Lecture notes: Topological phases

Phase transitions generics (informal)

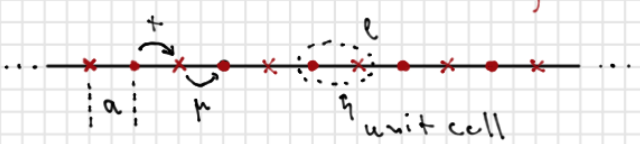
- Phases: distinct macro-states of matter
- distinct phases often distinguished by different local symmetries (cf. ferro-/paramagnetic phase)
- different symmetries characterized by local order parameters
- $\exists$ classical phase transitions and quantum phase transitions (changes in ground states at $T=0$ driven by control parameter)

Q: What happens if topology enters the game?

Will discuss three case studies:

- SSH-chain (a topological insulator)
- BKT-phase transition
- $\mathbb{Z}_2$- spin liquid

The Su-Schrieffer-Heeger (SSH) chain (1978)



(periodic b.c.)

Diagonalize problem in terms of 2-component Bloch wave functions

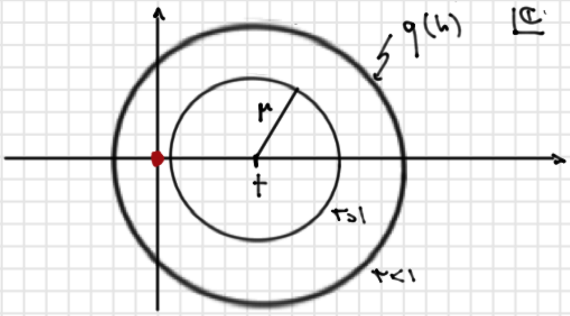
- $\psi_k(l) = \begin{pmatrix} \psi_\times \\ \psi_\bullet \end{pmatrix}_k e^{ikl}$ $\qquad k = 0, \frac{2\pi}{L}, \dots, \frac{\pi}{a}$ $\qquad L = N(2a) =$ chain length

- $\hat{H}_k = \begin{array}{c} \begin{pmatrix} & q_k \\ \bar{q}_k & \end{pmatrix} \end{array} \begin{array}{c} \times \\ \bullet \end{array}$ $\qquad q_k = t + e^{ik\cdot 2a}\mu$

- Eigenvalues: $\varepsilon_k^{\pm} = \pm |q_k| = \left(t^2 + \mu^2 + 2t\mu\cos(2ka)\right)^{1/2}$

assume $t, \mu \in \mathbb{R}$ for simplicity

- interpretation: $r \equiv \frac{t}{\mu} = 1$ marks quantum phase transition between two distinct insulating phases. No change in symmetry, no local order parameter.
- topological order: ground states of $r<1$ and $r>1$ carry distinct topological invariant

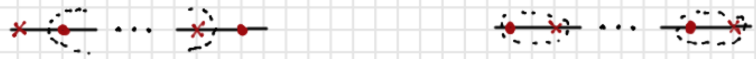


- curve $S^1 \to \mathbb{C} \setminus \{0\}$

  $k \mapsto q(k)$

  homotopically trivial/non-trivial for $r>1$ / $r<1$

- observable difference: cut system open depending on

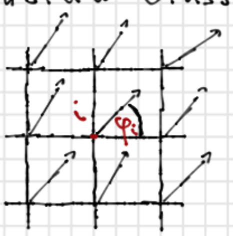


the system with $r>1$ has/has not two zero energy boundary states sharply ($\mathcal{O}(a)$) localized at system boundaries. The insulating $r>1$ phase has a 'conducting surface'.

- Summary: $\exists$ (second order) topological quantum phase transitions without symmetry changes / local order parameter.

The Berezinsky-Kosterlitz-Thouless (BKT) transition

- Consider classical xy-model in 2d



$$Z = \int \underset{\prod_i d\varphi_i}{\underset{\shortparallel}{\mathcal{D}\varphi}} \exp\left(-\mathfrak{J} \sum_{\langle i,j\rangle} \cos(\varphi_i - \varphi_j)\right) \qquad \mathfrak{J} = \overset{\text{ferro. exchange}}{\mathfrak{J}_f} / T > 0$$

- Mermin-Wagner Thm.: no symmetry breaking phase transition

  ... and yet the model has a phase transition

- Consider spin-spin correlation function $C_{i,j} = \langle e^{i\varphi_i} e^{-i\varphi_j} \rangle$

1) Low T: $\mathfrak{J} \gg 1$

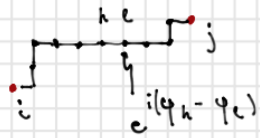
$$S[\varphi] \equiv \mathfrak{J} \sum_{\langle i,j\rangle} \cos(\varphi_i - \varphi_j) \simeq -\frac{\mathfrak{J}}{2} \int d^2x\, \nabla\varphi \nabla\varphi + \text{const.}$$

$$\langle (\varphi(x_i) - \varphi(x_j))^2 \rangle = \ln\left(|x_i - x_j| / \underset{\text{lattice spacing}}{a}\right) / \pi\mathfrak{J}$$

$$C_{i,j} = e^{-\frac{1}{2}\langle (\varphi(x_i) - \varphi(x_j))^2 \rangle} = \left(\frac{a}{|x_i - x_j|}\right)^{1/2\pi\mathfrak{J}}$$

$\leadsto$ power law correlations

2) High T, $\xi \ll 1$ ⇝ Expand action in ξ



$C_{ij} \sim \xi^{d_{ij}} \sim e^{-\ln\xi \cdot c \cdot |x_i - x_j|}$ exponentially decaying cov.

d_{ij}: Manhattan metric

Conclusion: There must be a finite T phase transition. What discriminates 1) from 2)?: 2) accounts for phase windings of compact var. φ.

Strategy: Teach 1) to include windings

Vortices
- cannot be individually destroyed, however
- vortices ('charges') and anti-vortices (anti-charges) may annihilate each other.

a vortex

a vortex/anti-vortex pair

- Individual vortex at r_0 described by, e.g., $\varphi(r) = \tan^{-1}\left(\frac{(r-r_0)_2}{(r-r_0)_1}\right) + \frac{\pi}{2}$ + small fluctuations $(|r-r_0| \gtrsim a)$
- have action cost $(r_0 = 0)$: • $\nabla\varphi = \frac{1}{r} e_\varphi$

$$S_v = \frac{\xi}{2}\int_a^L r\,dr \int_0^{2\pi} d\varphi\, \frac{1}{r^2} + S^{uv}(a) =$$

$S^{uv}(a)$: short distance action

$$= \pi\xi \ln\left(\frac{L}{a}\right) + S^{uv}(a)$$

⇝ A single vortex costs action ~ $\ln L$

- Q: Will vortices be present in the system?

Free energy of a vortex: $F_v = -T\ln Z_v = -T\ln e^{-S_v} \times \left(\frac{L}{a}\right)^2$

$\left(\frac{L}{a}\right)^2$: # of different vortex center coord.

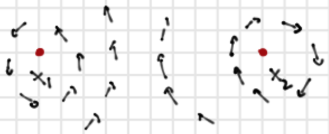
$$= \ln\left(\frac{L}{a}\right) \times \{\pi\xi_f - T\,2\} + \text{const.}$$

$\pi\xi_f$: vortex energy; $T\,2$: entropy

A: For $T \geq T_{BKT} = \frac{\pi}{2}\xi_f$ vortex formation becomes favorable.

• Q: What happens in high temperature phase?

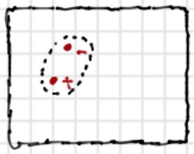
A: Estimate action of vortex anti-vortex pair



$$S(+,x_1;-,x_2) = 2S^{UV}(a) + 2\pi\xi \ln\left(|x_1-x_2|/a\right)$$

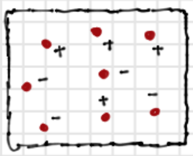
cf. action of two particles in 2d with electric charge $\pm q(a) \hat{=} \pm 2\pi\xi$ and fugacity $y(a) \hat{=} \exp(-S^{UV}(a))$

1) low $T < T_{BKT}$

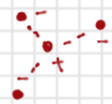


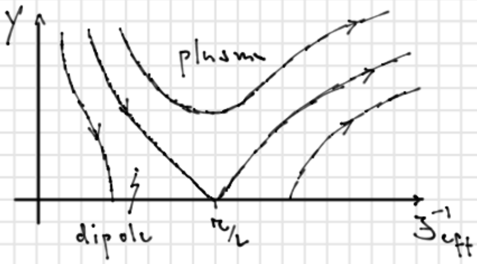
thermal fluctuations: tightly bound dipoles

2) high $T > T_{BKT}$



plasma of charged particles

• In depth analysis: two scaling variables $q(a) \sim \xi_{eff}(a)$, $y(a)$. These two flow (depend on) cutoff a. At lower resolution scales screening  leads to 'renormalization' →



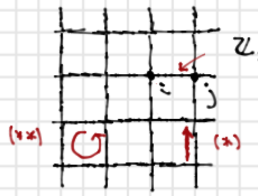
• Summary: • topological defects may behave like particles and
  • interact
  • They may drive a finite T classical phase transition
  • without symmetry breaking

$\mathbb{Z}_2$-lattice gauge theory (Wegner 71, review Kogut RMP 79)

- $\mathbb{Z}_2$-degrees of freedom frequently emerging in correlated fermion systems (Senthil & Fisher, PRB 62, 7850 (2000)): $\underset{\text{fermion}}{c^\dagger} \to \underset{\text{charge}}{e^{i\phi}} \cdot \underset{\text{neutral fermion}}{f^\dagger} = e^{i\phi}(-1)\times(-1)\, f^\dagger$

  ⇝ an emergent 'gauge degree of freedom'

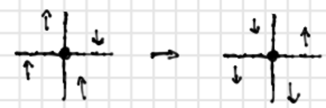
- Formulate $\mathbb{Z}_2$ gauge theory on a lattice. (Here 2d □-lattice for simplicity)



$\mathbb{Z}_2$- d.o.f. $|\pm\rangle_{ij}$ eigenstates of op $\sigma_{z,ij}$.

- local gauge trafo at i acts through $\prod_{n.n.\, i} \sigma_{x,ij}$

  ⇝ $|\pm\rangle_{ij} \to |\mp\rangle_{ij}$



- dynamical players of $\mathbb{Z}_2$ lattice gauge theory
  - gauge field along link $i \to j$: $\sigma_{z,ij}$ ($\hat{=}\ e^{iA_{ij}}$ in U(1) lattice ED)
  - electric flux through link $i \to j$: $\sigma_{x,ij}$ (*). $\quad \sigma_z \sigma_x \sigma_z = -\sigma_x \quad (\hat{=}\ e^{iA} E e^{-iA} = E+1)$
  - magnetic flux through plaquette: $\sigma_{z,ij}\sigma_{z,jk}\sigma_{z,kl}\sigma_{z,li}$ (*) $\quad (\hat{=}\ e^{iA_{ij}} \cdots e^{iA_{li}})$

  electric and magnetic flux are gauge invariant

- gauge invariant Hamiltonian

$$\hat{H} = -\gamma \sum_{\text{links}} \overset{(*)}{\sigma_{x,ij}} - \lambda \sum_{\square} \overset{(**)}{\sigma_{z,ij}\sigma_{z,jk}\sigma_{z,kl}\sigma_{z,li}}$$

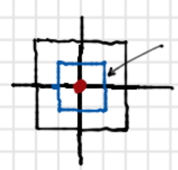
- system supports quantum phase transition driven by $r = \lambda/\gamma$

  1) confining phase $r \leq 1$
  2) topological phase $r \geq 1$

1) Confining phase

- Define 'charge density op': $\hat{\rho}_i = \sum_j \sigma_{x,ij}$

  Charge is locally conserved by $\hat{H}$: $[\hat{H}, \hat{\rho}_i] = 0$



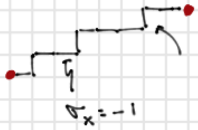
flip all spins cut by this dual lattice loop

- Ground state in charge neutral sector: $\forall i \ \ \hat{\rho}_i |\Psi\rangle = 0$:

  $\sigma_{x,ij} = 1$ globally

• Q: What is ground state in sector of Hilbert space with two charges sitting at i, j?

A: $(\lambda = 0)$

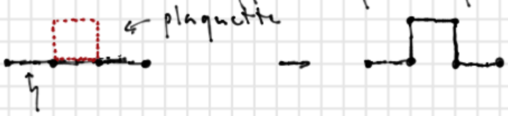


a minimal electric flux line. Costs energy $2\gamma\, \overbrace{d(i,j)}^{\text{Manhattan}} = \underbrace{\sigma}_{\text{string tension}}\, d(i,j)$

Energy grows linearly with distance: confinement

• Q: What is the effect of the flux term on this

A:



$\sim$ string fluctuations $\sigma \to 2\gamma - \overbrace{\lambda^2}^{\text{strength of pert.}} / \underbrace{4\gamma}_{\text{excitation energy}}$

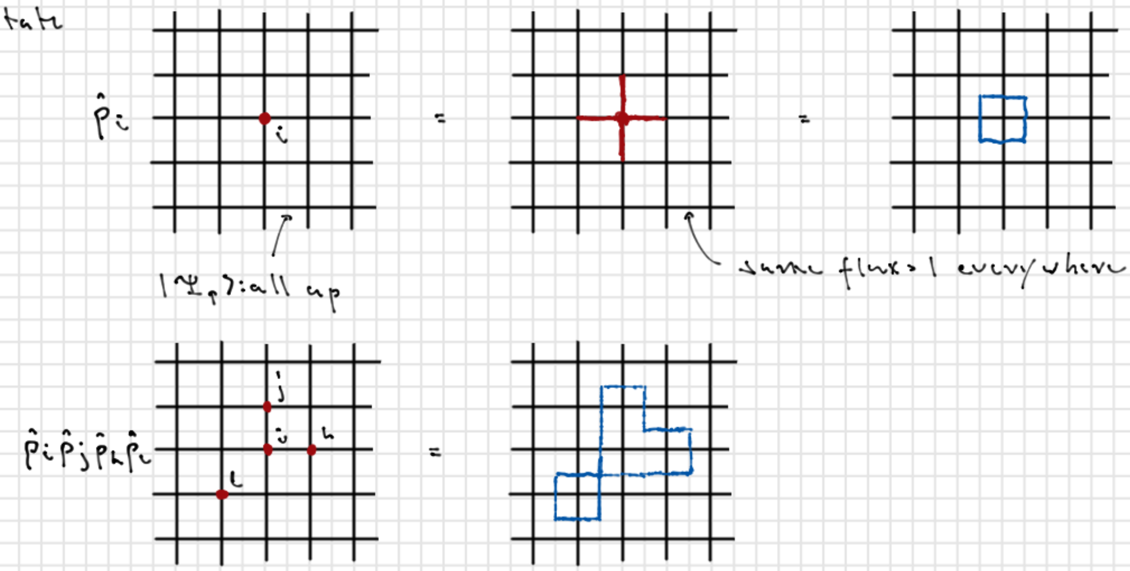
magnetic flux term softens string tension

2) Spin liquid phase

• Consider limit $\gamma = 0$. Ground state has flux $\prod_{\square} \sigma_z \dots \sigma_z$ on each plaquette. This is an implicit characterization.

• Q: How does the ground state in a sector of fixed charge look like?

A: Start from all spins $\sigma_{z,ij} = 1$ state $|\Psi_\uparrow\rangle$. This is not a charge eigenstate



ground state: state of all equal weight superposition of closed strings (cf. BCS), a string net condensate

Corollary: charges totally deconfined.

Q: Is the ground state unique?

A: Def.: V: no. of vertices, E: no. of edges, F: no. of faces

Compactify surface (for simplicity), e.g. . Counting:

E d.o.f. (the spins)

−(V−1) charge constraints (−1 is overall charge neutrality)

−(F−1) flux constraint (−1 is overall flux neutrality)

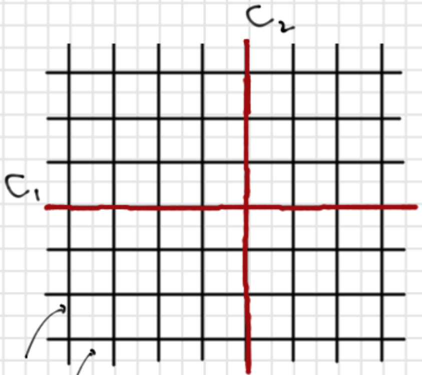
$\underbrace{-F+E-V}$ -2 qubit d.o.f. remain

(−) Euler-Characteristic $\chi = 2-2g$. Surface genus g.

~ ground state degeneracy: 2^{2g}. A hallmark of topological matter.

Q: How do we characterize different ground states?

A:

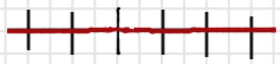
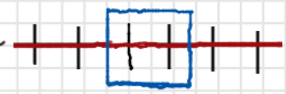


assume periodic boundary conditions a torus, T^2

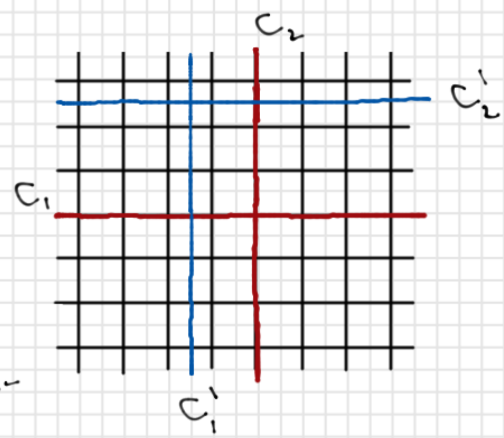
$\sigma_z^a = \prod_{C_a} \sigma_{z,ij}$.

or any other curve winding around T^2.

Claim: $\{\sigma_z^1, \sigma_z^2\}$ are topological invariants of each charge sector. E.g.

 and 

same C'



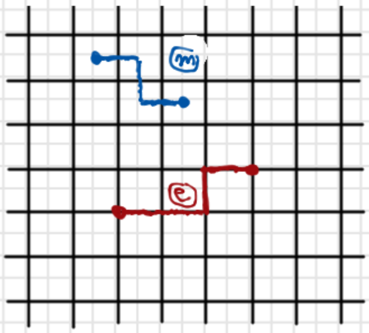
Invariants changed by nonlocal operators.

$\sigma_x^a = \prod_{C_a'} \sigma_{x,ij}$ $\qquad [\sigma_x^a, \sigma_z^a]_+ = 0$

Global qubits $\{\sigma_x^a, \sigma_z^a\}$ provide topological characterization of ground state ~ topological quantum computation.

Q: What are excitations of the system?

A:



m: a 'magnetic' excitation $= \prod_{\substack{\text{all links} \\ \text{cut by string}}} \sigma_{x,ij}$

changes flux of terminal plaquettes. Costs energy 2λ.

e: an 'electric excitation' $= \prod_{\substack{\text{all links along} \\ \text{path}}} \sigma_{z,ij}$

changes charge at terminal points. If system contains 'chemical potential' $\mu \cdot \sum_i \hat{p}_i$, energy cost 2μ.

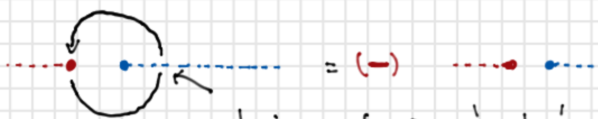
Think of e, m as quantum particles forming on top of (closed) string ground state.

Q: What is the statistics of these particles?

A: Find out what happens as we braid them around one other.

$\rightsquigarrow$ nothing happens : a system of bosons. The same with .

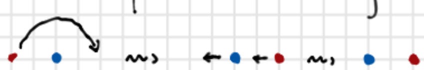
However:



$= (-)$

string of σ_x 'cuts' through one of σ_z : a minus sign

are semions relative to each other. Note: particle exchange: a π-rotation



	π-exch.	2π-braid
fermions	-1	1
semions	i	-1

Something interesting happens if we fuse m and e into a composite excitation:

$\rightsquigarrow$ are fermions relative to each other. Have generated fermions

as
- emergent particles of gauge theory
- terminal excitations of strings (cf. Jordan-Wigner in 1d)
- particles obeying strict parity conservation

$\Big\} \rightsquigarrow$ conceptual proximity to string theory

- Summary:
  - $\mathbb{Z}_2$ gauge theory has phase transition without local order parameter or symmetry breaking
  - $\mathbb{Z}_2$ topological fluid = a string condensate
  - Rigorous (L→∞) ground state degeneracy 2^g (the closest approx. to an 'order parameter')
  - Supports (abelian) quasi particles as excitations, including emergent fermions