FUNCTIONAL INTEGRAL APPROACH TO DISORDERED BOSONS

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The appearance of coherence and order is studied for ultracold bosonic atoms in the presence of additional disorder potentials. These arise either naturally like in current carrying wire traps, or artificially and controllably like in laser speckle fields. The description of such disordered bosons within a suitably generalized Bogoliubov theory, first given by Huang and Meng, is rederived here within a functional integral approach for replicated bosonic fields. The superfluidity in homogeneous Bose systems with condensates depleted by weak interactions and disorder can thereby be discussed.

Keywords: Bogoliubov theory; Disordered bosons; Functional integral approach.

1. Introduction

Usually, one studies ultracold quantum gases moving in a one-particle potential $V(\mathbf{x})$ which is fixed by an external magneto-optical trap. Here, however, we consider a different physical situation where the one-particle potential $V(\mathbf{x})$ is fluctuating at each space point $\mathbf{x}$. Such a frozen disorder potential, which consists of a random distribution of hills and valleys, was considered some time ago for modelling superfluid helium in porous media.[1] By doing so, one assumes that the pores can be modelled by statistically distributed local scatterers so that the ensemble average of the disorder potential vanishes

$$\overline{V(\mathbf{x}_1)} = 0 \tag{1}$$

and their correlation function

$$\overline{V(\mathbf{x}_1)V(\mathbf{x}_2)} = R^{(2)}(\mathbf{x}_1, \mathbf{x}_2)\,. \tag{2}$$

decays with a characteristic correlation length ξ which models the average pore size. Nowadays such random potentials can be created artificially and controllably by laser speckles,[2–5] incommensurate lattices,[6] or different localized atomic species.[7] However, they can also arise naturally via the spatial fluctuations of the electric currents generating the magnetic wire traps.[8–10]

Thus, one can raise the fundamental question how the phenomenon of Bose-Einstein condensation is affected by an additional weak disorder. To this end Huang and Meng in 1992 generalized the Bogoliubov theory of ultracold Bose gases[11] for frozen random potentials.[12–16] It turned out that the formation of local condensates in the minima of the random potential reduces the superfluid component of the fluid even at zero temperature, where, in the absence of disorder, the whole fluid would be superfluid.[17] Although this finding agrees qualitatively with the earlier experiments in porous media, the predictions of this Bogoliubov theory for disordered bosons have not yet been tested experimentally in a more quantitative manner. Recently, we have worked out a proposal to test this theory by measuring the disorder-induced shifts in the frequencies of the collective excitations in trapped Bose-Einstein condensed gases,[18] as these can be measured with an accuracy of a few fractions of a percent.[19]

Here we study the Bogoliubov theory of disordered bosons within a functional integral approach. In order to perform ensemble averages with respect to the random potential, we apply the replica method which turned out to be quite useful for studying spin glasses.[20,21] In particular, we find that a replica-symmetric ansatz for the background of the replicated bose fields is sufficient to completely rederive the Huang-Meng theory. There is no indication that replica-symmetry breaking occurs for disordered bosons in contrast to the replica theory of spin glasses. This result coincides with the general physical reasoning that the tunneling of bosons between the local minima of the frozen disorder potential leads to a quantum state with a well-defined global phase where any frustration is absent. Contrary to that replica-symmetry breaking occurs within the classical theory of spin glasses as the spins trapped in the respective local minima of the energy landscape are frustrated.

2. Theoretical Description

We start with the thermodynamical properties of a disordered homogeneous Bose gas. The functional integral for the grand-canonical partition function reads

$$\mathcal{Z} = \oint \mathcal{D}\psi^* \oint \mathcal{D}\psi \, e^{-\mathcal{A}[\psi^*,\psi]/\hbar}\,, \tag{3}$$

where the integration is performed over all Bose fields $\psi^*(\mathbf{x},\tau),\psi(\mathbf{x},\tau)$ which are periodic in imaginary time τ. The euclidean action is given by

$$\mathcal{A}[\psi^*,\psi] = \int_0^{\hbar\beta} d\tau \int d^D x \left\{ \psi^*(\mathbf{x},\tau) \left[\hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2}{2M} \mathbf{\Delta} + V(\mathbf{x}) - \mu \right] \psi(\mathbf{x},\tau) + \frac{g}{2}\, \psi^{*2}(\mathbf{x},\tau)\psi^2(\mathbf{x},\tau) \right\}, \tag{4}$$

where M denotes the particle mass, μ the grand-canonical potential, and g the strength of the contact interaction. As the grand-canonical partition function represents a functional of the disorder potential $V(\mathbf{x})$, the corresponding thermodynamic potential follows from the expectation value

$$\Omega = -\frac{1}{\beta}\, \overline{\ln \mathcal{Z}}\,. \tag{5}$$

In general it is not possible to explicitly evaluate expression (5), as the averaging with respect to the disorder potential $V(\mathbf{x})$ and the nonlinear function of the logarithm do no commute:

$$\overline{\ln \mathcal{Z}} \neq \ln \overline{\mathcal{Z}}\,. \tag{6}$$

An important method to perform the averaging procedure prescribed by (5) is provided by investigating the Nth power of the grand-canonical partition function $\mathcal{Z}$ in the limit $N \to 0$. Indeed, from

$$\mathcal{Z}^N = e^{N \ln \mathcal{Z}} = 1 + N\, \ln \mathcal{Z} + \dots \tag{7}$$

we deduce for the thermodynamic potential (5):

$$\Omega = -\frac{1}{\beta} \lim_{N \to 0} \frac{\overline{\mathcal{Z}^N} - 1}{N}\,. \tag{8}$$

The N-fold replication of the disordered Bose gas (3), (4) and a subsequent averaging with respect to the disorder potential $V(\mathbf{x})$ corresponds to the characteristic functional

$$I[j] = \overline{\exp\left\{ i \int d^D x\, j(\mathbf{x}) V(\mathbf{x}) \right\}} \tag{9}$$

with the auxiliary current field

$$j(\mathbf{x}) = \frac{i}{\hbar} \int_0^{\hbar\beta} d\tau \sum_{\alpha=1}^{N} \psi_\alpha^*(\mathbf{x},\tau)\psi_\alpha(\mathbf{x},\tau)\,. \tag{10}$$

Due to the above assumptions (1) and (2), the characteristic functional is of the general form

$$I[j] = \exp\left\{ \sum_{n=2}^{\infty} \frac{i^n}{n!} \int d^D x_1 \cdots \int d^D x_n R^{(n)}(\mathbf{x}_1,\dots,\mathbf{x}_n)\, j(\mathbf{x}_1) \cdots j(\mathbf{x}_n) \right\} \tag{11}$$

By definition the cumulant functions $R^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ are symmetric with respect to their arguments $\mathbf{x}_1,\ldots,\mathbf{x}_n$. Therefore, the disordered Bose gas is described by the disorder averaged, replicated grand-canonical partition function

$$\overline{\mathcal{Z}^N} = \left\{ \prod_{\alpha=1}^{N} \oint \mathcal{D}\psi_\alpha^* \oint \mathcal{D}\psi_\alpha \right\} e^{-\mathcal{A}^{(N)}[\psi^*,\psi]/\hbar} \tag{12}$$

with the replica action

$$\mathcal{A}^{(N)}[\psi^*,\psi] = \int_0^{\hbar\beta} d\tau \int d^D x \sum_{\alpha=1}^{N} \left\{ \psi_\alpha^*(\mathbf{x},\tau) \left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2M}\mathbf{\Delta} - \mu \right] \psi_\alpha(\mathbf{x},\tau) \right.$$
$$\left. + \frac{g}{2}\, |\psi_\alpha(\mathbf{x},\tau)|^4 \right\} + \sum_{n=2}^{\infty} \frac{1}{n!} \left(\frac{-1}{\hbar} \right)^{n-1} \int_0^{\hbar\beta} d\tau_1 \cdots \int_0^{\hbar\beta} d\tau_n \int d^D x_1 \cdots \int d^D x_n$$
$$\int d^D x_n \sum_{\alpha_1=1}^{N} \cdots \sum_{\alpha_n=1}^{N} R^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n)\, |\psi_{\alpha_1}(\mathbf{x}_1,\tau_1)|^2 \cdots |\psi_{\alpha_n}(\mathbf{x}_n,\tau_n)|^2\ . \tag{13}$$

Thus, in leading order $n = 2$ the random potential leads to a residual attractive interaction between the replica fields $\psi_\alpha^*(\mathbf{x},\tau)$, $\psi_\alpha(\mathbf{x},\tau)$ which is, in general, bilocal in both space and imaginary time.

3. Bogoliubov Theory

Now we apply the field-theoretic background method[23–25] in order to derive the effective potential for the replicated action (13) within a Bogoliubov theory.[11] As we restrict ourselves to a homogeneous Bose gas, the background fields are assumed to be independent of space and imaginary time. Furthermore, as the replica action (13) has a global $U(1) \times O(N)$-symmetry, we assume that the background fields are replica symmetric. Thus, we arrive at the decomposition

$$\psi_\alpha(\mathbf{x},\tau) = \sqrt{n_0} + \delta\psi_\alpha(\mathbf{x},\tau)\,, \tag{14}$$

where n_0 denotes the condensate density. Inserting this decomposition in the replica action (13), we have only to take into account the fluctuation fields $\delta\psi_\alpha^*(\mathbf{x},\tau)$, $\delta\psi_\alpha(\mathbf{x},\tau)$ in zeroth and second order due to the background method. Performing the replica limit $N \to 0$ according to (8), then yields the effective potential defined by

$$V_{\rm eff}(n_{\mathbf{0}}) = -\frac{1}{V\beta} \lim_{N\to 0} \frac{\ln \overline{Z^N}}{N}\,. \tag{15}$$

A lengthy but straight-forward calculation shows that, within the Bogoliubov theory, all higher contributions $n > 2$ of the cumulant functions $R^{(n)}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ do not contribute to the effective potential due to the replica limit $N \to 0$. Thus, we end up with

$$V_{\text{eff}} = -\mu n_{\mathbf{0}} + \frac{g}{2} n_{\mathbf{0}}^2 + \int \frac{d^D k}{(2\pi)^D} \left\{ \frac{E(\mathbf{k})}{2} + \frac{1}{\beta} \ln \left[1 - e^{-\beta E(\mathbf{k})} \right] - \frac{\epsilon(\mathbf{k}) - \mu + g n_{\mathbf{0}}}{E(\mathbf{k})^2} R(\mathbf{k}) n_{\mathbf{0}} \right\} \tag{16}$$

with the Fourier transformed

$$R(\mathbf{k}) = \int d^D x \, e^{-i\mathbf{k}\mathbf{x}} R^{(2)}(\mathbf{0}, \mathbf{x}) \tag{17}$$

as well as the dispersions $\epsilon(\mathbf{k}) = \hbar^2 \mathbf{k}^2 / 2M$ and

$$E(\mathbf{k}) = \sqrt{\left[\epsilon(\mathbf{k}) - \mu + g n_{\mathbf{0}}\right]^2 + 2 \left[\epsilon(\mathbf{k}) - \mu + g n_{\mathbf{0}}\right] g n_{\mathbf{0}}} \, . \tag{18}$$

4. Condensate Density

Extremizing the effective potential (16) with respect to the condensate density $n_{\mathbf{0}}$ then yields the particle density via $n = -\partial V_{\text{eff}} / \partial \mu$:

$$n = n_{\mathbf{0}} + \int \frac{d^D k}{(2\pi)^D} \left\{ \frac{\epsilon(\mathbf{k}) + gn}{E(\mathbf{k})} \left[\frac{1}{2} + \frac{1}{e^{\beta E(\mathbf{k})} - 1} \right] + \frac{n R(\mathbf{k})}{\left[\epsilon(\mathbf{k}) + 2gn\right]^2} \right\} . \tag{19}$$

Here $E(\mathbf{k})$ denotes the Bogoliubov dispersion

$$E(\mathbf{k}) = \sqrt{\epsilon(\mathbf{k})^2 + 2\epsilon(\mathbf{k}) gn} \, , \tag{20}$$

which does not depend on the disorder. This finding of the original Huang-Meng theory[12] here stems from the fact that the disorder average in (13) is taken before analyzing the quantum fluctuations. From (19) we read off at $T = 0$ that the depletion of the condensate density consists of two terms. The interaction-induced depletion is given by the UV-divergent expression

$$\Delta n_{\mathbf{0}}^{(\text{int})} = - \int \frac{d^D k}{(2\pi)^D} \frac{\epsilon(\mathbf{k}) + gn}{2 \sqrt{\epsilon(\mathbf{k})^2 + 2\epsilon(\mathbf{k}) gn}} \, , \tag{21}$$

which is calculated by applying dimensional regularization.[26] In $D = 3$ dimension Eq. (21) reduces with $g = 4\pi\hbar^2 a / M$ and the s-wave scattering length a to the well-known result of Bogoliubov:[11]

$$\Delta n_{\mathbf{0}}^{(\text{int})} = - \frac{8}{3\sqrt{\pi}} \, (an)^{3/2} \, . \tag{22}$$

Correspondingly, the disorder-induced depletion reads

$$\Delta n_{\mathbf{0}}^{(\mathrm{dis})} = -\int \frac{d^D k}{(2\pi)^D} \frac{R(\mathbf{k})n}{\left[\epsilon(\mathbf{k}) + 2gn\right]^2} \,, \tag{23}$$

which can be further evaluated for a specific Fourier-transformed disorder correlation $R(\mathbf{k})$.[14,18]

5. Superfluid Density

In order to determine the superfluid density of a disordered Bose gas, we transform our system to an inertial frame with moves uniformly with the velocity $\mathbf{u}$ with respect to the laboratory.[27] A corresponding Galilei boost leads to the additional action

$$\delta\mathcal{A} = \int_0^{\hbar\beta} d\tau \int d^D x \psi^*(\mathbf{x},\tau)\, \mathbf{u}\frac{\hbar}{i}\boldsymbol{\nabla}\psi(\mathbf{x},\tau)\,. \tag{24}$$

Going again through the replica and the Bogoliubov formalism yields the effective potential

$$V_{\mathrm{eff}} = -\mu n_{\mathbf{0}} + \frac{g}{2}\, n_{\mathbf{0}}^2 + \int \frac{d^D k}{(2\pi)^D}\Bigg\{ \frac{E(\mathbf{k})}{2} + \frac{1}{\beta}\ln\left[1 - e^{-\beta E(\mathbf{k})}\right] - R(\mathbf{k})n_{\mathbf{0}} \tag{25}$$
$$\times \frac{\epsilon(\mathbf{k}) - \mu + g n_{\mathbf{0}}}{E(\mathbf{k})^2} - \left[\frac{\beta\hbar^2\mathbf{k}^2\, e^{\beta E(\mathbf{k})}}{2D\left[e^{\beta E(\mathbf{k})} - 1\right]^2} + \frac{R(\mathbf{k})n_{\mathbf{0}}\hbar^2\mathbf{k}^2\left[\epsilon(\mathbf{k}) - \mu + g n_{\mathbf{0}}\right]}{DE(\mathbf{k})^4}\right]\mathbf{u}^2\Bigg\}$$

Its extremum with respect to the condensate density coincides with the grand-canonical potential Ω whose explicit dependence on the boost velocity $\mathbf{u}$ defines the momentum $\mathbf{p} = -\partial\Omega/\partial\mathbf{u}$ of the system. This yields

$$\mathbf{p} = MVn\mathbf{v} + V\int \frac{d^D k}{(2\pi)^D}\left\{ \frac{\hbar^2\beta\mathbf{k}^2 e^{\beta E(\mathbf{k})}}{D\left[e^{\beta E(\mathbf{k})} - 1\right]^2} + \frac{2\hbar^2 R(\mathbf{k})n\mathbf{k}^2\epsilon(\mathbf{k})}{DE(\mathbf{k})^4}\right\}\mathbf{u}\,. \tag{26}$$

Thus, the system momentum (26) is of the form $\mathbf{p} = MVn_n\mathbf{u} + \ldots$, which defines the normal density n_n. With this we obtain

$$n_n = \int \frac{d^D k}{(2\pi)^D}\left\{ \frac{\hbar^2\beta\mathbf{k}^2 e^{\beta E(\mathbf{k})}}{DM\left[e^{\beta E(\mathbf{k})} - 1\right]^2} + \frac{4R_0 n}{D\left[\epsilon(\mathbf{k}) + 2gn\right]^2}\right\}, \tag{27}$$

whose complement defines the superfluid density $n_s = n - n_n$. At $T = 0$ we read off from (27) that the superfluid density has no interaction-induced depletion[17] but a depletion due to disorder

$$\Delta n_s^{(\mathrm{dis})} = -\frac{4n}{D}\int \frac{d^D k}{(2\pi)^D} \frac{R(\mathbf{k})}{\left[\epsilon(\mathbf{k}) + 2gn\right]^2}\,. \tag{28}$$

A comparison of (23) with (28) shows that the disorder-induced depletions of the condensate and the superfluid density are related via

$$\Delta n_s^{(\mathrm{dis})} = \frac{4}{D}\,\Delta n_{\mathbf{0}}^{(\mathrm{dis})} \tag{29}$$

irrespective of the Fourier-transformed disorder correlation $R(\mathbf{k})$. This means in $D = 3$ dimensions that the depletion of the superfluid density exceeds the corresponding one of the condensate density by a factor $4/3$. On general physical grounds it is understandable that this factor is larger than 1. The localized condensates, which form in the randomly distributed minima of the random potential, do not contribute to and thereby "hamper" the superfluid motion.

6. Conclusion and Outlook

Here we have rederived the Huang and Meng theory[12] for a homogeneous three-dimensional hard-sphere Bose gas in a random external potential within a functional integral approach. In qualitative agreement with the experiments in porous media,[1] the formation of local condensates in the minima of the random potential reduces the superfluid component of the fluid even at zero temperature, where, in the absence of disorder, the whole fluid would be superfluid. The recent experimental advances in trapping Bose-Einstein condensates in a disordered medium[3,4] make it interesting to test in a more quantitative manner the predictions of the model considered by Huang and Meng. In Ref. 18 we, therefore, extended the latter approach and the approach of the present work to include a harmonic trapping potential in addition to the weak external random potential. We considered there a condensate in the limit of a large number of particles N and in the presence of disorder with a correlation length shorter than the healing length of the superfluid. These conditions allowed for a simple hydrodynamical formulation of the problem similar to the theory of wave propagation in random elastic media.[28]

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