Collective Motion of Polarized Dipolar Fermi Gases in the Hydrodynamic Regime

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Abstract

Recently, a seminal STIRAP experiment allowed the creation of ⁴⁰K⁸⁷Rb molecules in the rovibrational ground state [K.-K. Ni et al., Science 322, 231 (2008)]. In order to describe such a polarized dipolar Fermi gas in the hydrodynamic regime, we work out a variational time-dependent Hartree-Fock approach. With this we calculate dynamical properties of such a system as, for instance, the frequencies of the low-lying excitations and the time-of-flight expansion. We find remarkable effects of a strong dipole-dipole interaction such as anisotropic breathing oscillations in momentum space and a suppression of the aspect-ratio inversion after release of the harmonic trap.

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Even before the realization of Bose-Einstein condensation (BEC) with $^{52}$Cr [1], much experimental and theoretical interest has been dedicated to ultracold quantum gases interacting through the long-range and anisotropic dipole-dipole interaction (DDI) [2]. For bosonic dipolar particles, the starting point of the theoretical investigations was the construction of a corresponding pseudo-potential by Yi and You [3]. After that, an exact solution of the Gross-Pitaevskii equation in the Thomas-Fermi regime was found for cylinder-symmetric traps [4]. Moreover, the DDI has been shown to shift the BEC critical temperature in a characteristic way in polarized systems [5] and to give rise to the Einstein-de-Haas effect, when spinorial degrees of freedom are considered [6]. From the experimental point of view, time-of-flight (TOF) techniques demonstrated both the first DDI-signature through small mechanical effects [7] as well as strong dipolar effects in quantum ferrofluids [8]. Furthermore, the shape of the trap was manipulated to stabilize a purely dipolar BEC against collapse [9].

Concerning fermionic dipolar systems, recent theoretical studies have considered interesting properties of homogeneous gases such as zero sound [10], Berezinskii-Kosterlitz-Thoules phase transition [11], and nematic phases [12]. In harmonically trapped systems, amazing predictions like anisotropic superfluidity [13], fractional quantum Hall physics [14], and Wigner crystallization [15] have been made. With respect to experimental investigations, the most promising atomic candidate is the fermionic chromium isotope $^{53}$Cr [16], which has a magnetic moment of $m = 6 m_B$ with $m_B$ denoting the Bohr magneton. For these atoms, calculations of equilibrium properties have shown that the DDI is only a small perturbation [17, 18]. However, by applying a stimulated Raman adiabatic passage (STIRAP) process, it has recently been achieved to cool and trap $^{40}$K$^{87}$Rb molecules into their rovibrational ground-state, where they possess an electric dipole moment of $d = 0.566$ Debye [19–22]. Due to the resulting strong DDI a considerable deformation of the momentum distribution is expected [17, 18]. Once these systems would have been further cooled into the quantum degenerate regime, the main task will be to identify unambiguously the presence of the DDI. In this respect, TOF experiments and oscillation frequency measurements represent the most fundamental diagnostic tools in the field of ultracold quantum gases. Their outcomes reveal important information on the nature of the system under investigation. They differ drastically depending on whether the system is in the collisionless regime, where collision rates are small, or in the hydrodynamic regime, where collisions take place so often that they lead to
local equilibrium. To date, investigations of dynamical properties of trapped dipolar Fermi gases have either been restricted to the collisionless regime [23] or excluded a deformation of the momentum distribution in the hydrodynamic regime [24]. Since the experiments with ultracold polar molecules are performed under strong dipolar interactions, one should expect them to lead the system into the hydrodynamic regime, and thus an analysis allowing for an anisotropy in the momentum distribution has to be carried out. In this letter, we shall use a variational time-dependent Hartree-Fock approach to address this question.

Consider $N$ spin-polarized fermionic dipoles of mass $M$ trapped in a cylinder-symmetric harmonic potential $U_{tr}(x) = M\omega_z^2 (x^2 + y^2 + \lambda^2 z^2)/2$ with trap anisotropy $\lambda$ at $T = 0$. Since the Pauli principle inhibits a contact interaction, they interact dominantly through $\pi$-dipoles of mass $M$. For electric dipole moments $d$ the DDI is characterized by $C_{dd} = \mu_0 m^2$, whereas for electric dipole moments $d$ we have $C_{dd} = 4\pi d^2$. In this letter we work out a time-dependent Hartree-Fock approach by extremizing the action $A = \int dt \langle \Psi | i\hbar \frac{\partial}{\partial t} - \hat{H} | \Psi \rangle$, where $\Psi(x_1,\cdots,x_N,t) = \langle x_1,\cdots,x_N | \Psi \rangle$ is a Slater determinant of one-particle orbitals and $\hat{H}$ denotes the underlying Hamilton operator. In order to describe the hydrodynamic regime, we follow a standard procedure of nuclear physics [25] and assume that frequent particle collisions result in the same phase $\chi(x,t)$ for all one-particle orbitals. Thus, we define a time-even Slater determinant through $\Psi_0(x_1,\cdots,x_N,t) = e^{-iM \sum_{i=1}^{N} \chi(x_i,t)/\hbar} \Psi(x_1,\cdots,x_N,t)$, which yields a time-even one-body density matrix $\rho_0(x,x';t) = e^{-iM [\chi(x,t)-\chi(x',t)]/\hbar} \rho(x,x';t)$. With this the action reduces to

$$A = -M \int dt \int d^3 x \left\{ \chi(x,t) \rho_0(x,t) + \frac{\rho_0(x,t)}{2} [\nabla \chi(x,t)]^2 \right\} - \int dt \langle \Psi_0 | \hat{H} | \Psi_0 \rangle,$$

where $\rho_0(x;t) = \rho_0(x,x;t)$ denotes the particle density and $\langle \Psi_0 | \hat{H} | \Psi_0 \rangle$ consists of the kinetic energy $E_{ki}$, the trapping potential $E_{tr}$, and the interaction, the latter being divided into the direct or Hartree term $E_{dd}^{D}$ as well as the exchange or Fock term $E_{dd}^{E}$. Due to the exchange term, the ground-state energy $\langle \Psi_0 | \hat{H} | \Psi_0 \rangle$ is not a function of the particle density $\rho_0(x;t)$ alone, but also contains the non-diagonal part $\rho_0(x,x';t)$.

As it is not possible to solve analytically the resulting Euler-Lagrange equations for $\chi(x,t)$ and $\rho_0(x,x';t)$, we propose here a variational extremization of the action. To this end, we express each energy contribution in terms of the Wigner transform of the one-body density
matrix \( \nu_0(\mathbf{X}, \mathbf{k}; t) = \int d^3 s \rho_0 (\mathbf{X} + \frac{s}{2}, \mathbf{X} - \frac{s}{2}; t) e^{-i\mathbf{k} \cdot \mathbf{s}} \). The kinetic and trapping energy are then given by

\[
E_{kl/tr} = \int \frac{d^3 x d^3 k}{(2\pi)^3} \nu_0 (\mathbf{x}, \mathbf{k}; t) \epsilon_{kl/tr} (\mathbf{x}, \mathbf{k})
\]  

(2)

with \( \epsilon_{kl} = \hbar^2 k^2 / 2M \) and \( \epsilon_{tr} = U_{tr}(\mathbf{x}) \), respectively. The direct term, which accounts for the deformation of the particle density, and the exchange term, which is related to the momentum space deformation, read

\[
E_{dd}^D = \int \frac{d^3 x d^3 k d^3 k' d^3 x' d^3 k'}{(2\pi)^6} \nu_0 (\mathbf{x}, \mathbf{k}; t) \nu_0 (\mathbf{x}', \mathbf{k}'; t) V_{dd}(\mathbf{x} - \mathbf{x}') e^{-i\mathbf{s} \cdot (\mathbf{k} - \mathbf{k}')}
\]

\[
E_{dd}^E = \int \frac{d^3 x d^3 k d^3 s d^3 k'}{(2\pi)^6} \nu_0 (\mathbf{x}, \mathbf{k}; t) \nu_0 (\mathbf{x}', \mathbf{k}'; t) V_{dd}(\mathbf{s}) e^{i\mathbf{s} \cdot (\mathbf{k} - \mathbf{k}')}
\]

(3)

At this point, we adopt the variational ansatz \( \chi(x, t) = [\alpha_x(t)(x^2 + y^2) + \alpha_z(t) z^2] / 2 \) for the phase and \( \nu_0 (\mathbf{x}, \mathbf{k}; t) = \Theta \left( 1 - \frac{x^2 + y^2}{R_x(t)^2} - \frac{z^2}{R_z(t)^2} - \frac{k_x^2 + k_y^2}{K_x(t)^2} - \frac{k_z^2}{K_z(t)^2} \right) \) for the Wigner phase space function with \( \Theta(\cdot) \) being the step function. With this we are now in the position to extremize the action (1) with respect to the time-dependent variational parameters \( \alpha_i(t) \) for the global phase as well as \( R_i(t) \) and \( K_i(t) \) for the Thomas-Fermi radii and the Fermi momenta. At first, one obtains \( \alpha_i = \dot{R}_i / R_i \), which is used to eliminate the parameters \( \alpha_i \) from the rest of the formalism. Under conservation of the particle number

\[
\dot{R}_x^2 \dot{R}_y K_x^2 \dot{K}_x = 1
\]  

(4)
FIG. 2: (Color Online) The lower (red) curve shows the ratio of the amplitudes $\zeta_x/\zeta_z$ as a function of $\epsilon_{dd}$ for $\lambda = 5$. For comparison, the equilibrium aspect ratio in momentum space against $\epsilon_{dd}$ for $\lambda = 5$ is depicted by the upper (blue) curve.

The equations of motion for the Thomas-Fermi radii read

$$\frac{1}{\omega_x^2} \frac{d^2 \hat{R}_x}{dt^2} = -\hat{R}_x + \frac{2 \tilde{K}_x^2 + \tilde{K}_z^2}{3 \hat{R}_x} + \epsilon_{dd} A(\hat{R}_x, \hat{R}_z, \tilde{K}_x, \tilde{K}_z),$$

$$\frac{1}{\omega_z^2} \frac{d^2 \hat{R}_z}{dt^2} = -\hat{R}_z + \frac{2 \tilde{K}_x^2 + \tilde{K}_z^2}{3 \hat{R}_z} + \epsilon_{dd} B(\hat{R}_x, \hat{R}_z, \tilde{K}_x, \tilde{K}_z).$$

Here we use $\bullet$ to represent the parameter $\bullet$ expressed in units of the non-interacting Thomas-Fermi radius $R_i^{(0)} = \sqrt{2E_F/M \omega_i^2}$ and the Fermi momentum $K_F = \sqrt{2E_F/\hbar^2}$ with the Fermi energy $E_F = (6N)^{1/3} \hbar \omega$. Furthermore the average trap frequency is $\bar{\omega} = (\omega_x^2 \omega_z^2)^{1/3}$ and the dipolar strength reads $\epsilon_{dd} = C_{dd} \left( \frac{M^2 \omega^5}{\hbar^2} \right) N^{1/3}$. The auxiliary functions are defined according to

$$A = -\frac{c_3}{\hat{R}_x \hat{R}_z} \left[ 1 - \frac{3 \hat{R}_x^2 \lambda^2 f_s \left( \hat{R}_x \lambda / \hat{R}_z \right)}{2 \left( \hat{R}_z^2 - \hat{R}_x^2 \lambda^2 \right)} - f_s \left( \frac{\tilde{K}_z}{\tilde{K}_x} \right) \right],$$

$$B = -\frac{c_3}{\hat{R}_x \hat{R}_z} \left[ -2 + \frac{3 \hat{R}_z^2 f_s \left( \hat{R}_z \lambda / \hat{R}_x \right)}{\left( \hat{R}_z^2 - \hat{R}_x^2 \lambda^2 \right)} - f_s \left( \frac{\tilde{K}_z}{\tilde{K}_x} \right) \right]$$

with the numerical constant $c_3 = \frac{2^{38/3}}{3^{23/3} \pi^{7/2}} \approx 0.2791$. Furthermore, the anisotropy function

$$f_s(x) = \begin{cases} \frac{2x^2 + 1}{1-x^2} - \frac{3x^2 \text{artanh} \sqrt{1-x^2}}{(1-x^2)^{3/2}}; & x \neq 1 \\ 0; & x = 1 \end{cases},$$

decreases monotonically from 1 at $x = 0$ to $-2$ at $x = \infty$, passing through zero at $x = 1$ and is commonly found in the dipolar BEC literature [4, 5]. In addition, the variational parameters are restricted to obey a further constraint at time $t$

$$\tilde{K}_z^2 - \tilde{K}_x^2 = \epsilon_{dd} C \left( \hat{R}_x, \hat{R}_z, \tilde{K}_x, \tilde{K}_z \right),$$

5
FIG. 3: (Color Online) Excitation frequencies for $\lambda = 5$ as functions of the DDI-strength $\epsilon_{dd}$. The upper blue (lower red) curve represents the monopole (quadrupole) frequency $\Omega_+$ ($\Omega_-$). The dashed (dotted) horizontal line represents the monopole (quadrupole) frequency of the non-interacting gas from Ref. [26].

with $C = \frac{3c_0}{R^2 R_z} \left[ 1 - \frac{(2\tilde{K}^2_z + \tilde{K}^2_x) f_s(\tilde{K}_z/\tilde{K}_x)}{2(\tilde{K}^2_z - \tilde{K}^2_x)} \right]$. This equation can be traced back to the exchange term and shows explicitly that a non-zero $\epsilon_{dd}$ implies a deformed momentum distribution $\tilde{K}_z \neq \tilde{K}_x$ for finite $\tilde{R}_x$, $\tilde{R}_z$ as was first pointed in Ref. [17]. Equations (4)–(6), (8) govern the static as well as dynamic properties of a polarized dipolar Fermi gas in the hydrodynamic regime and are the main result of this letter. They determine the temporal evolution of both the spatial and the momentum distribution of a dipolar Fermi gas which are directly experimentally accessible via TOF techniques. The static solutions agree precisely with the ones obtained before in Refs. [17, 18]. In Fig. 1 we present our findings for the spatial aspect ratio as a function of the dipolar strength $\epsilon_{dd}$. The characteristic feature is that a minimal value of $\lambda$ is required for stabilizing a system with a given $\epsilon_{dd}$. In the current set-up of Ref. [22] one has $4 \cdot 10^4 \, ^{40}\text{K}^{87}\text{Rb}$ molecules with a radial trapping frequency of $\omega_x = \omega_y \approx 2\pi \cdot 175$ Hz. Assuming an average trap frequency of that value yields for the dipole interaction strength $\epsilon_{dd} \approx 5.3$. Thus, for future experiments in the quantum degenerate regime one should choose the experimental anisotropy $\lambda$ to be much larger than the minimal value $\lambda_{\text{min}} \approx 3.89$ in order to render the system stable against collapse. Amazingly, the minimum value of $\lambda$, which supports a stable gas, diminishes very slowly so that $\lambda = 0.05$ still supports stable samples for $\epsilon_{dd} \lesssim 1.6$. Having summarized the most important aspects of the static solutions, we turn now to the dynamical properties of the system, which are the main subject of this letter. In a cylinder-symmetric system the mono- and quadrupole low-lying oscillation modes couple to
FIG. 4: (Color Online) Mono- (blue) and quadrupole (red) oscillation frequencies of the dipolar 
Fermi gas normalized by the non-interacting values from Ref. [26] as functions of the trap aspect 
ratio $\lambda$ for different values of the dipolar strength $\epsilon_{dd}$. The dashed curves are for $\epsilon_{dd} = 0.5$ and the 
solid ones for $\epsilon_{dd} = 1.0$.

each other. In order to obtain the frequency of these modes in the hydrodynamic regime, 
we expand the radii and momenta around their respective equilibrium values according to 
$\tilde{R}_i = \tilde{R}_i(0) + \eta_i e^{i\Omega t}$, $\tilde{K}_i = \tilde{K}_i(0) + \zeta_i e^{i\Omega t}$, where $\eta_i$ ($\zeta_i$) denotes a small oscillation amplitude 
in the $i$-th direction in real (momentum) space and $\Omega$ represents the oscillation frequency. 
Inserting these into the equations of motion (4)–(6), (8), a linearization yields at first for 
the ratio of the momentum amplitudes

$$\frac{\zeta_x}{\zeta_z} = \frac{\tilde{K}_x \tilde{K}_z^2 + \tilde{K}_z^2 - \epsilon_{dd} \tilde{K}_z \partial C / \partial \tilde{K}_z}{2\tilde{K}_z^2 - \epsilon_{dd} \tilde{K}_z \partial C / \partial \tilde{K}_z},$$  

(9)

where all terms are evaluated at equilibrium. This quantity is plotted against $\epsilon_{dd}$ for $\lambda = 5$ in 
the red (lower) curve in Fig. 2 and is compared with the corresponding equilibrium momentum 
aspect ratio (blue, upper curve). Setting $C = 0$, i. e., removing the exchange term, one 
has $\zeta_x = \zeta_z$, whereas for $0 < \epsilon_{dd} < \epsilon_{dd}^{\text{crit}} \approx 7.34$, the ratio $\zeta_x/\zeta_z$ decreases monotonically from 
1 to about 0.28. This shows that the exchange term gives rise to characteristic anisotropic breathing oscillations in momentum space, which can be regarded as a trademark sign of 
dipolar effects in fermionic quantum gases.

Eliminating the momentum amplitudes $\zeta_i$ yields a reduced linear homogeneous system for 
the spatial amplitudes $\eta_i$. Demanding non-trivial solutions yields an explicit but lengthy 
result for the monopole (quadrupole) oscillation frequency $\Omega_+$ ($\Omega_-$) which depends via the 
equilibrium values of the Thomas-Fermi radii $\tilde{R}_x, \tilde{R}_z$ and the Fermi momenta $\tilde{K}_x, \tilde{K}_z$ upon 
the trap anharmonicity $\lambda$ and the dipolar strength $\epsilon_{dd}$. In the special case of an ideal Fermi 
gas, i. e. $\epsilon_{dd} = 0$, the oscillation frequencies $\Omega_\pm$ reduce to the correct non-interacting values
FIG. 5: (Color Online) Cloud aspect ratio in TOF expansion for $\lambda = 0.75$ with $\epsilon_{dd} = 0, 0.3, \text{ and } 1.8$ (continuous, top to bottom). The dotted curves depict the momentum aspect ratio for $\epsilon_{dd} = 0, 0.3, \text{ and } 1.8$ (bottom to top). Notice that increasing $\epsilon_{dd}$ prevents the real space aspect ratio to become larger than one, while it always becomes asymptotically unity in momentum space. The inlet shows the asymptotic aspect ratio as a function of $\epsilon_{dd}$.

$$\Omega_{\pm}^{(0)} = \omega_x^2 \left( 5 + 4\lambda^2 \pm \sqrt{25 - 32\lambda^2 + 16\lambda^4} \right) / 3,$$

which were first obtained in Ref. [26]. Fig. 3 shows the oscillation frequencies of the mono- (blue) and quadrupole (red) modes plotted against $\epsilon_{dd}$ for $\lambda = 5$. As $\epsilon_{dd}$ becomes larger, we find that the monopole frequency increases and that the quadrupole frequency decreases, becoming imaginary at $\epsilon_{dd}^{\text{crit}} \approx 7.34$, the same value for which the system becomes unstable (see Fig. 1). In Fig. 4 we have also studied how the frequencies depend on the anisotropy $\lambda$ for $\epsilon_{dd} = 0.5$ (dashed) and $\epsilon_{dd} = 1.0$ (continuous). It turns out that the quadrupole frequencies are larger than in the non-interacting case for $\lambda < 1$ and smaller for $\lambda > 1$, while the contrary is true for the monopole modes. This behaviour agrees qualitatively with dipolar BECs [4].

It remains to study the TOF expansion of a dipolar Fermi gas. This is done by numerically solving the Eqs. (4)–(6), (8), while removing the trap frequencies. The results are presented in Fig. 5, where the spatial and momentum aspect ratios are plotted as functions of time (in units of $\omega_x^{-1}$) at $\lambda = 0.75$ for different $\epsilon_{dd}$. For a non-interacting gas we find that the aspect ratio is inverted during the time evolution. This remains true for weak interactions, but strong dipolar interactions eventually prevent this inversion in close analogy with the physics of dipolar BECs [8] (See inlet of Fig. (5)). This has to be compared with the aspect ratio in momentum space (dotted curves) which becomes asymptotically isotropic as a consequence of local equilibration.
In the present letter we have investigated both low-lying oscillation frequencies and TOF expansion data for a polarized dipolar Fermi gas in the hydrodynamic regime. Our findings have revealed different characteristic fingerprints for the presence of a strong DDI. Therefore, we are confident that our results will play a crucial role for detecting DDI, once $^{40}\text{K}^{87}\text{Rb}$ molecules will have been cooled into the quantum degenerate regime.

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