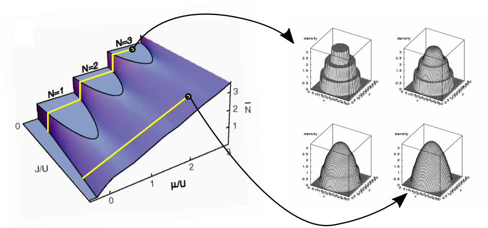
An insulator is incompressible :
Within a Mott lobe, changing the chemical potential does not change the density.

Consequence of the gap for producing particle/hole excitations, which vanishes at the phase boundaries.

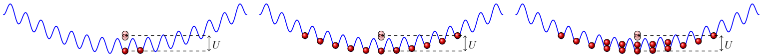
Consequence : non-uniform density profile in a trap

Density profile in the LDA given by $n_{\rm uniform}\left[\mu - V_{\rm h}(\mathbf{r})\right]$.

- **Superfluid :** density changes smoothly from the center of the cloud to its edge
- **Mott insulator :** density changes abruptly; plateaux with uniform density



Simple picture in 1D :



Munich experimental setup (Sherson *et al.* 2010) :

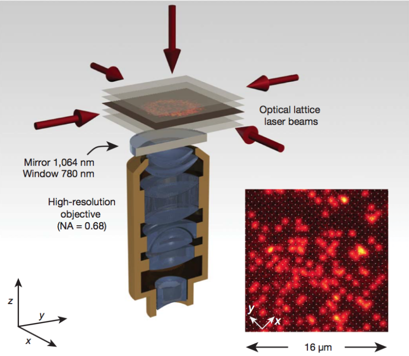
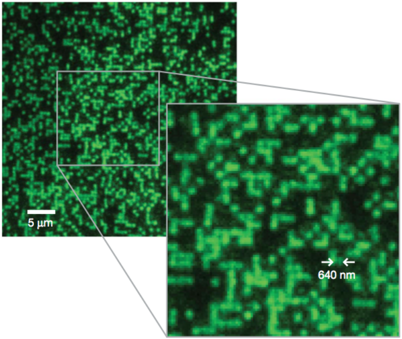
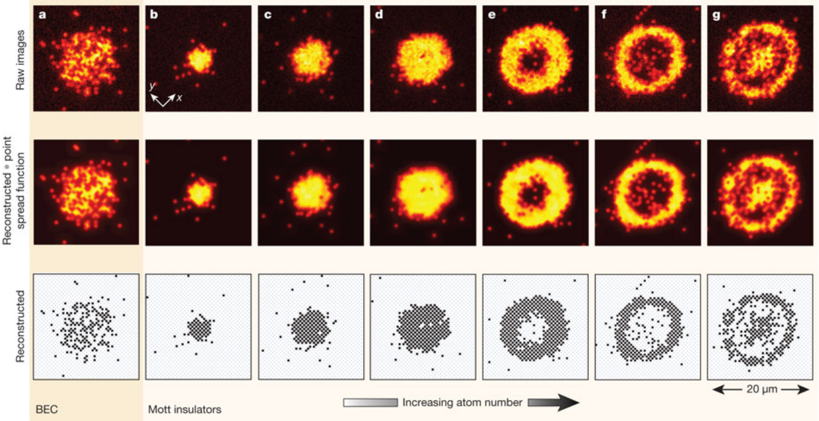


image of a dilute gas

image of a Bose-Einstein condensate in a 2D lattice [Bakr *et al.*, 2010]:

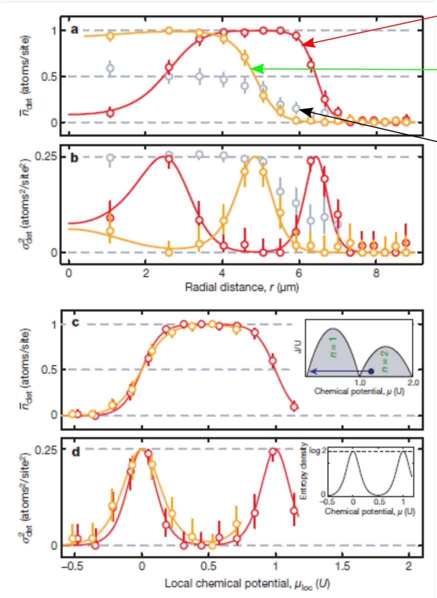


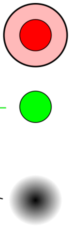
In-situ images of a BEC and of Mott insulators [Sherson *et al.*, Nature 2010]:



Total atom number (or chemical potential) increases from left to right.
Lowest row : reconstructed map of the atom positions, obtained by deconvolution of the raw images to remove the effect of finite imaging resolution.

Mott shells and LDA

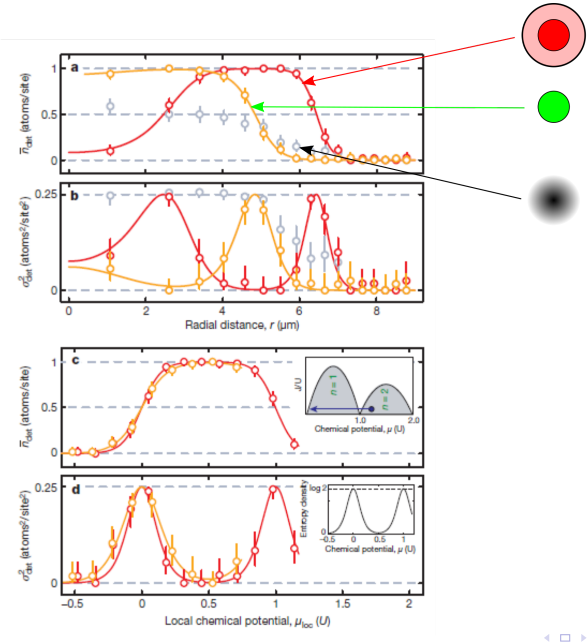




Mott insulator,
n_0=1 and n_0=2 shells

Mott insulator,
n_0=1 and n_0=2 shells

BEC
n_{max}~1

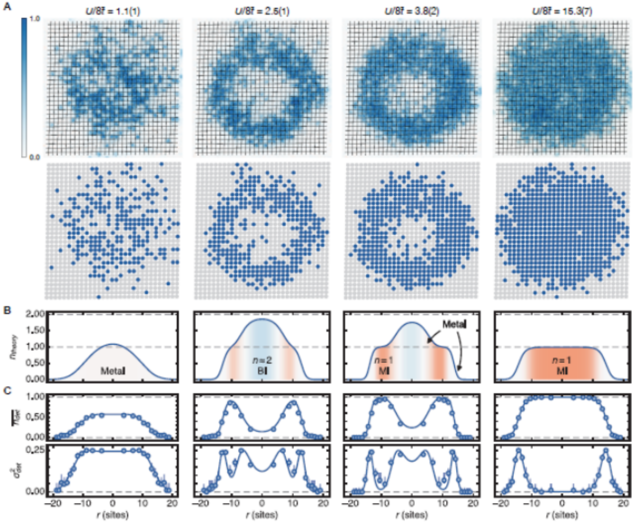


Mott insulator,
n_0=1 and n_0=2 shells

Mott insulator,
n_0=1 and n_0=2 shells

BEC
n_{max}~1

Two-component fermions with repulsive interactions: Fermionic Mott insulator



[Greif *et al.*, arxiv1511.06366 (2015)]
Fermionic quantum gas microscopes also demonstrated in : Haller *et al.*, arxiv1503.02005 (2015); Cheuk *et al.*, arxiv1503.02648 (2015); Omran *et al.*, arxiv1510.04599 (2015).

Optical lattices are an essential element in the experimental toolkit of modern atomic physics. We covered a couple of applications :

- Coherent manipulation of atomic wavepackets : interferometry and metrology,
- Realization of new strongly correlated states of matter, *e.g.* a Mott insulator state.

Many other examples and prospects for the future :

- Spectroscopy without Doppler broadening : optical atomic clocks,
- Dynamical optical lattices in optical cavities,
- Realization of artificial magnetic fields and topological phases of matter,
- Fermionic quantum gases and Hubbard models,
- long-range interactions (dipole-dipole),
- ...

- current experiments achieve $S/Nk_B \sim 1$,
- many interesting phases are awaiting below that scale
- example : two-component repulsive Hubbard model :
  - antiferromagnetic Néel phase below $S_c/Nk_B \lesssim 0.5$
  - d-wave superconductors ? unknown but certainly much lower entropy.
- limits of the current "cooling then adiabatic transfer" technique,

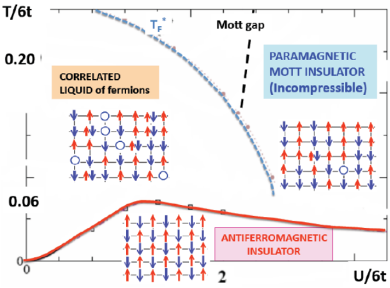


figure from [Georges & Giamarchi, arxiv1308.2684 (2014)]

New methods to cool atoms directly in the lattice badly needed.

See reviews for a more detailed discussion :

[McKay & DeMarco, , Rep. Prog. Phys. **74**, 0544401 (2011)],

[Georges & Giamarchi, arxiv1308.2684 (2014)]

Thank you for your attention !