Quantum Field Theory of Two-Dimensional Spin Liquids

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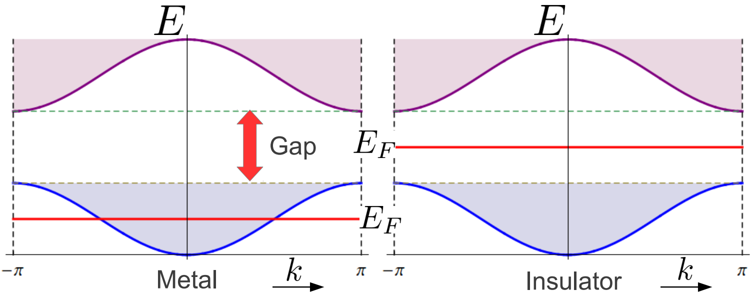
Hamburg, Oct. 2013

Outline:

- Mott insulators and the Hubbard model
- spin liquids and $U(1)$ gauge theories
- Chiral spin liquids
- Quantum dimer model

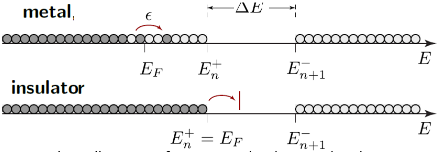
Lecture notes with full details of the derivations will be provided. Some additional material not included here will be also discussed there. First (incomplete) version will become available at the homepage of the student seminar next week (possibly on Monday).

Band theory of solids



E_F = Fermi energy

k = wavevector

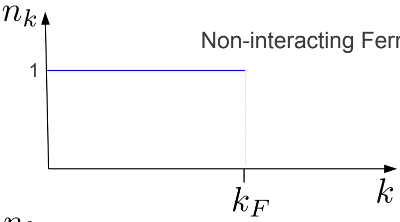


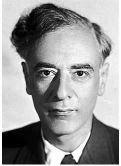
Response to small external perturbation

Landau Fermi liquid theory

- Effective quantum field theory of interacting Fermi systems
- Quasi-particle concept: one-to-one correspondence between particles in a free electron gas and elementary excitations in the interacting Fermi systems

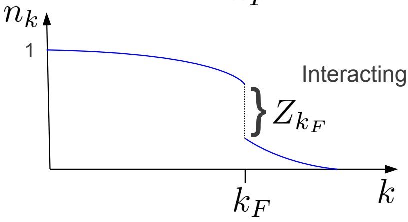
Non-interacting Fermi system at T=0





Lev Landau

Interacting Fermi system at T=0



Mott insulators

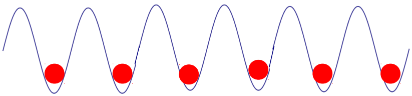
- Band structure calculations predict metallic behavior in some materials, which is contradicted by experiments
- Failure of one-particle theory $\Longrightarrow$ Many-body theory of interacting systems is necessary
- Landau Fermi liquid theory describes well the metallic behavior of interacting systems, but is insufficient to deal with Mott insulators
- Paradigmatic model for Mott insulators: Hubbard model

$$H = -t \sum_{\langle i,j \rangle} \sum_{\sigma} f^{\dagger}_{i\sigma} f_{j\sigma} - \mu \sum_{i,\sigma} n_{i\sigma} + U \sum_{i} n_{i\uparrow} n_{i\downarrow}$$

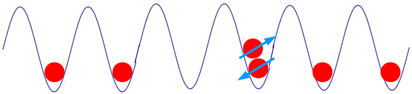
$n_{i\sigma} \equiv f^{\dagger}_{i\sigma} f_{i\sigma}$ $\qquad U > 0$

Mott insulators

- Schematic picture of half-filled band (or, more generally, one electron per unit cell):



- Band theory does not forbid double occupancy, provided electrons have opposite spin:

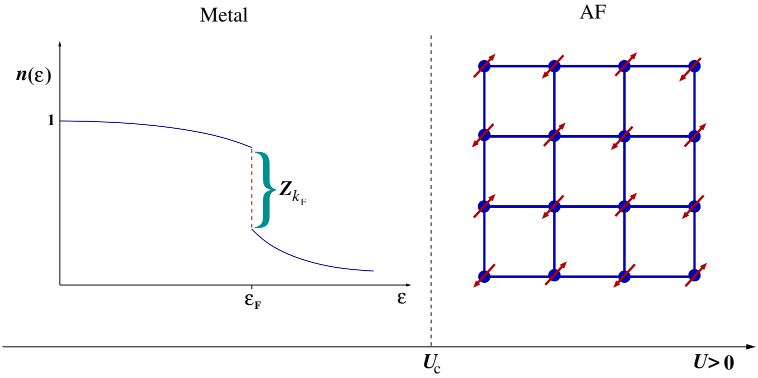


$\Longrightarrow$ predicts metallic behavior

- Double occupancy is actually forbidden for $U \gg t$

Mott-Hubbard metal-insulator transition

Typically a Mott insulator is magnetically ordered



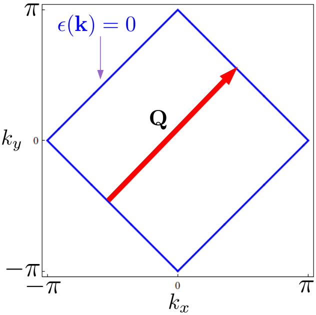
$$U_c \sim t$$

Mott insulators

- Fermionic Hubbard model at half-filling $\Longrightarrow$ Metal-Mott insulator transition
- Bosonic Hubbard model at integer filling $\Longrightarrow$ Superfluid-Mott insulator transition
- Insulating phase: Interaction driven gapped excitations, unbroken $U(1)$ symmetry
- Superfluid phase: Interaction driven gapless excitations, broken $U(1)$ symmetry
- Metallic phase: Fermi surface, unbroken $U(1)$ symmetry
- Spin liquid: Mott insulator without broken symmetries and with fractionalized excitations

Symmetries of the Hubbard model

- Particle-hole symmetry at half-filling, i.e., $\frac{1}{L}\sum_{j,\sigma}\langle n_{j\sigma}\rangle = 1$, and bipartite lattices:



$$f_{j\sigma} \to e^{i\mathbf{Q}\cdot\mathbf{R}_j} f_{j\sigma}^{\dagger}$$

$$f_{j\sigma}^{\dagger} \to e^{i\mathbf{Q}\cdot\mathbf{R}_j} f_{j\sigma}$$

$$\mathbf{Q} = (\pi, \pi)$$

$$\epsilon(\mathbf{k}+\mathbf{Q}) = -\epsilon(\mathbf{k})$$

$$\epsilon(\mathbf{k}) = -2t(\cos k_x + \cos k_y)$$

- $U(1)$ symmetry: $f_{j\sigma} \to e^{i\theta} f_{j\sigma}$, $f_{j\sigma}^{\dagger} \to e^{i\theta} f_{j\sigma}^{\dagger}$

Symmetries of the Hubbard model

- $SU(2)$ spin symmetry: $\mathbf{S} = \sum_j \mathbf{S}_j$, where $\mathbf{S}_j = \frac{1}{2}\psi_j^\dagger \vec{\sigma} \psi_j$, with $\psi_j = [f_{j\uparrow} \;\; f_{j\downarrow}]^T$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \Longrightarrow [\mathbf{S}, H] = 0$
- $SU(2)$ pseudo-spin symmetry (valid for bipartite lattices): $\mathbf{J} = \sum_j e^{\mathbf{Q}\cdot\mathbf{R}_j} \mathbf{J}_j$, where $\mathbf{J}_j = \frac{1}{2}\eta_j^\dagger \vec{\sigma} \eta_j$, with $\eta_j = [f_{j\uparrow} \;\; f_{j\downarrow}^\dagger]^T$ $\Longrightarrow [\mathbf{J}, H] = 0$
- Full symmetry of the Hubbard model is $SO(4) = SU(2) \times SU(2)$
- The $SO(4)$ symmetry allowed to complete the exact solution for the one-dimensional Hubbard model by obtaining the full excitation spectrum; see book by Essler, Frahm, Göhmann, Klümper, and Korepin, *The one-dimensional Hubbard model* (Cambridge University Press, 2005)
- Strong-coupling ($U \gg t$) limit of the Hubbard model: Heisenberg antiferromagnet $\Rightarrow H = \frac{2t^2}{U} \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$

Symmetries of the Hubbard model

- Hubbard model in bipartite lattices at half-filling: $\mu = U/2$ (*exact*)
- Proof: particle-hole transformation $f_{i\sigma} \to e^{i\mathbf{Q}\cdot\mathbf{R}_i} f_{i\sigma}^\dagger$, $f_{i\sigma}^\dagger \to e^{i\mathbf{Q}\cdot\mathbf{R}_i} f_{i\sigma}$,

$$H' = U - 2\mu - t\sum_{\langle i,j\rangle}\sum_\sigma f_{i\sigma}^\dagger f_{j\sigma} + (\mu - U)\sum_\sigma n_{i\sigma} + U\sum_i n_{i\uparrow}n_{i\downarrow}$$

$\mu = U/2 \Longrightarrow H' = H$. If $\mathcal{F}$ and $\mathcal{F}'$ are the free energy densities associated to Hamiltomians H and H',

$$n = -\frac{\partial\mathcal{F}}{\partial\mu}, \qquad 2 - n = -\frac{\partial\mathcal{F}'}{\partial\mu}$$

$\mu = U/2 \Longrightarrow n = 2 - n \Longrightarrow n = 1$

Mean-field theory for the Hubbard model

- At half-filling the Hubbard Hamiltonian can be rewritten as

$$H = -t \sum_{\langle i,j \rangle} \sum_{\sigma} f^{\dagger}_{i\sigma} f_{j\sigma} - \frac{2U}{3} \sum_{i} \mathbf{S}_i^2$$

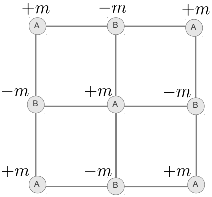
- Magnetic (mean-field) ground states: $\langle \mathbf{S}_j \rangle = \mathbf{m}$ (FM) or $\langle \mathbf{S}_j \rangle = e^{i\mathbf{Q}\cdot\mathbf{R}_j}\mathbf{m}$ (AF)
- Due to nesting, AF instabilities arise at half-filling, so an AF ordered ground state is favored (lower energy) over a FM ground state. Away from half-filling a FM ground state is favored
- Spin liquid mean-field ground states (more later) arise in a square lattice with nearest neighbor hopping and half-filling only when generalizing $SU(2) \to SU(N)$, with N sufficiently large.

- Hubbard-Stratonovich transformation:

$$H=-t\sum_{\langle i,j\rangle}\sum_{\sigma}f^{\dagger}_{i\sigma}f_{j\sigma}-U\sum_{i}\mathbf{m}_i\cdot\mathbf{S}_i+\frac{3U}{8}\sum_{i}\mathbf{m}_i^2$$

- Staggered magnetization: $\mathbf{m}_i=e^{i\mathbf{Q}\cdot\mathbf{R}_i}\mathbf{m}$. The $SU(2)$ symmetry allows us to choose $\mathbf{m}=m\hat{\mathbf{z}}$ without loss of generality.

$$f_{i\sigma}=\begin{cases}c_{i\sigma}, & i\in A\\ \bar{c}_{i\sigma}, & i\in B\end{cases}$$



- Mean-field Hamiltonian: $H_{\mathrm{MF}}=\sum_{\mathbf{k},\sigma}\psi^{\dagger}_{\mathbf{k}\sigma}M_{\mathbf{k}\sigma}\psi_{\mathbf{k}\sigma}+\frac{3UL}{8}m^2$

$$\psi_{\mathbf{k}\sigma}=\begin{bmatrix}c_{\mathbf{k}\sigma}\\ \bar{c}_{\mathbf{k}\sigma}\end{bmatrix},\qquad \psi^{\dagger}_{\mathbf{k}\sigma}=\begin{bmatrix}c^{\dagger}_{\mathbf{k}\sigma} & \bar{c}^{\dagger}_{\mathbf{k}\sigma}\end{bmatrix},\qquad M_{\mathbf{k}\sigma}=\begin{bmatrix}-\frac{\sigma Um}{2} & \varepsilon_{\mathbf{k}}\\ \varepsilon_{\mathbf{k}} & \frac{\sigma Um}{2}\end{bmatrix}$$

- Energy spectrum: $E^{\pm}_{\mathbf{k}} = \pm\sqrt{\varepsilon^2_{\mathbf{k}} + \Delta^2}$, where $\Delta^2 \equiv U^2 m^2/4$
- Ground state energy density: $E_0 = -\frac{2}{L}\sum'_{\mathbf{k}} E^{+}_{\mathbf{k}} + \frac{3}{2U}\Delta^2$
- $\frac{\partial E_0}{\partial m} = 0 \Longrightarrow \frac{3}{2U} = \int \frac{d^2k}{(2\pi)^2}\frac{1}{\sqrt{\varepsilon^2_{\mathbf{k}}+\Delta^2}} = \int_{-4t}^{0} d\varepsilon \frac{1}{\sqrt{\varepsilon^2+\Delta^2}}$
- Approximate form of the density of states in two dimensions: $\rho(\varepsilon) \approx \frac{\ln(t/\varepsilon)}{4\pi^2 t}$ (see lecture notes)
- Solution of gap equation:

$$\boxed{\Delta = 2\pi^2 t \exp\left(-\sqrt{\frac{12t}{U}}\right)}$$

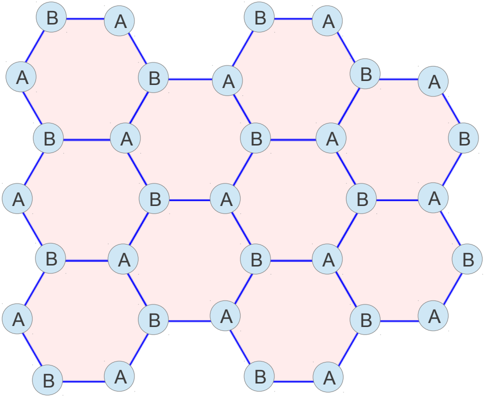
- At half-filling mean-field theory yields an AF ground state for all $U > 0 \Longrightarrow$ *no metal-insulator transition at finite U, i.e.,* $U_c = 0$

Disordering Mott insulators

- Our mean-field theory fails to describe a metal-insulator transition with $U_c \neq 0$. Quantum fluctuations around the mean-field solution are expected to destroy the AF order at weak-coupling. We will give an example with long-range Coulomb interaction in a honeycomb lattice (i.e., interacting graphene), where the gap has the form $\Delta \sim \exp\left(-\text{const}\sqrt{\frac{t}{U-U_c}}\right)$, such that the system becomes metallic for $U < U_c$, when the gap vanishes. However, such a system features an excitonic condensate for $U > U_c$ rather than an AF phase (this is a consequence of the long-range Coulomb interaction). In this case it is not the $SU(2)$ symmetry that is being spontaneously broken, but the so-called chiral symmetry. The chiral symmetry will also be important in our study of $U(1)$ spin liquids.
- Important question: can a Mott insulator also be disordered in the strong-coupling regime?

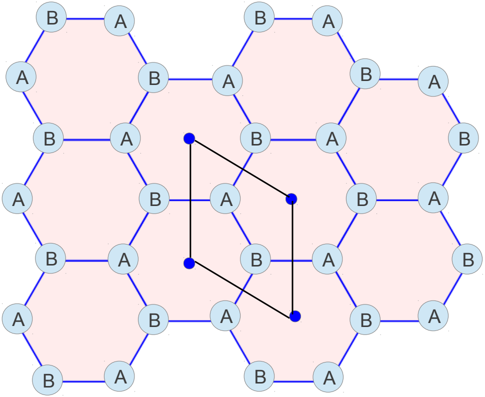
Disordering Mott insulators

Tight-binding in a honeycomb lattice:



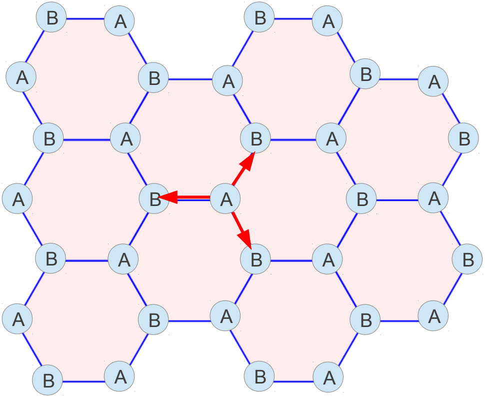
Disordering Mott insulators

Tight-binding in a honeycomb lattice:



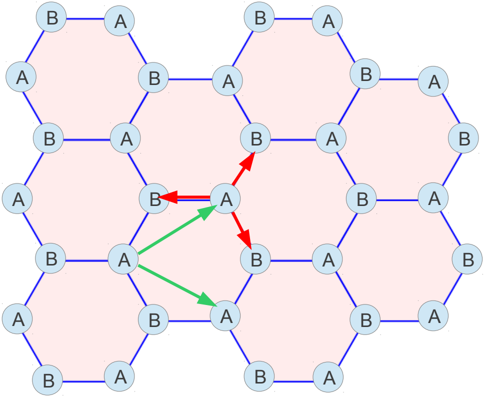
Disordering Mott insulators

Tight-binding in a honeycomb lattice:



Disordering Mott insulators

Tight-binding in a honeycomb lattice:



Disordering Mott insulators

Tight-binding in a honeycomb lattice:

$$f_{i\sigma} = \begin{cases} c_{i\sigma}, & i \in A \\ \bar{c}_{i\sigma}, & i \in B \end{cases} \qquad \mathbf{a}_1 = \hat{\mathbf{y}}, \qquad \mathbf{a}_2 = \frac{\sqrt{3}}{2}\hat{\mathbf{x}} - \frac{1}{2}\hat{\mathbf{y}}$$

$$H_0 = \textstyle\sum_{\mathbf{k},\sigma} \psi^\dagger_{\mathbf{k}\sigma} M_{\mathbf{k}} \psi_{\mathbf{k}\sigma} \qquad \mathbf{a}_3 = -\frac{\sqrt{3}}{2}\hat{\mathbf{x}} - \frac{1}{2}\hat{\mathbf{y}}$$

$$\psi_{\mathbf{k}\sigma} = \begin{bmatrix} c_{\mathbf{k}\sigma} \\ \bar{c}_{\mathbf{k}\sigma} \end{bmatrix}, \qquad \psi^\dagger_{\mathbf{k}\sigma} = \begin{bmatrix} c^\dagger_{\mathbf{k}\sigma} & \bar{c}^\dagger_{\mathbf{k}\sigma} \end{bmatrix}, \qquad M_{\mathbf{k}} = \begin{bmatrix} 0 & -t\varphi_{\mathbf{k}} \\ -t\varphi^*_{\mathbf{k}} & 0 \end{bmatrix}$$

$$\varphi_{\mathbf{k}} = \textstyle\sum_{i=1}^3 e^{i\mathbf{k}\cdot\mathbf{a}_i} = e^{ik_y} + 2\cos\left(\frac{\sqrt{3}}{2}k_x\right) e^{-ik_y/2}$$

Energy eigenvalues:

$$\boxed{E_{\mathbf{k}} = \pm t\sqrt{1 + 4\cos^2\left(\frac{\sqrt{3}}{2}k_x\right) + 4\cos\left(\frac{\sqrt{3}}{2}k_x\right)\cos\left(\frac{3}{2}k_y\right)}}$$

- The spectrum $E_{\mathbf{k}}$ has (independent) nodes at $\mathbf{k}_1 = \frac{4\pi}{3\sqrt{3}}\hat{\mathbf{x}}$ and $\mathbf{k}_2 = \frac{2\pi}{3\sqrt{3}}\hat{\mathbf{x}} - \frac{2\pi}{3}\hat{\mathbf{y}}$
- Expanding the tight-binding Hamiltonian around the nodes:

$$H_0 \approx -t \sum_{\mathbf{k},\sigma} \sum_{i=1,2} \psi^\dagger_{\mathbf{k}+\mathbf{k}_i,\sigma} M_{\mathbf{k}+\mathbf{k}_i} \psi_{\mathbf{k}+\mathbf{k}_i,\sigma}$$

$$\varphi_{\mathbf{k}+\mathbf{k}_1} \approx \frac{3}{2}(-k_x + ik_y), \qquad \varphi^*_{\mathbf{k}+\mathbf{k}_1} \approx \frac{3}{2}(-k_x - ik_y),$$

$$\varphi_{\mathbf{k}+\mathbf{k}_2} \approx \frac{3e^{i\pi/3}}{2}(-k_x - ik_y), \qquad \varphi^*_{\mathbf{k}+\mathbf{k}_2} \approx \frac{3e^{-i\pi/3}}{2}(-k_x + ik_y)$$

- Define $\psi_{1\sigma}(\mathbf{k}) = [c_{\mathbf{k}+\mathbf{k}_1,\sigma} \;\; \bar{c}_{\mathbf{k}+\mathbf{k}_1,\sigma}]^T$ and $\psi_{2\sigma}(\mathbf{k}) = [e^{i\pi/3}\bar{c}_{\mathbf{k}+\mathbf{k}_2,\sigma} \;\; c_{\mathbf{k}+\mathbf{k}_2,\sigma}]^T$

$$\begin{aligned} \Longrightarrow H_0 &\approx \frac{3t}{2} \sum_{\mathbf{k},\sigma} \sum_{i=1,2} \psi^\dagger_{i\sigma}(\mathbf{k}) \begin{bmatrix} 0 & k_x - ik_y \\ k_x + ik_y & 0 \end{bmatrix} \psi_{i\sigma}(\mathbf{k}) \\ &= v_F \sum_{\mathbf{k},\sigma} \sum_{i=1,2} \psi^\dagger_{i\sigma}(\mathbf{k}) \mathbf{k} \cdot \vec{\sigma} \psi_{i\sigma}(\mathbf{k}) \end{aligned}$$

Here $v_F \equiv 3t/2$ and $\vec{\sigma} = \sigma_x \hat{\mathbf{x}} + \sigma_y \hat{\mathbf{y}}$

- Four-component Dirac fermion representation: $\Psi_\sigma = [\psi_{1\sigma} \;\; \psi_{2\sigma}]^T$, $\bar{\Psi}_\sigma = \Psi^\dagger \gamma^0$
- Dirac γ matrices:

$$\gamma^0 = \begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} i\sigma_y & 0 \\ 0 & -i\sigma_y \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} -i\sigma_x & 0 \\ 0 & i\sigma_x \end{pmatrix}$$

- Dirac Lagrangian: $\mathcal{L}_0 = \bar{\Psi} i \not{\partial} \Psi$ (Dirac "slash" notation: $\not{Q} = \gamma^\mu Q_\mu$); $\partial_\mu = (\partial_t, v_F \nabla)$ and $\not{\partial} = \gamma^0 \partial_t - v_f \vec{\gamma} \cdot \nabla$
  $\Longrightarrow$ *massless Dirac fermions*
- Action including Coulomb interaction:

$$\mathcal{H}_{\text{Coulomb}} = \frac{U}{2} \sum_{\alpha,\beta} \int d^2r \int d^2r' \bar{\Psi}_\alpha(\mathbf{r}) \gamma^0 \Psi_\alpha(\mathbf{r}) \frac{1}{|\mathbf{r} - \mathbf{r}'|} \bar{\Psi}_\beta(\mathbf{r}') \gamma^0 \Psi_\beta(\mathbf{r}')$$

$U \equiv e^2/\epsilon$

- $\Longrightarrow 1/r$ interaction (like in 3D) in 2D, instead of $\ln r$. We are assuming that the 2D system is embedded in a 3D world and feels electromagnetic forces of it (e.g., interacting graphene). Later we will show (in the context of spin liquids) that even if a $\ln r$ is used, screening effects caused by quantum fluctuations ultimately make it $1/r$
- Fourier transform of $1/r$ in 2D: $\mathcal{F}(1/r) = 2\pi/|\mathbf{k}|$
- Lagrangian after a Hubbard-Stratonovich transformation:

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - U\gamma^0 A_0)\Psi + \frac{1}{2}A^0\sqrt{-\nabla^2}A^0$$

  $\Longrightarrow$ Fermionic sector like in QED$_{2+1}$ with vanishing spatial components of gauge field: $A^\mu = (A^0, 0, 0)$.
  Electromagnetic energy is nonlocal: $\propto \mathbf{E}\cdot\frac{1}{\sqrt{-\nabla^2}}\mathbf{E}$ instead of $\propto \mathbf{E}^2$
- We will see later that a $U(1)$ spin liquid features a QED$_{2+1}$ having all components of the (emergent) gauge field nonzero.

- The Lagrangian density has a chiral symmetry

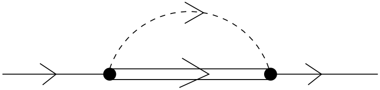
$$\psi \to e^{i\gamma_{3,5}\theta}\psi$$

$$\gamma_3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

  I is a 2×2 identity matrix.

- $\Longrightarrow$ Metal-insulator transition induced by spontaneous chiral symmetry breaking
- In the insulator phase there is an excitonic condensate, $\langle\bar{\Psi}_\sigma\Psi_\sigma\rangle = \sum_{i=1,2}\langle(c^\dagger_{\mathbf{k}+\mathbf{k}_i,\sigma}c_{\mathbf{k}+\mathbf{k}_i,\sigma} - \bar{c}^\dagger_{\mathbf{k}+\mathbf{k}_i,\sigma}\bar{c}_{\mathbf{k}+\mathbf{k}_i,\sigma})\rangle$
- This can be viewed as a pseudo-spin condensate for each actual spin σ: $a_{i\sigma,\uparrow} = c_{\mathbf{k}+\mathbf{k}_i,\sigma} \qquad a_{i\sigma,\downarrow} = \bar{c}_{\mathbf{k}+\mathbf{k}_i,\sigma}$

- Schwinger-Dyson equation: $G^{-1} = G_0^{-1} + \Sigma$

$\Sigma =$ 

- Analytical form in imaginary time:

$$
\begin{aligned}
&i\gamma_0 p_0 Z_0(p) + v_F i\vec{\gamma}\cdot\mathbf{p} Z_1(p) + \Sigma(p) \\
&= i\not{p} + g^2 \int \frac{d^3k}{(2\pi)^3} |k| \frac{\gamma_0[\Sigma(k-p) + i\gamma_0(k_0-p_0)Z_0(k-p) + v_F i\vec{\gamma}\cdot(\mathbf{k}-\mathbf{p})Z_1(k-p)]\gamma_0}{[Z_0^2(k-p)(k_0-p_0)^2 + Z_1^2(k-p)v_F^2(\mathbf{k}-\mathbf{p})^2 + \Sigma^2(k-p)]|\mathbf{k}|}.
\end{aligned}
$$

$$g^2 = 2\pi U$$

- self-consistent equations:

$$\Sigma(p) = g^2 \int \frac{d^3k}{(2\pi)^3} \frac{|k+p|\Sigma(k)}{[Z_0^2(k)k_0^2 + Z_1^2(k)v^2\mathbf{k}^2 + \Sigma^2(k)]|\mathbf{k}+\mathbf{p}|},$$

$$Z_0(p) = 1 + \frac{g^2}{p_0} \int \frac{d^3k}{(2\pi)^3} \frac{|k+p|k_0}{[Z_0^2(k)k_0^2 + Z_1^2(k)v^2\mathbf{k}^2 + \Sigma^2(k)]|\mathbf{k}+\mathbf{p}|},$$

$$Z_1(p) = 1 - \frac{g^2}{\mathbf{p}^2} \int \frac{d^3k}{(2\pi)^3} \frac{|k+p|\mathbf{p}\cdot\mathbf{k}}{[Z_0^2(k)k_0^2 + Z_1^2(k)v^2\mathbf{k}^2 + \Sigma^2(k)]|\mathbf{k}+\mathbf{p}|}.$$

- Approximation: $Z_0(k) \approx 1$ and $Z_1(k) \approx 1$ inside the integrals $\Longrightarrow$ self-consistency only for $\Sigma(k)$
- Approximation to solve the gap equation: $\Sigma(k) \approx \Sigma(0)$

$$\Sigma(0) = g^2 \Sigma(0) \int \frac{d^3k}{(2\pi)^3} \frac{1}{[k_0^2 + v_F^2 \mathbf{k}^2 + \Sigma^2(0)]|\mathbf{k}|}$$

- Solution: $\Sigma(0) = \Lambda e^{-2\pi v_F/U} = \Lambda e^{-3\pi t/U} \Longrightarrow$ non-analytic in U; once more, $U_c = 0$
- *Too naive!* $\Longrightarrow$ This would mean that any interaction, no matter how small, would make graphene an insulator...
- Better approximation: $\Sigma(p) \approx \Sigma(0, \mathbf{p})$, i.e., we ignore the frequency dependence of the selfenergy.

- After integrating over the frequency, the gap equation becomes

$$\Sigma(0,\mathbf{p}) = \frac{g^2}{8\pi^2}\int d^2k \frac{\Sigma(0,\mathbf{k})}{\sqrt{\mathbf{k}^2+\Sigma^2(0,\mathbf{k})}|\mathbf{k}+\mathbf{p}|}$$

- Notation: $\sigma(k) \equiv \Sigma(0,\mathbf{k})$, where should be understood as $|\mathbf{p}|$ and not as $|p| = \sqrt{p_0^2 + v_F^2\mathbf{p}^2}$ as before
- Thus, after performing the angular integrations,

$$\sigma(p) = \frac{g^2}{4\pi p}\int_0^p dk \frac{k\sigma(k)}{\sqrt{k^2+\sigma^2(k)}} + \frac{g^2}{4\pi}\int_p^\Lambda dk \frac{\sigma(k)}{\sqrt{k^2+\sigma^2(k)}}$$

- Converting in a differential equation:

$$\frac{d}{dp}\left[p^2\frac{d\sigma(p)}{dp}\right] = -\frac{2\lambda p\sigma(p)}{\sqrt{p^2+\sigma^2(p)}}, \qquad \lambda \equiv g^2/(8\pi v_F)$$

- Linearized regime:

$$\frac{d}{dp}\left[p^2\frac{d\sigma(p)}{dp}\right] = -2\lambda\sigma(p)$$

- Ansatz: $\sigma(p) = Ap^\alpha$
- Boundary condition: $p\frac{d\sigma}{dp}\Big|_{p=\Lambda} = -\sigma(\Lambda)$
- Solution:

$$\boxed{\sigma(p) = \frac{A}{\sqrt{p}}\sin\left\{\frac{\sqrt{8\lambda-1}}{2}\ln\left[\frac{p}{\sigma(0)}\right]\right\}}$$

$\lambda \equiv g^2/(8\pi)$

- $\sigma(0) = \Lambda\exp\left(-\frac{2\pi}{\sqrt{8\lambda-1}}\right) = \Lambda\exp\left(-\pi\sqrt{\frac{3t}{U-3t/4}}\right)$
  $\Longrightarrow U_c = 3t/4$

Remarks:

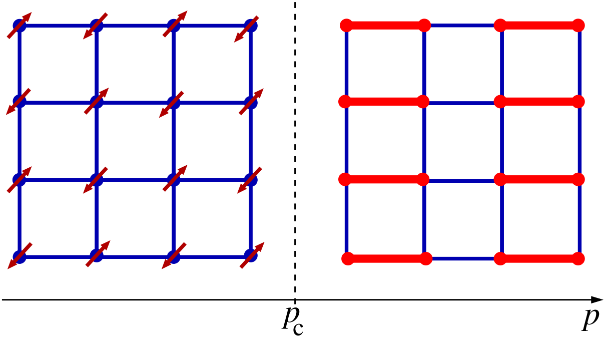
- $U \leq U_c \Longrightarrow$ semi-metal phase (Dirac cones)
  $U > U_c \Longrightarrow$ excitonic insulator phase
- Similar to the inverse correlation length (gap) in the Berezhinsky-Kosterlitz-Thouless transition:
  $\xi^{-1} \sim \exp\left(-\frac{\mathrm{const}}{\sqrt{T-T_c}}\right)$
- $\sigma(0) \sim \langle \bar{\Psi}_\alpha \Psi_\alpha \rangle$
- By computing fluctuation corrections to the Coulomb interaction, the value of U_c gets modified to (see lecture notes)

$$U_c = \frac{6t}{8 - \pi N}$$

  Here N comes from generalizing the number of spin degrees of freedom from 2 to N. The above result implies that a gap can only be generated
  if $N < N_c = 8/\pi \approx 2.55$

Disordering Mott insulators

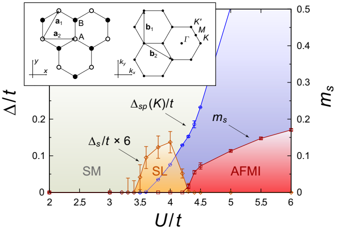
- Spin liquid: a Mott insulator with no broken symmetries that has fractionalized excitations
- Theoretically subtle: Mott insulators tend to order at low temperatures. Sometimes Mott insulators can be tuned to a paramagnetic state by competing interactions, but break the symmetries of the lattice. Ex: a valence-bond solid (Read and Sachdev, 1989)



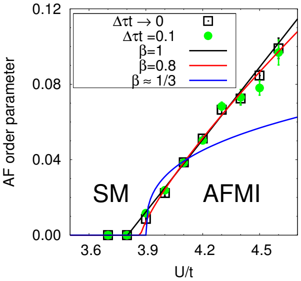
- Experimentally elusive, despite some promising recent experiments

Disordering Mott insulators

- Conflicting numerical (Monte Carlo) results for the (short-range interacting) Hubbard model on a honeycomb lattice:



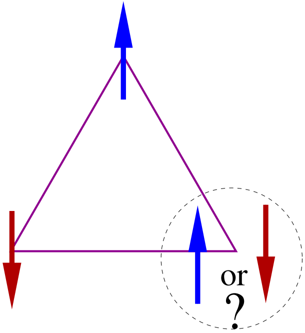
Meng et al., Nature (London) **464**, 847 (2010)



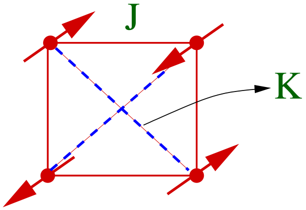
S. Sorella, Y. Otsuka, and S. Yunoki, Sci. Rep. **2**, 992 (2012)

Disordering Mott insulators

- Types of frustration in a Heisengerg AF:



Geometric frustration



Competing exchange interactions

- Group theoretic frustration: $SU(2) \longrightarrow SU(N)$, with N large enough to stabilize the spin liquid

$U(1)$ spin liquid

- Heisenberg AF on a square lattice:

$$H = J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$

$\mathbf{S}_i = \frac{1}{2} f^\dagger_{i\alpha} \vec{\sigma}_{\alpha\beta} f_{i\beta}$ Local constraint: $f^\dagger_{i\alpha} f_{i\alpha} = 1$

- $\sigma^a_{\alpha\beta}\sigma^a_{\gamma\delta} = 2\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta}$
- $H = -\frac{J}{2} \sum_{\langle i,j \rangle} f^\dagger_{i\alpha} f_{j\alpha} f^\dagger_{j\beta} f_{i\beta} + \text{const}$
- $SU(N)$ generalization: rescaling $J \to J/N$, constraint $f^\dagger_{i\alpha} f_{i\alpha} = N/2$
- Hubbard-Stratonovich transformation:

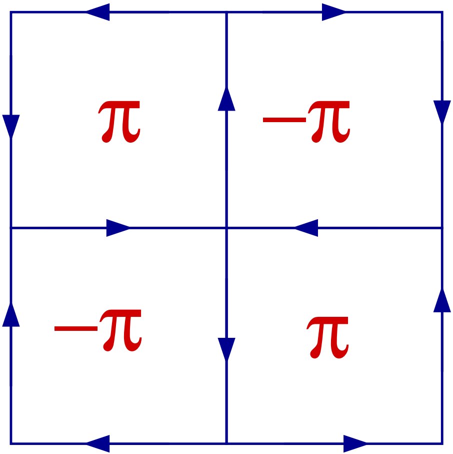
$$H = \sum_{\langle i,j \rangle} \left(\frac{N}{J} |\chi_{ij}|^2 - \chi_{ij} f^\dagger_{i\alpha} f_{j\alpha} + \text{h.c.} \right)$$

- Lagrangian in imaginary time:
  $L = \sum_j [f_{j\sigma}(\partial_\tau + i\lambda_j) f_{j\sigma} - (N/2) i\lambda_j] + H$
- Local gauge invariance: $f_{j\sigma} \to e^{i\theta_j} f_{j\sigma}, \quad \chi_{ij} \to e^{i(\theta_i - \theta_j)} \chi_{ij},$
  $\lambda_j \to \lambda_j - \partial_\tau \theta_j$

$U(1)$ spin liquid

- Saddle-point solution at large N (π-flux phase):

$$\chi_{ij} = \chi_0 e^{i\theta_{ij}} \qquad \sum_{\mathcal{P}=\{ijkl\}} \theta_{\mathcal{P}} = \pi$$



$U(1)$ spin liquid

- For any plaquette:

$$\begin{aligned} \chi_1 &= \chi_{j,j+\hat{e}_x}, \qquad \chi_2 = \chi_{j+\hat{e}_x,j+\hat{e}_y}, \\ \chi_3 &= \chi_{j+\hat{e}_x+\hat{e}_y,j+\hat{e}_y}, \qquad \chi_4 = \chi_{j+\hat{e}_y,j} \end{aligned}$$

- Rewriting the MF Hamiltonian:

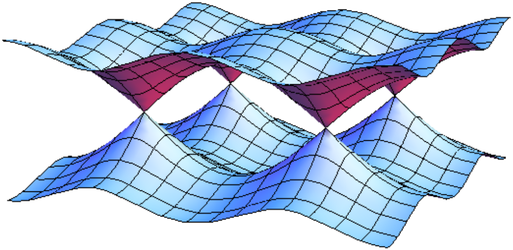
$$\begin{aligned} H &= -\sum_{j\in A}\sum_{\sigma}\left(\chi_1 \bar{c}^\dagger_{j+\hat{e}_x,\sigma} c_{j\sigma} + \chi_4^* \bar{c}^\dagger_{j+\hat{e}_y,\sigma} c_{j\sigma} + \text{h.c.}\right) \\ &- \sum_{j\in B}\left(\chi_3^* c^\dagger_{j+\hat{e}_x,\sigma} \bar{c}_{j\sigma} + \chi_2 c^\dagger_{j+\hat{e}_y,\sigma} \bar{c}_{j\sigma} + \text{h.c.}\right) \\ &+ \frac{NL}{J}\left(|\chi_1|^2 + |\chi_2|^2 + |\chi_3|^2 + |\chi_4|^2\right). \end{aligned}$$

$U(1)$ *spin liquid*

- In momentum space:

$$H = \sum_{\mathbf{k}}{}' \begin{bmatrix} c^\dagger_{\mathbf{k}\sigma} & \bar{c}^\dagger_{\mathbf{k}\sigma} \end{bmatrix} \begin{bmatrix} 0 & |\chi|\left(\cos k_1 + i\cos k_2\right) \\ |\chi|\left(\cos k_1 - i\cos k_2\right) & 0 \end{bmatrix} \begin{bmatrix} c_{\mathbf{k}\sigma} \\ \bar{c}_{\mathbf{k}\sigma} \end{bmatrix}$$

- Spectrum: $E_\pm(\mathbf{k}) = \pm|\chi|\sqrt{\cos^2 k_x + \cos^2 k_y}$



$U(1)$ *spin liquid*

- Linearizing near the nodes $(\pi/2, \pi/2)$ (1) and $(-\pi/2, \pi/2)$ (2):

$$H \simeq \sum_{\mathbf{k}}{}' \begin{bmatrix} c^\dagger_{1\mathbf{k}\sigma} & \bar{c}^\dagger_{1\mathbf{k}\sigma} & c^\dagger_{2\mathbf{k}\sigma} & \bar{c}^\dagger_{2\mathbf{k}\sigma} \end{bmatrix} \left\{ -|\chi| k_1 \begin{bmatrix} \tau_1 & 0 \\ 0 & -\tau_1 \end{bmatrix} + |\chi| k_2 \begin{bmatrix} \tau_2 & 0 \\ 0 & \tau_2 \end{bmatrix} \right\} \begin{bmatrix} c_{1\mathbf{k}\sigma} \\ \bar{c}_{1\mathbf{k}\sigma} \\ c_{2\mathbf{k}\sigma} \\ \bar{c}_{2\mathbf{k}\sigma} \end{bmatrix}$$

$\Longrightarrow$ Dirac fermions once more!

- Phase fluctuations: $\chi_{lm} = \chi e^{iA_{lm}}$
- Effective quantum field theory: $\mathcal{L} = \sum_{\alpha=1}^{N} \bar{\psi}_\alpha (\not{\partial} + i\not{A}) \psi_\alpha$

  $\Longrightarrow$ QED in 2+1 spacetime dimensions

$U(1)$ *spin liquid*

- The mean-field theory yields a Mott insulator without broken symmetries (no local order parameter) and fractionalized excitations represented by the Dirac fermions (spinons)
- It is fractionalized in the sense that, differently from an AF-ordered Mott insulator, the (spinon) excitations have spin 1/2. In contrast, magnons in an antiferromagnet are spin 1 excitations $\Longrightarrow$ AF magnons fractionalize into spinon excitations. The spin falls apart.
- If true, the magnetization is composed by more fundamental building blocks (like quarks in QCD), the spinons, which deconfine in the large N limit
- Confinement/deconfinement physics arises because the $U(1)$ group is compact (Polyakov, 1977), since the gauge field is a phase, and therefore must be periodic.
- Question: can spinons still be deconfined at lower values of N? This is the problem of the stability of $U(1)$ spin liquids [Hermele et al., PRB **70**, 214437 (2004); Nogueira and Kleinert, PRL **95**, 176406 (2005); PRB **77**, 045107 (2008)]

$U(1)$ *spin liquid*

- Note that the gauge field in spin liquid QED (or better, "quantum spinodynamics") is emergent
- Just like in our discussion of interacting graphene, the field theory of $U(1)$ spin liquid has also a chiral symmetry
- The key to the stability of the $U(1)$ spin liquid relies on two important aspects:
  1. Chiral symmetry breaking: if there is a regime where the chiral condensate, $\langle\bar{\psi}_\alpha\psi_\alpha\rangle$ is nonzero, then the staggered magnetization $\langle\bar{\psi}_\alpha\vec{\sigma}_{\alpha\beta}\psi_\beta\rangle$ is also nonzero.
  2. Compactness of the gauge field and the nature of spinon confinement

$U(1)$ *spin liquid*

- Integrating out the fermions: $S_{\text{eff}} = -N\,\mathrm{Tr}\ln(\not{\partial} + i\not{A})$
- A_μ-propagator is obtained by expanding the effective action up to quadratic order in A_μ (assuming d spacetime dimensions):

$$S_{\text{eff}} \approx \frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\Sigma_{\mu\nu}(p)A_\mu(p)A_\nu(-p),$$

where

$$\Sigma_{\mu\nu}(p) = -N\int \frac{d^d k}{(2\pi)^d}\mathrm{tr}[\gamma_\mu G_0(k)\gamma_\nu G_0(p-k)]$$

Fermionic propagator: $G_0(p) = -\frac{i\not{p}}{p^2}$

$U(1)$ *spin liquid*

- Current conservation implies $p_\mu \Sigma_{\mu\nu}(p) = 0$. Thus,

$$\Sigma_{\mu\nu}(p) = p^2 \Pi(p) \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right)$$

- After some calculation detailed in the lecture notes, we obtain

$$\Pi(p) = N c(d) |p|^{d-4}, \qquad c(d) = \frac{8\Gamma(2 - d/2)\Gamma^2(d/2)}{(4\pi)^{d/2}\Gamma(d)}$$

For $d = 2 + 1$,

$$\Pi(p) = \frac{N}{8|p|}$$

- Gauge field propagator in the Landau gauge:

$$D_{\mu\nu}(p) = \frac{1}{N c(d) |p|^{d-2}} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right)$$

U(1) *spin liquid*

- Wilson loop:

$$\mathcal{W}_C = \left\langle \exp\left(-ie_0 \oint_C dx_\mu A_\mu(x)\right)\right\rangle$$

- In the lattice this corresponds to $\sim \langle \chi_{ij}\chi_{jk}\chi_{kl}\chi_{li}\rangle$ around a closed loop
- Asymptotic behavior for a large time span T: $\mathcal{W} \sim e^{-TV(R)}$
- Integrating out A_μ yields

$$\ln \mathcal{W}(C) = -\frac{(d-2)\Gamma(d)}{4\Gamma^3(d/2)\Gamma(2-d/2)N}\oint_{C\times C}\frac{dx_\mu dy_\mu}{|x-y|^2}$$

- For large T the main contribution comes from the integral,

$$\begin{aligned}\int_0^T dt \int_0^T du \frac{1}{(t-u)^2+R^2} &= 2T\left[\frac{\arctan(T/R)}{R} + \frac{R}{T}\ln\left(\frac{R}{\sqrt{R^2+T^2}}\right)\right] \\ &\underset{T\gg R}{\approx} \frac{\pi T}{R} + 2\ln\left(\frac{R}{T}\right).\end{aligned}$$

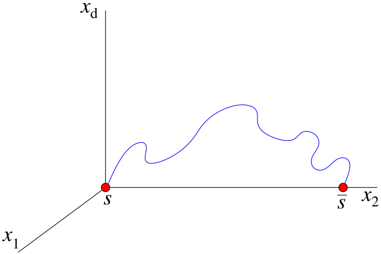
- $$\Longrightarrow NV(R) \approx -\frac{\pi(d-2)a(d)}{24R}\left[1+\frac{2R}{T}\ln\left(\frac{R}{T}\right)\right],$$

  $a(d) = \frac{6\Gamma(d)}{\Gamma^3(d/2)\Gamma(2-d/2)}$
  For $d = 2+1$ we have $a(3) = 96/\pi^2 \approx 9.73$

- Comparison with Lüscher's string model [M. Lüscher, Nucl. Phys. B **180**, 317 (1981)]:

  $$V(R) \approx \tau_s R - \frac{(d-2)\pi}{24R}$$

  $\tau_s =$ string tension



- Only the coefficient of $1/R$ is universal: just make a scale transformation for T and R to see this

Spontaneous chiral symmetry breaking

- Schwinger-Dyson equation:

$$G^{-1}(p) = G_0^{-1}(p) + \int \frac{d^3k}{(2\pi)^3}\gamma_\mu G(p-k)\gamma_\nu D_{\mu\nu}(k),$$

where

$$G(p) = \frac{1}{i\not{p}Z(p) + \Sigma(p)} = \frac{\Sigma(p) - i\not{p}Z(p)}{Z^2(p)p^2 + \Sigma^2(p)}$$

- Self-consistent equations:

$$\Sigma(p) = \frac{16}{N}\int \frac{d^3k}{(2\pi)^3}\frac{\Sigma(k)}{[Z^2(k)k^2 + \Sigma^2(k)]|k+p|}$$

$$Z(p) = 1 - \frac{8}{Np^2}\int \frac{d^3k}{(2\pi)^3}\frac{[k^2 - p^2 + (k+p)^2](k+p)\cdot pZ(k)}{[Z^2(k)k^2 + \Sigma^2(k)]|k+p|^3}$$

- Differential equations:

$$\frac{d}{dp}\left[p^2\frac{d\Sigma(p)}{dp}\right] = -\frac{8}{\pi^2 N}\frac{p^2\Sigma(p)}{Z^2(p)p^2+\Sigma^2(p)}$$

$$\frac{d}{dp}\left[p^4\frac{dZ(p)}{dp}\right] = \frac{8}{\pi^2 N}\frac{p^4 Z(p)}{Z^2(p)p^2+\Sigma^2(p)}$$

- Boundary conditions:

$$\lim_{p\to 0} p\Sigma(p) = 0, \qquad p\frac{d\Sigma(p)}{dp}\bigg|_{p=\Lambda} = -\Sigma(\Lambda)$$

$$p\frac{dZ(p)}{dp}\bigg|_{p=\Lambda} = 3[1 - Z(\Lambda)]$$

Positivity of the spectral representation implies
$0 < Z(0) \leq 1$

- Approximate DE for the self-energy:

$$\frac{d}{dp}\left[p^2\frac{d\Sigma(p)}{dp}\right] = -\frac{8}{\pi^2 N}\frac{p^2\Sigma(p)}{p^2+\Sigma^2(0)}$$

- Approximate solution:

$$\Sigma(p) = \Sigma(0)_2F_1\left[\frac{1}{4}-\frac{\mathrm{i}}{4}\gamma, \frac{1}{4}+\frac{\mathrm{i}}{4}\gamma; \frac{3}{2}; -\frac{p^2}{\Sigma^2(0)}\right]$$

$$\gamma = \sqrt{\frac{32}{\pi^2 N}-1}$$

- Solution in the regime $p^2 \gg \Sigma^2(0)$:

$$\Sigma(p) \approx \frac{|C|}{4}\sqrt{\frac{\pi\Sigma^3(0)}{p}}\cos\left\{\frac{\gamma}{2}\ln\left[\frac{p}{\Sigma(0)}\right]+\theta\right\}$$

$$\theta = \arccos\left(\frac{C+C^*}{2|C|}\right), \qquad C = \frac{\Gamma(\mathrm{i}\gamma/2)(1+\mathrm{i}\gamma)}{\Gamma^2\left(\frac{5}{4}+\mathrm{i}\frac{\gamma}{4}\right)}$$

- The boundary conditions imply

$$\Sigma(0) = \Lambda \exp\left(-\frac{2\pi}{\gamma}\right)$$

$\Longrightarrow$ vanishes for $N \geq N_c = 32/\pi^2 \approx 3.24$
chiral symmetry is broken for $N = 2 \Longrightarrow$ no stable spin liquid in the $SU(2)$ case $\Longrightarrow$ AF state

- Approximate solution for $Z(p)$ is more easily obtained from its integral equation, which is equivalent to the DE
- Approximate form of the integral equation:

$$Z(p) \approx 1 - \frac{8}{3\pi^2 N}\left[\int_p^\Lambda \frac{dkk}{k^2+\Sigma^2(0)} + \frac{1}{p^3}\int_0^p \frac{dkk^4}{k^2+\Sigma^2(0)}\right]$$

- Solution:

$$\begin{aligned} Z(p) &= 1 + \frac{8}{3\pi^2 N}\left\{\ln\left[\frac{\sqrt{p^2+\Sigma^2(0)}}{\Lambda}\right] - \frac{1}{3}\right. \\ &+ \left.\frac{\Sigma^2(0)}{p^2} - \frac{\Sigma^3(0)}{p^3}\arctan\left[\frac{p}{\Sigma(0)}\right]\right\} \end{aligned}$$

$U(1)$ *spin liquid and Compact QED*

- Chiral symmetry breaking gives a mass to the spinons. This change the dynamics of the emergent gauge field.
- Vacuum polarization for massive Dirac fermions:

$$\Pi(p) = \frac{N}{4\pi p^2}\left[m + \frac{(p^2 - 4m^2)}{2|p|}\arctan\left(\frac{|p|}{2m}\right)\right]$$

$m \equiv \Sigma(0)$

- Low-energy Maxwell theory:

$$\mathcal{L}_M = \frac{1}{4g^2}F_{\mu\nu}^2$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
$g^2 \equiv 12\pi m/N$

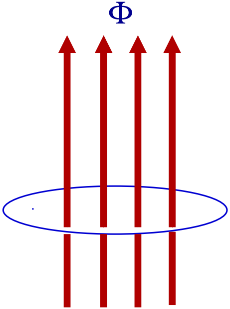
- Recall that the gauge field is supposed to be compact
$\Longrightarrow$ Compact QED_{2+1}

$U(1)$ spin liquid and Compact QED

What does compact $U(1)$ in continuum limit means?

Flux quantization (open surface)

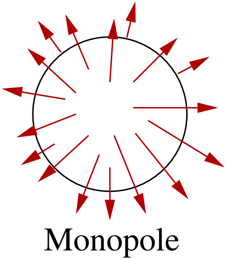
$$\int_S d\vec{S} \cdot (\nabla \times \vec{A}) = 2\pi n/e$$



Example: Superconductor, Abrikosov vortices

Flux quantization (closed surface)

$$\oint_S d\vec{S} \cdot (\nabla \times \vec{A}) = 2\pi n/e$$



Example: Magnetic monopoles

$U(1)$ *spin liquid and Compact QED*

- Action of compact QED$_{2+1}$ in the lattice (Polyakov,1977):

$$\mathcal{S} = -\frac{1}{2g^2} \sum_{i,\mu,\nu} \cos(F_{i\mu\nu})$$

- For a lattice superfluid with lattice action $\mathcal{S}_{\mathrm{SF}} = \sum_\tau \left[\frac{1}{2} \sum_i (\Delta_\tau \theta_i(\tau))^2 - \rho_s \sum_{i,j} \cos(\theta_i(\tau) - \theta_j(\tau)) \right]$ the topological defects are vortices, which at zero temperature are point defects in $(1+1)D$ and line loops in $(2+1)D$. For compact QED$_{2+1}$ the topological defects in are points called instantons (= magnetic monopoles in spacetime)
- In contrast to Polyakov's case, in our case only the spatial components of the gauge field are periodic, since the A_0 (time) component corresponds to the Lagrange multiplier enforcing the single occupancy constraint. Thus, $-\infty < A_{i0} < \infty$, $0 \leq A_{ij} \leq 2\pi$. However, we will show that both theories lead to the same dual theory describing a field theory of instantons. This will allow us to make A_0 also periodic.

$U(1)$ spin liquid and Compact QED

- Villain form of the action $\Longrightarrow$ write the Fourier representation:

$$e^{c\cos x} = \sum_{n=-\infty}^{\infty} I_n(c) e^{inx}$$

and use $I_n(c) \sim e^{-n^2/(2c)}$ for $c \to \infty$

$\Longrightarrow e^{c\cos x} \sim \sum_{n=-\infty}^{\infty} e^{-n^2/(2c)+inx} \sim \sum_{m=-\infty}^{\infty} e^{-\frac{c}{2}(x-2\pi m)^2}$ with help of the Poisson formula

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dy f(y) e^{i2\pi m y}$$

- Rewriting the action:

$$\mathcal{S} = \frac{1}{2g^2} \sum_{i,l} \left[(\Delta_0 A_{il} - \Delta_l A_{i0} - 2\pi N_{il})^2 + \left(\sum_{i,j} \epsilon_{ij} \Delta_i A_{lj} - 2\pi M_l \right)^2 \right]$$

where N_{il} and M_l are integer lattice fields

- The Villain form is relatively easy to dualize

$U(1)$ *spin liquid and Compact QED*

- First step to the dual transformation:

$$\mathcal{S} = \sum_l \left[\frac{g^2}{2} h_{lj}^2 - \mathrm{i} h_{lj} (\Delta_0 A_{lj} - \Delta_j A_{l0} - 2\pi N_{lj}) + \frac{g^2}{2} n_l^2 - \mathrm{i} n_l \sum_{i,j} \epsilon_{ij} \Delta_i A_{lj} \right]$$

- Integrating A_{l0} out yields the constraint $\sum_j \Delta_j h_{ij} = 0$
- Solving the constraint: $h_{ij} = \sum_l \epsilon_{jl} \Delta_l b_i$. We obtain,

$$\begin{aligned} \mathcal{S} &= \sum_i \left[\frac{g^2}{2} \sum_j \left(\sum_l \epsilon_{jl} \Delta_l b_i \right)^2 + \frac{g^2}{2} n_i^2 - \mathrm{i} \sum_{j,l} \epsilon_{jl} \Delta_l b_i (\Delta_0 A_{ij} - 2\pi N_{ij}) \right. \\ &- \left. \mathrm{i} n_i \sum_{l,j} \epsilon_{lj} \Delta_l A_{ij} \right] \end{aligned}$$

- Scalar field b_i can be promoted to an integer field L_i by means of the Poisson formula, making N_{ij} disappear

$U(1)$ spin liquid and Compact QED

- $$\begin{aligned}\mathcal{S} &= \sum_i \left[\frac{g^2}{2}\sum_j\left(\sum_l \epsilon_{jl}\Delta_l L_i\right)^2 + \frac{g^2}{2}n_i^2 - \mathrm{i}\sum_{j,l}\epsilon_{jl}\Delta_l L_i \Delta_0 A_{ij}\right.\\ &- \left.\mathrm{i}n_i\sum_{l,j}\epsilon_{lj}\Delta_l A_{ij}\right]\end{aligned}$$
- Integrating out A_{ij} yields constraint: $\sum_l \epsilon_{jl}\Delta_0\Delta_l L_i = \sum_l \epsilon_{jl}\Delta_l n_i$, which solves to $\Delta_0 L_i = n_i$.

$$\mathcal{S} = \sum_i \left[\frac{g^2}{2}\sum_j\left(\sum_l \epsilon_{jl}\Delta_l L_i\right)^2 + \frac{g^2}{2}(\Delta_0 L_i)^2\right]$$

- Poissonizing once more converts L_i to a scalar field φ:

$$\mathcal{S} = \sum_i \left[\frac{g^2}{2}\sum_{j,l}(\Delta_l\varphi_i)^2 + \frac{g^2}{2}(\Delta_0\varphi_i)^2 - 2\pi i N_i \varphi_i\right]$$

$U(1)$ spin liquid and Compact QED

- Dual theory is given by a sine-Gordon lattice action describing the quantum dynamics of instantons:

$$\mathcal{S}_{\text{SG}} = \sum_i \left[\frac{g^2}{8\pi^2} (\Delta_\mu \varphi_i)^2 - 2z \cos \varphi_i \right]$$

$-\infty < \varphi_i < \infty$ (φ_i is non-compact)

$z = e^{-2\pi^2 c_0/g^2}$ with $c_0 = 0.2527$ is the fugacity of the instanton gas.

- In the continuum limit we have

$$S_{SG} = \left(\frac{g}{2\pi}\right)^2 \int d^3x \left[\frac{1}{2}(\partial_\mu \varphi)^2 - M^2 \cos \varphi\right]$$

$M^2 = (8\pi^2/g^2)z$

- The excitations of the sine-Gordon theory in (2+1)D are always gapped $\Longrightarrow$ no phase transition!
- *Instantons are condensed* $\Longleftrightarrow$ *spinons are permanently confined*
- String tension $\Longrightarrow$ $\tau_s = 2g^2 M/\pi^2$

$U(1)$ *spin liquid and Compact QED*

- The dual theory is equivalent to a London model that features instantons, rather than vortices.
- London model with instantons is a Mott insulator, while the London model of vortices is a superconductor

$$\mathcal{L}_{\text{London}} = \frac{1}{2}(\boldsymbol{\nabla} \times \mathbf{h})^2 + \frac{m_h^2}{2}\mathbf{h}^2$$

satisfying the constraint $\boldsymbol{\nabla} \cdot \mathbf{h} = \frac{2\pi}{m_h g}\rho_I(\mathbf{x})$
Instanton density: $\rho_I(\mathbf{x}) = \sum_i q_i \delta^3(\mathbf{x} - \mathbf{x}_i)$ $\qquad q_i = \pm q \in \mathbb{Z}$

- In contrast to the London model of superconductors, m_h here is an arbitrary mass scale, which reflects the topological nature of the problem
$\Longrightarrow$ Dual Meissner effect independent of (penetration) length scale. We have here an example of topological dual Meissner effect

$U(1)$ spin liquid and Compact QED

- Proof of equivalence to the sine-Gordon theory:
  Partition function for a given monopole density,

$$Z(\rho_I) = \int \mathcal{D}\mathbf{h}\mathcal{D}\sigma \ \exp\left(-\int d^3x\mathcal{L}\right),$$

where

$$\mathcal{L} = \frac{1}{2}(\boldsymbol{\nabla} \times \mathbf{h})^2 + \frac{m_h^2}{2}\mathbf{h}^2 + i\sigma(\boldsymbol{\nabla} \cdot \mathbf{h} - 2\pi\rho_I/m_h g),$$

Integrating out $\mathbf{h}$ yields

$$S = \frac{1}{m_h^2}\int d^3x \left[\frac{1}{2}(\boldsymbol{\nabla}\sigma)^2 - i\frac{2\pi m_h}{g}\sigma\rho_I\right]$$

Defining $\sigma = g m_h \varphi/2\pi$ and summing over all instanton configurations yields the sine-Gordon theory once more

$U(1)$ spin liquid and Compact QED

- Partition function:

$$Z = \int \mathcal{D}\varphi \exp\left[-\frac{1}{2}\left(\frac{g}{2\pi}\right)^2 \int d^3r(\nabla\varphi)^2\right] \sum_N \sum_{\{q_j\}} \frac{z^N}{2^N N!} \int \left(\prod_{j=1}^{N} d^3\mathbf{r}_j\right) \exp\{i \sum_l q_l \varphi(\mathbf{r}_l)\}$$

- Keeping only ± 1 instantons:

$$\begin{aligned} Z &= \int \mathcal{D}\chi \exp\left[-\frac{1}{2}\left(\frac{g}{2\pi}\right)^2 \int d^3r(\nabla\varphi)^2\right] \sum_N \frac{(z)^N}{N!}\left[\int d^3r \cos\varphi\right]^N \\ &= \int \mathcal{D}\varphi \exp\left\{-\int d^3r \left[\frac{1}{2}\left(\frac{g}{2\pi}\right)^2 (\nabla\varphi)^2 - z\cos\varphi\right]\right\}. \qquad (1) \end{aligned}$$

$U(1)$ spin liquid and Compact QED

- Debye-Hückel approximation:

$$\mathcal{L}_{\rm SG} \approx \frac{1}{2}\left(\frac{g}{2\pi}\right)^2 (\boldsymbol{\nabla}\varphi)^2 + z\varphi^2$$

- Correlation functions:

$$\langle \varphi(p)\varphi(-p)\rangle = \frac{(2\pi/g)^2}{p^2+M^2}$$

$$\langle \rho_I(p)\rho_I(-p)\rangle = \left(\frac{g}{2\pi}\right)^2 \frac{M^2p^2}{p^2+M^2}$$

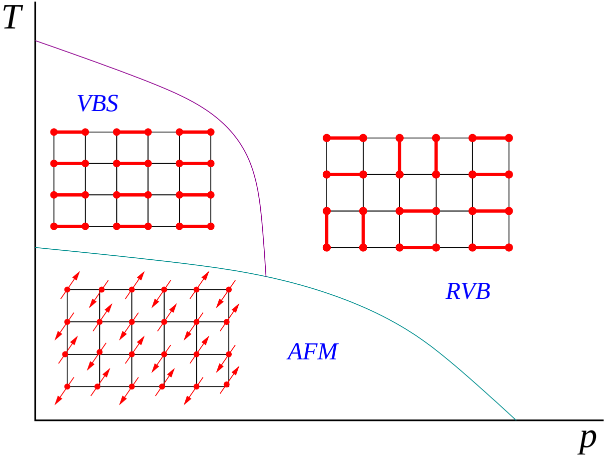
$$\langle h_\mu(p)h_\nu(-p)\rangle = \frac{1}{p^2+m_h^2}\left[\delta_{\mu\nu} - \left(1-\frac{M^2}{m_h^2}\right)\frac{p_\mu p_\nu}{p^2+M^2}\right]$$

- Physical magnetic correlation function (Polyakov, 1977):

$$\langle H_\mu(p)H_\nu(-p)\rangle = \lim_{m_h\to\infty} g^2 m_h^2 \langle h_\mu(p)h_\nu(-p)\rangle = g^2\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2+M^2}\right)$$

$U(1)$ spin liquid and Compact QED

Ways to stabilize the spin liquid:



$$|\bullet\!\!-\!\!\bullet\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

$U(1)$ spin liquid and Compact QED

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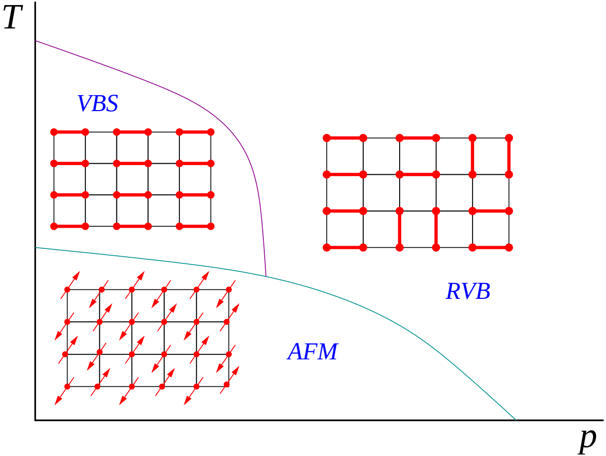

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$U(1)$ *spin liquid and Compact QED*

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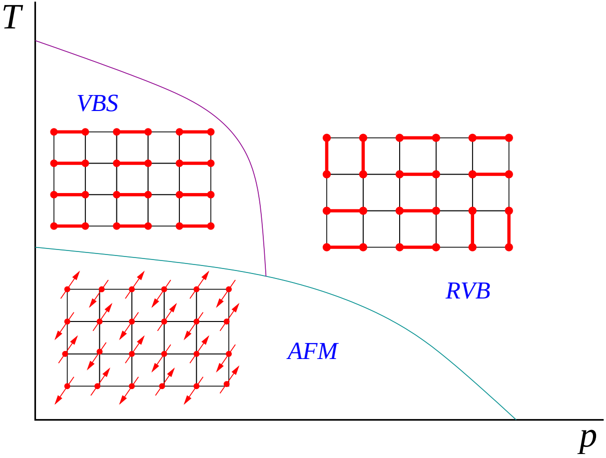
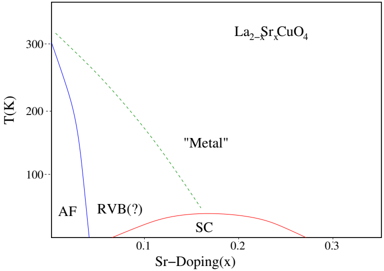


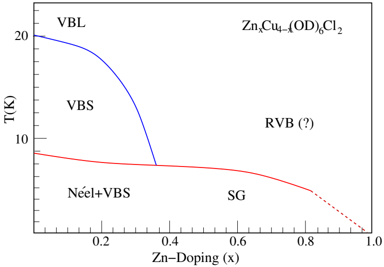
$$|\bullet\!\!-\!\!\bullet\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

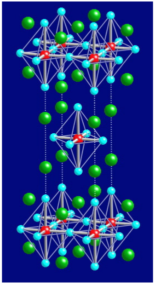
$U(1)$ spin liquid and Compact QED

Bednorz und Müller (1986)

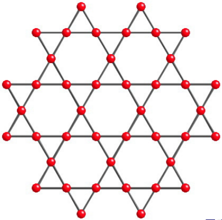


S.-H. Lee *et al.*, Nature Materials (August 2007)





CuO-plane



Kagome lattice

$U(1)$ *spin liquid and Compact QED*

Compact Abelian Higgs model in (2+1)D:

$$S = -\beta \sum_{i,\mu} \cos(\Delta_\mu \theta_i - q A_{i\mu}) - \kappa \sum_{i,\mu,\nu} \cos(F_{i\mu\nu})$$

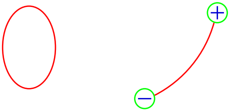
$q \in \mathbb{N}$, $\quad \kappa \equiv 1/g^2$

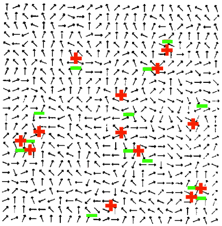
- $q = 1 \Longrightarrow$ no phase transition
- $q = 2 \Longrightarrow$ spinon pairing, Ising universality class
  Z_2 spin liquid
- $\kappa \to \infty \Longrightarrow XY$ model
- $\beta \to \infty \Longrightarrow Z_q$ gauge theory ($q > 1$)

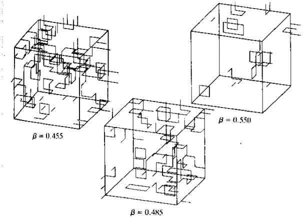
$U(1)$ spin liquid and Compact QED

Duality in 2+1 dimensions:

- θ_i periodic $\Longrightarrow$ vortex loops
- $A_{i\mu}$ periodic $\Longrightarrow$ instantons
- Dual theory $\Longrightarrow$ field theory for a gas of vortex loops and vortex lines with instantons attached at the ends
- θ_i periodic, $A_{i\mu}$ non-periodic $\Longrightarrow$ vortex loops, <u>no instantons</u> $\Longrightarrow$ superconductor universality class, no matter the value of q

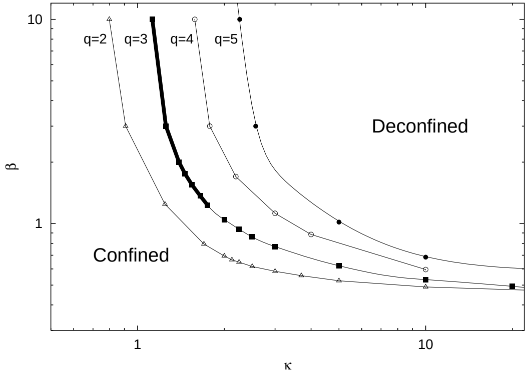






$U(1)$ spin liquid and Compact QED

Phase diagram: [Smiseth,Smørgrav,Nogueira,Hove,Sudbø, PRL 89, 226403 (2002); PRB 67, 205104 (2003)]



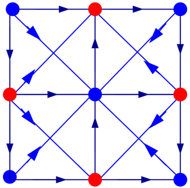
- Confined phase: ordinary Mott insulator
- Deconfined (Higgs) phase: fractionalized insulator
- $q = 3 \Longrightarrow$ tricritical point

Chiral spin liquid

- Hamiltonian: $H = J_1 \sum_{\langle i,j\rangle} \mathbf{S}_i \cdot \mathbf{S}_j + J_2 \sum_{\langle\langle i,j\rangle\rangle} \mathbf{S}_i \cdot \mathbf{S}_j$
- Hubbard-Stratonovich transformation:

$$H = \sum_{\langle i,j\rangle} \left(\frac{N}{J_1} |\chi_{ij}|^2 - \chi_{ij} f^\dagger_{i\alpha} f_{j\alpha} + \text{h.c.} \right) + \sum_{\langle\langle i,j\rangle\rangle} \left(\frac{N}{J_2} |\chi_{ij}|^2 - \chi_{ij} f^\dagger_{i\alpha} f_{j\alpha} + \text{h.c.} \right)$$

- Mean-field Ansatz: π flux in plaquettes, $\pi/2$ flux in triangles



$$\chi_{i,i+\hat{x}} = \mathrm{i}\chi_1$$
$$\chi_{i,i+\hat{y}} = \mathrm{i}(-1)^{i_x}\chi_1$$
$$\chi_{i,i+\hat{x}+\hat{y}} = -\mathrm{i}(-1)^{i_x}\chi_2$$
$$\chi_{i,i+\hat{x}-\hat{y}} = \mathrm{i}(-1)^{i_x}\chi_2$$

- $E_\pm(\mathbf{k}) = \pm\sqrt{4\chi_1^2(\sin^2 k_x + \sin^2 k_y) + 16\chi_2^2 \cos^2 k_x \cos^2 k_y}$

Chiral spin liquid

- MF solution yields $\chi_1 \neq 0$ and $\chi_2 = 0$ if $J_2/J_1 < 1/2$. Both χ_1 and χ_2 are nonzero when $J_2/J_1 > 1/2$
- Chiral spin liquid ($\chi_2 \neq 0$) breaks time-reversal symmetry $\Longrightarrow$ orbital ferromagnetism
- Gapped Dirac spectrum
- The two-component Dirac spinors ψ_1 and ψ_2 have the same mass $m = 4\chi_2$
- Dirac γ matrices: $\gamma^0 = \sigma_z$, $\gamma^1 = -i\sigma_x$, and $\gamma^2 = i\sigma_y$
- Effective Dirac Lagrangian:

$$\mathcal{L} = \sum_{i=1,2} \sum_{\alpha} \bar{\psi}_{i\alpha}(i\partial\!\!\!/ - J\!\!\!/ a\!\!\!/ - m)\psi_{i\alpha}$$

- mass term breaks both parity and TR.
  Parity transf.: $\psi \to \gamma^1\psi$, $\bar{\psi} \to -\bar{\psi}\gamma^1 \Longrightarrow \bar{\psi}\psi \to -\bar{\psi}\psi$.
  TR transf.: $\psi \to \gamma^2\psi$, $\bar{\psi} \to -\bar{\psi}\gamma^2 \Longrightarrow \bar{\psi}\psi \to -\bar{\psi}\psi$

Chiral spin liquid

- Integrating out the fermions generate a Chern-Simons term $\Longrightarrow$ topological field theory. Low-energy form of the CS term:

$$\mathcal{L}_{\mathrm{CS}} = \frac{N}{4\pi}\frac{m}{|m|}\epsilon_{\mu\nu\lambda}a^{\mu}\partial^{\nu}a^{\lambda}$$

- Vacuum polarization:

$$\Sigma_{\mu\nu}(p) = -2NJ^2\int\frac{d^3k}{(2\pi)^3}\mathrm{tr}[\gamma_\mu G(k)\gamma_\nu G(p+k)]$$

$$G(k) = \frac{1}{i\not{k}+m} = \frac{m-i\not{k}}{k^2+m^2}$$

- CS term arises due to $\mathrm{tr}(\gamma_\mu\gamma_\nu\gamma_\lambda) = 2i\epsilon_{\mu\nu\lambda}$
- The CS term is a topological term, since it is independent of the metric

Quantum dimer model

- Hamiltonian (Kivelson, Rokhsar, and Sethna, 1987):

$$
\begin{aligned}
H &= \sum_{\square} \Big[-t\left(|\,\|\,\rangle\langle\,=\,| + |\,=\,\rangle\langle\,\|\,|\right) \\
&+ v\left(|\,\|\,\rangle\langle\,\|\,| + |\,=\,\rangle\langle\,=\,|\right)\Big]
\end{aligned}
$$

- Useful property (not valid for actual spin singlet states):

$$
\begin{aligned}
&\left(|\,\|\,\rangle\langle\,=\,| + |\,=\,\rangle\langle\,\|\,|\right)^2 \\
&= |\,\|\,\rangle\langle\,\|\,| + |\,=\,\rangle\langle\,=\,|
\end{aligned}
$$

- Set $B_\square = |\,\|\,\rangle\langle\,=\,| + |\,=\,\rangle\langle\,\|\,|$

$$
\Longrightarrow H = \sum_{\square}(-tB_\square + vB_\square^2)
$$

Quantum dimer model

- Introduce σ^z_{ij} such that $\sigma^z_{ij} = +1$ when a dimer is present in the bond (i,j), and $\sigma^z_{ij} = -1$ when it is absent. Raising/lowering operators: $\sigma^{\pm}_{ij} = \frac{1}{2}(\sigma^x_{ij} \pm i\sigma^y_{ij})$
- Rewriting Hamiltonian:

$$H = -t\sum_{\square}(W_{\square} + W^{\dagger}_{\square}) + v\sum_{\square}(W_{\square}W^{\dagger}_{\square} + W^{\dagger}_{\square}W_{\square})$$

$$W_{\square} = \sigma^+_{ij}\sigma^-_{jk}\sigma^+_{kl}\sigma^-_{li}$$

- Gauge field: $\sigma^{\pm}_{ij} = \frac{e^{\pm iA_{ij}}}{\sqrt{2}}$

$$\Longrightarrow H = \sum_{i,m,n}\left[-\frac{t}{2}\cos(F_{imn}) + \frac{v}{8}\cos(2F_{imn})\right]$$

- $[\sigma^z_{ij}, \sigma^{\pm}_{ij}] = \pm 2\sigma^{\pm}_{ij} Longrightarrow\ \sigma^z_{ij}$ conjugate to A_{ij}

$$S = is\sum_{\tau,j,n}\sigma^z_{jn}\nabla_{\tau}A_{jn} + H.$$

Quantum dimer model

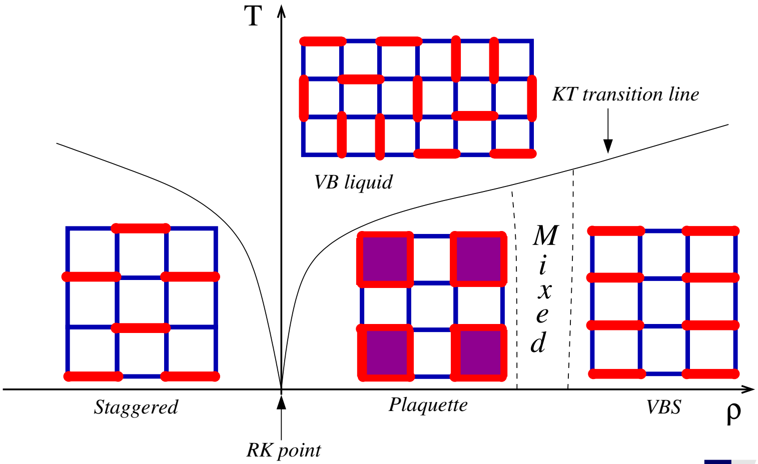
- Lattice gauge theory action (valid both in $d = 2 + 1$ and $d = 3 + 1$):

$$
\begin{aligned}
S &= -\sum_{i,\tau,n} \ln\left[\cos(s\nabla_\tau A_{in})\right] \\
&+ \sum_{i,\tau,m,n}\left[-\frac{t}{2}\cos(F_{imn}) + \frac{v}{8}\cos(2F_{imn})\right].
\end{aligned}
$$

$(F_{imn} = \nabla_m A_{in} - \nabla_n A_{im})$
[F. S. Nogueira and Z. Nussinov, PRB **80**, 104413 (2009)]

Quantum dimer model

Phase diagram for $d = 2 + 1$ ($\rho = t - v$):



Quantum dimer model

- Dual model ($d = 2 + 1$, $\rho = t - v$):

$$\mathcal{L} = \frac{c}{2}(\partial_\tau h)^2 + \frac{\rho}{2}(\nabla h)^2 + \frac{1}{2K}(\nabla^2 h)^2 - z\cos(2\pi h).$$

  Kosterlitz-Thouless-like phase transition at $T = 0$ and for $\rho = 0$ ($t = v$); no transition for $T > 0$ and $\rho = 0$. VBS state for $\rho > 0$ and staggered VBS for $\rho < 0$; KT transition for $T > 0$ and $\rho > 0$

- Dual model ($d = 3 + 1$) at the Rokhsar-Kivelson ($t = v$) point:

$$\tilde{\mathcal{L}} = \frac{K}{2}(\partial_\tau \mathbf{a})^2 + \frac{1}{2c}(\nabla \times \nabla \times \mathbf{a})^2 + |(\nabla - 2\pi i \mathbf{a})\psi|^2 + r|\psi|^2 + \frac{u}{2}|\psi|^4$$

  First-order phase transition at $T = 0$; Second-order transition for $T > 0$

- Dual model ($d = 3 + 1$) above the RK point ($t > v$):

$$\tilde{\mathcal{L}}_{\rho>0} = \frac{1}{2}(\partial_\tau \mathbf{a})^2 + \frac{\rho}{2c}(\nabla \times \mathbf{a})^2 + |(\nabla - 2\pi i\sqrt{\rho}\,\mathbf{a})\psi|^2 + r|\psi|^2 + \frac{u}{2}|\psi|^4$$

  $T = 0$: First-order transition; $T > 0$: Second-order transition

Conclusion

- Is there a spin liquid?
- In theory, yes, at least in some special models on frustrated lattices, or regimes (large N, strong interactions) in well-known models (Hubbard)
- Experiments: promising
- High-T_c superconductors? After more than 20 years, the spin liquid lost some influence here...

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