

## Mean-field phase diagram of disordered bosons in a lattice at nonzero temperature

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**Abstract.** Bosons in a periodic lattice with on-site disorder at low but nonzero temperatures are considered within a mean-field theory. The criteria used for the definition of the superfluid, Mott insulator and Bose glass are analysed. Since the compressibility never vanishes at nonzero temperatures, it cannot be used as a general criterion. We show that the phases are unambiguously distinguished by the superfluid density and the density of states of the low-energy excitations. The phase diagram of the system is calculated. It is shown that even a tiny temperature leads to a significant shift of the boundary between the Bose glass and superfluid.

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**Contents**

<b>1. Introduction</b>	<b>2</b>
<b>2. Hamiltonian</b>	<b>3</b>
<b>3. Boundary between the superfluid and non-superfluid phases</b>	<b>4</b>
3.1. Pure case. . . . .	5
3.2. Disorder with homogeneous distribution . . . . .	5
<b>4. Boundary between the Mott-insulator and Bose-glass phases</b>	<b>6</b>
4.1. Green's function . . . . .	7
4.2. Density of states . . . . .	8
<b>5. Compressibility</b>	<b>10</b>
<b>6. Conclusions</b>	<b>11</b>
<b>Acknowledgment</b>	<b>13</b>
<b>References</b>	<b>13</b>

**1. Introduction**

The remarkable experimental control over ultracold atomic gases in optical lattices acquired in the last couple of years [1]–[6] has opened up completely new lines of interest in the field of Bose–Einstein condensation. One of these is ultracold atoms in optical lattices with disorder, which can be created by several methods. One of the possibilities is to use a laser speckle field [7]–[9]. An alternative way of creating a disorder potential is the introduction of a tiny fraction of a second atomic species which is strongly localized on random sites [10]–[12]. Tuning to a Feshbach resonance, these random scatterers can even make the disorder very strong. The random potentials for atoms can be also created via the spatial fluctuations of the electric currents generating the magnetic wire traps [13]–[15] or with the aid of the incommensurate lattices [16, 17]. Disordered lattices for ultracold rubidium atoms have been recently created by superimposing a regular periodic optical potential on the speckle field [18] and on the incommensurate lattice [19].

These very recent experimental developments prompt the theoretical question of what the behaviour of ultracold atomic gases in lattice potentials with disorder might be. Typical for the modern area of Bose–Einstein condensation (BEC) in particular and ultracold atomic gases in general is that such basic theoretical questions have been asked before in a different context. Indeed, there is an enormous body of literature on ultracold Fermi gases in disordered lattices referring, first of all, to electrons in amorphous solids, either in the normal or in the superconducting state [20]. The possibility to study fermionic atomic gases from this point of view is a very active and interesting field with which we shall not deal here, however. Suffice it to say that completely new questions appear like the BEC–BCS (Bardeen–Cooper–Schrieffer) crossover [21, 22]. The influence of disorder on this problem is still completely unknown.

Here, we shall focus exclusively on bosons in potentials with disorder. This is also known as the ‘dirty boson problem’. It first came up in the pre-BEC area in the context of experimental investigations of superfluidity of  $^4\text{He}$  in the random pores of Vycor. The surprising finding that, for sufficiently low coverage of the pores, the  $^4\text{He}$  superfluidity would disappear even in an extrapolation to zero temperature [23]–[25] prompted many theoretical studies.

These were based on Hartree–Fock theory [26, 27], generalizations of the Bogoliubov and the Beliaev theories [28] for random potentials [29]–[36], field-theoretical considerations [37, 38], mean-field theory [38, 39], renormalization group theory [40]–[44] and quantum Monte Carlo simulations [45]–[54] as well as numerical diagonalizations [55]. The consensus which developed from these studies is that in a disordered lattice at temperature  $T = 0$ , two new phases of bosons may exist besides the superfluid phase which is theoretically defined by the presence of the off-diagonal long-range order. One is the Mott-insulator phase, which only exists at commensurate fillings of the lattice, and is distinguished by the absence of the off-diagonal long-range order, a nonzero energy gap and a vanishing compressibility. The other is the Bose-glass phase, which is distinguished again by the absence of the off-diagonal long-range order, a non-vanishing density of states at zero energy, and a nonzero compressibility. A more recent suggestion towards identifying the Bose-glass phase has been made in [56].

While these operational definitions of a Mott-insulator phase and a Bose-glass phase at temperature  $T = 0$  are precise and clear-cut, they run into the obvious difficulty that experiments and quantum Monte Carlo simulations are never performed at  $T = 0$ . It is therefore necessary to examine the extent to which these or similar definitions can be applied at least at small nonzero temperatures. This is the goal of the present paper. In order to achieve our goal, we have to investigate the low-lying states of a suitable model of strongly interacting bosons in a lattice with disorder. We shall choose for this purpose a Bose–Hubbard model with on-site disorder of bounded variation. For our purpose of defining and distinguishing the various phases at nonzero temperature, it is sufficient to analyse the basic model within a mean-field theory which allows us to detect all the phases and to see what happens if the system parameters are changed, although it does not provide precise conditions for various phase transitions encountered in the system.

The paper is organized as follows. In section 2, the model and the mean-field approach to its analysis are defined. Then, in section 3, the phase boundary between the superfluid and the two non-superfluid phases is derived from the condition of vanishing off-diagonal long-range order. This is first done for the pure case without disorder, reproducing a well-known result, and then generalizing it to the case with disorder. In section 4 follows the definition of the Bose-glass and Mott-insulator phases at nonzero temperature and the examination of the phase boundary between them. A common criterion for the distinction of these two phases at  $T = 0$  is the compressibility. How well this quantity serves this purpose at finite temperature is therefore examined in section 5. In section 6, the paper ends with some final conclusions.

## 2. Hamiltonian

We consider a system of spinless bosons in a homogeneous infinitely extended lattice of dimension  $d = 1, 2, 3$  described by the Bose–Hubbard Hamiltonian (in units of  $\hbar = 1$ )

$$H_{\text{BH}} = -J \sum_{\langle i,j \rangle} a_i^\dagger a_j + \frac{U}{2} \sum_i a_i^\dagger a_i^\dagger a_i a_i - \sum_i (\mu + \epsilon_i) a_i^\dagger a_i, \quad (1)$$

where  $J$  is the tunnelling matrix element,  $U$  is the on-site interaction constant, and  $\mu$  is the chemical potential. In this study, we assume that the random on-site energies  $\epsilon_i$  at different sites are uncorrelated and equally distributed with the probability density  $p(\epsilon)$ .

We introduce the superfluid order parameter  $\psi = \langle a_i \rangle$ , where  $\langle \dots \rangle = \text{Tr}[\dots \exp(-\beta H)] / \text{Tr}[\exp(-\beta H)]$  and  $\dots = \prod_i \int_{-\infty}^{+\infty} \dots p(\epsilon_i) d\epsilon_i$  denote quantum-mechanical and disorder

averaging, respectively, and  $\beta = 1/(kT)$ . Making use of the decoupling mean-field approximation in the hopping term [57]–[59] which is valid for sufficiently high-dimensional systems, we obtain the following on-site Hamiltonian

$$H = -2dJ(\psi a^\dagger + \psi^* a) + 2dJ|\psi|^2 + \frac{U}{2}a^\dagger a^\dagger a a - (\mu + \epsilon)a^\dagger a, \quad (2)$$

where we have omitted the site index.

The phase diagram of the system can be obtained in the following manner. First of all, one has to calculate the disorder-averaged free energy of the system  $\overline{F(\psi)}$  corresponding to the Hamiltonian (2). Then minimizing it with respect to  $\psi$  to determine  $\psi = \psi_m$ , one can distinguish the superfluid ( $\psi_m \neq 0$ ) and non-superfluid ( $\psi_m = 0$ ) regions of the parameter space. Applying a small phase gradient to the atomic matter-field operator and calculating the corresponding correction to the free energy in a manner similar to [60], one can show that the superfluid density following from the Hamiltonian (2) equals  $|\psi_m|^2$ . In the non-superfluid region, one has to work out the disorder average of the static superfluid susceptibility  $\overline{\chi}$  or the density of states  $\overline{\rho(\omega)}$  for the single-particle excitations [38]. In the region where  $\overline{\rho(\omega)} = 0$  in the interval  $0 \leq \omega < \omega_g$  we have, by the definition we apply, the Mott-insulator phase with the energy gap  $\omega_g$ . On the other hand, again by definition, the Bose-glass phase occurs when  $\lim_{\omega \rightarrow 0} \overline{\rho(\omega)} \neq 0$  which corresponds to a divergent superfluid susceptibility [38].

The form of the mean-field Hamiltonian (2) implies that, in our approximation, the properties of the Mott-insulator phase as well as the Bose-glass phase, where  $\psi$  vanishes, do not depend on the tunnelling matrix element  $J$ . This is consistent with the fact that the boundary between these phases occurs only for small values of  $J$ . The transition to superfluidity, where  $\psi_m \neq 0$  starts to appear, also depends on  $J$  in our approximation.

### 3. Boundary between the superfluid and non-superfluid phases

In order to calculate the free energy of the system, one has to solve the eigenvalue problem for the Hamiltonian (2). This can be done exactly by means of numerical calculations. However, the boundary between the superfluid and non-superfluid phases can also be determined with high accuracy treating the first term in the Hamiltonian (2) as a perturbation. The free energy can only depend on  $|\psi|^2$  since a change of the phase in  $\psi$  can be undone by the unitary transformation  $a \rightarrow \exp(-i\varphi)a$ ,  $a^\dagger \rightarrow \exp(i\varphi)a^\dagger$ . Indeed, the calculations show that the result has the following structure:

$$\overline{F(\psi)} = \overline{a_0} + \overline{a_2}|\psi|^2 + \overline{a_4}|\psi|^4 + \dots \quad (3)$$

The explicit form of  $a_4$  as well as  $a_0$  and  $a_2$  for  $T = 0$  was obtained in [59]. The generalization to  $T \neq 0$  and the average over disorder needed here is straightforward. Since  $\overline{a_4}$  turns out to be always positive and  $\overline{a_2}$  can be either positive or negative, the superfluid/non-superfluid transition is of second order. The equation  $\overline{a_2} = 0$  determines the phase boundary which is given by

$$\int_{-\infty}^{+\infty} \frac{d\mu' p(\mu - \mu')}{Z_0(\mu')} \sum_{m=0}^{\infty} \left[ \frac{m}{\mu' - U(m-1)} + \frac{m+1}{Um - \mu'} \right] e^{-\beta E_m(\mu')} = \frac{1}{2dJ}, \quad (4)$$

with

$$Z_0(\mu) = \sum_{m=0}^{\infty} e^{-\beta E_m(\mu)}, \quad E_m(\mu) = \frac{U}{2}m(m-1) - \mu m, \quad (5)$$

being the partition function without hopping.

### 3.1. Pure case

In the pure case, we have  $p(\epsilon) = \delta(\epsilon)$ , so we get from (4) for the phase boundary [61]

$$\frac{2dJ}{Z_0(\mu)} \sum_{m=0}^{\infty} \left[ \frac{m}{\mu - U(m-1)} + \frac{m+1}{Um - \mu} \right] e^{-\beta E_m(\mu)} = 1. \quad (6)$$

In the zero-temperature limit, this equation reduces to the well-known result for the boundary between the superfluid and Mott-insulator [58, 59]

$$2dJ = \frac{[\mu - U(n-1)][Un - \mu]}{\mu + U}. \quad (7)$$

Here,  $n$  denotes the positive integer at which  $E_m(\mu)$  is minimal with respect to  $m$ . This fixes  $n$  as the smallest integer larger than or equal to  $\mu/U$ .

### 3.2. Disorder with homogeneous distribution

In the following, we choose for simplicity a homogeneous disorder distribution in the interval  $\epsilon \in [-\Delta/2, \Delta/2]$

$$p(\epsilon) = \frac{1}{\Delta} [\Theta(\epsilon + \Delta/2) - \Theta(\epsilon - \Delta/2)], \quad (8)$$

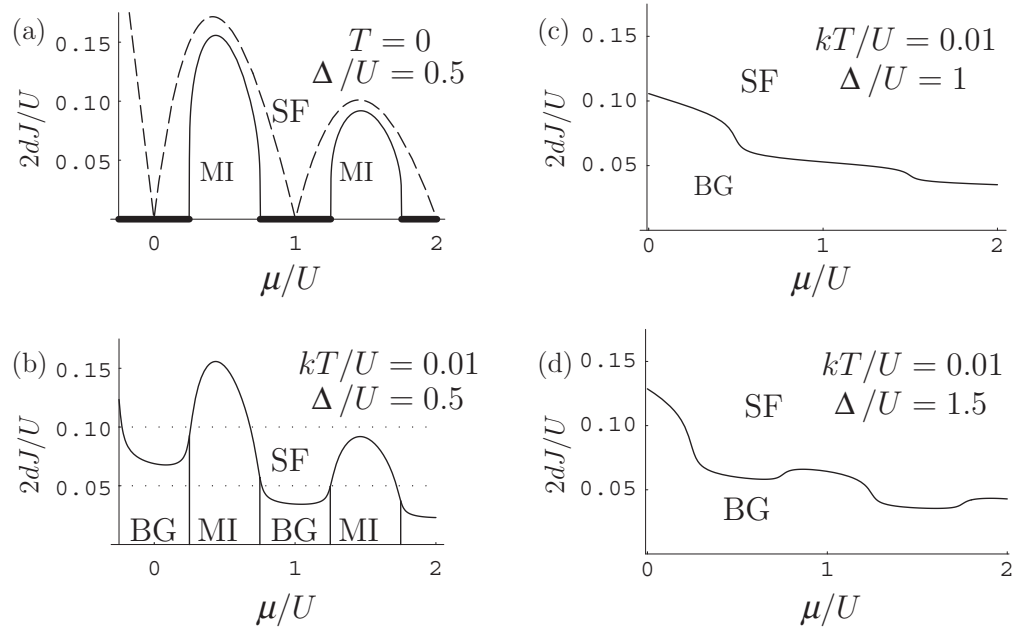
so we have for the phase boundary (4)

$$\frac{2dJ}{\Delta} \int_{\mu-\Delta/2}^{\mu+\Delta/2} \frac{d\mu'}{Z_0(\mu')} \sum_{m=0}^{\infty} \left[ \frac{m}{\mu' - U(m-1)} + \frac{m+1}{Um - \mu'} \right] e^{-\beta E_m(\mu')} = 1. \quad (9)$$

We discuss first the special case  $T = 0$  and consider  $\Delta < U$ . It is assumed that  $\mu \in [U(n-1), Un]$ ,  $n = 1, 2, \dots$ . Equation (9) then gives

$$2dJ = \Delta \left[ n \ln \frac{\mu - U(n-1) + \Delta/2}{\mu - U(n-1) - \Delta/2} - (n+1) \ln \frac{Un - \mu - \Delta/2}{Un - \mu + \Delta/2} \right]^{-1}, \quad (10)$$

if  $\mu \in [U(n-1) + \Delta/2, Un - \Delta/2]$ , otherwise  $2dJ = 0$ . In the limit  $\Delta \rightarrow 0$ , equation (10) reduces to (7). The phase boundary at  $T = 0$  following from equation (10) is shown in figure 1(a) for typical parameter values. It has a lobe structure and the size of the lobes decreases with increasing  $\Delta$ . If  $T = 0$  but  $\Delta > U$ , we obtain  $2dJ = 0$  as the transition line for any value of  $\mu$ , i.e., the lobes disappear and the superfluid phase appears as soon as  $J$  is turned on [38].

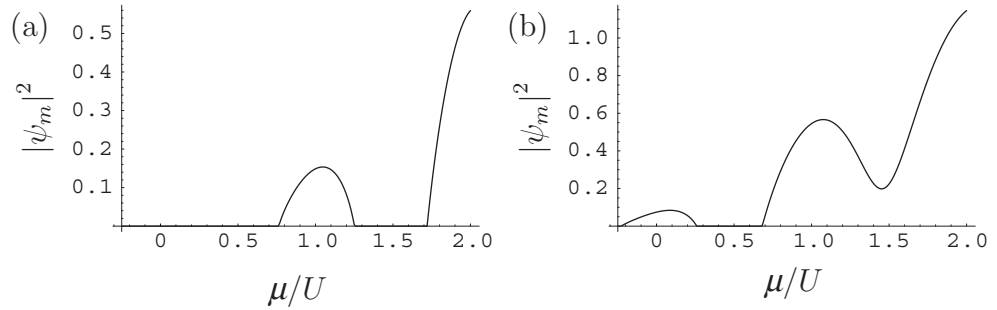


**Figure 1.** Mean-field  $(\mu, J)$  phase diagram for the homogeneous disorder distribution (8). At  $T = 0$  (a), the Bose-glass phase exists only for  $J = 0$  (bold lines). The dashed line shows the boundary (7) between the superfluid phase and the Mott-insulator phase in the pure case. At  $T \neq 0$  (b)–(d), the Bose-glass phase exists at  $J \neq 0$  as well. If the disorder becomes strong (c) and (d), the Mott-insulator phase disappears. MI, Mott insulator with energy gap; BG, Bose glass with non-vanishing density of states at zero energy; SF, superfluid with non-vanishing superfluid density. The dotted lines in (b) indicate the two values of  $J$  used to obtain the plots in figures 2, 7 and 8.

In the case of nonzero temperature, equation (9) always gives non-vanishing values of  $2dJ$ . The boundary between the superfluid and non-superfluid phases for small  $T$  and different values of  $\Delta$  is shown in figures 1(b)–(d). The plots obtained from equation (9) and that obtained by numerically diagonalizing the Hamiltonian (2) and minimizing the free energy with respect to  $\psi$  are indistinguishable. If the temperature increases, the boundary between the superfluid and non-superfluid phases goes upwards and the size of the non-superfluid region grows. The presence of even a small temperature ( $kT/U = 0.01$ ) changes rather strongly the phase boundary. This is in contrast to the pure case where values of  $kT/U \sim 1$  are required in order to get a noticeable shift of the boundary between the Mott-insulator and the superfluid phases [62]. The superfluid density  $|\psi_m|^2$  obtained by the numerical diagonalization for the same parameters as in figure 1(b) and the values of  $J$  indicated by dotted lines in figure 1(b) is plotted in figure 2.

#### 4. Boundary between the Mott-insulator and Bose-glass phases

In the Mott-insulator as well as in the Bose-glass phase, the superfluid order parameter  $\psi = \psi_m$  vanishes, which implies that  $J$  disappears from the mean-field Hamiltonian (2). Therefore, the properties of these two phases do not depend on  $J$  in the mean-field approximation. This is related



**Figure 2.** Superfluid density for  $kT/U = 0.01$ ,  $\Delta/U = 0.5$ ,  $2dJ/U = 0.05$  (a),  $0.1$  (b) (along the dotted lines in figure 1(b)).

to the fact that the two phases are localized and, therefore, the dependence on  $J$  should be weak. However, more accurate calculations beyond the mean-field theory should give some dependence on  $J$ , in particular close to the boundary with the superfluid phase. The Mott-insulator phase is characterized by the gap in the excitation spectrum and it has a finite superfluid susceptibility. The Bose-glass phase has no gap and the superfluid susceptibility diverges. All this is directly related to the properties of Green's functions and the density of states.

#### 4.1. Green's function

The bosonic single-particle Green's function  $G(t)$  is defined as [63]

$$G(t) = -i[\Theta(t)G_{>}(t) + \Theta(-t)G_{<}(t)], \quad G_{>}(t) = \langle a(t)a^\dagger(0) \rangle, \quad G_{<}(t) = \langle a^\dagger(0)a(t) \rangle, \quad (11)$$

where  $a(t)$  is the annihilation operator in the Heisenberg representation. Straightforward calculations lead to the result

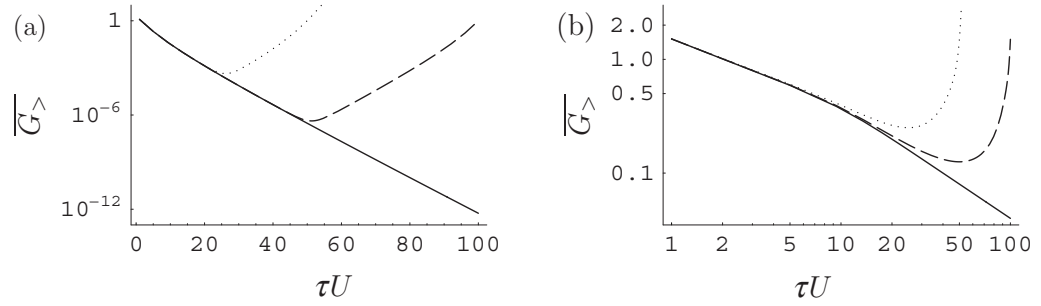
$$G_{>}(t) = \frac{1}{Z_0(\mu')} \sum_{m=0}^{\infty} (m+1) e^{(\mu' - Um)it - \beta E_m(\mu')}, \quad G_{<}(t) = \frac{1}{Z_0(\mu')} \sum_{m=0}^{\infty} m e^{[\mu' - U(m-1)]it - \beta E_m(\mu')}, \quad (12)$$

where  $\mu' = \mu + \epsilon$  denotes the random local chemical potential. One can easily show that the imaginary-time Green's functions satisfy the periodicity condition  $G_{>}(\tau + \beta) = G_{<}(\tau)$ , where  $\tau = it$  is the imaginary time.

At  $T = 0$ , the imaginary-time Green's function takes the form

$$\overline{G_{>}(\tau)} = \frac{(n_- + n_+ + 1)(n_+ - n_-)}{2\tau\Delta} + \frac{1}{\tau\Delta} [(n_+ + 1) e^{-(Un_+ - \mu - \Delta/2)\tau} - (n_- + 1) e^{-(Un_- - \mu + \Delta/2)\tau} - \frac{1}{2}(n_- + n_+ + 3)(n_+ - n_-) e^{-U\tau}], \quad (13)$$

where  $n_{\pm}$  is the smallest integer greater than or equal to  $(\mu \pm \Delta/2)/U$ . If  $n_+ = n_-$  which corresponds to  $\mu \in [U(n-1) + \Delta/2, Un - \Delta/2]$  for  $\Delta < U$ , the first term in equation (13) vanishes and the superfluid susceptibility defined by the integral over the real-time Green's function  $\overline{\chi} = \int_0^\infty \overline{G_{>}(t)} dt$  is a finite quantity. This means that we do not have the Bose-glass phase,



**Figure 3.** Correlation function  $\overline{G}_>$  for  $\Delta/U = 0.5$  in the Mott-insulator ( $\mu/U = 0.5$  (a)) and Bose-glass ( $\mu/U = 1$  (b)) phase.  $T = 0$  (solid lines),  $kT/U = 0.01$  (dashed lines),  $kT/U = 0.02$  (dotted lines). Note that the scale of  $\overline{G}_>$  is logarithmic but the scale of  $\tau$  is linear in (a) and logarithmic in (b), respectively.

i.e., we are in the Mott-insulator phase (see figure 1(a)). If  $n_+ > n_-$  which corresponds to  $\mu \in [U(n-1), U(n-1) + \Delta/2] \cup [Un - \Delta/2, Un]$  for  $\Delta < U$  or arbitrary  $\mu$  for  $\Delta > U$ , the first term in equation (13) survives and renders  $\bar{\chi}$  divergent which is the distinguishing property of the Bose-glass phase. In the case  $\Delta > U$ , the lobes in figure 1(a) disappear completely which means that the Mott-insulator phase is destroyed by the disorder [38].

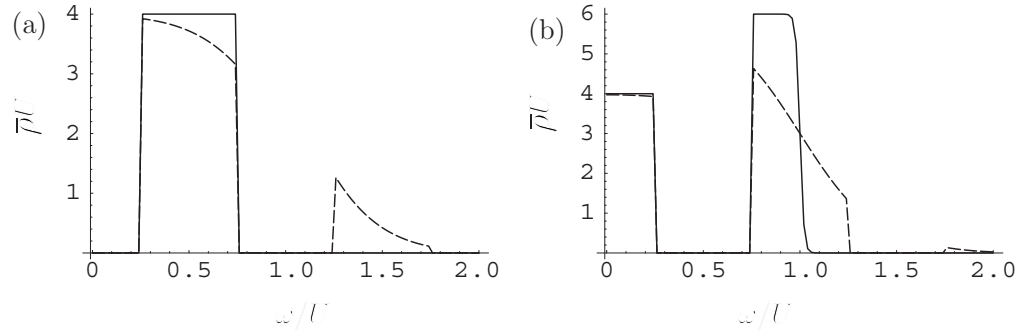
At nonzero temperature, it is more difficult to analyse the structure of the disorder averaged Green's function. Expanding equation (12) for large but finite values of  $\beta$  shows that Green's function has a similar structure as equation (13) but the explicit expressions become very long and we do not display them. Typical  $\tau$ -dependences of  $\overline{G}_>$  in the Mott-insulator as well as in the Bose-glass phase are shown for different temperatures in figure 3. Due to the different scales of  $\tau$ , figures 3(a) and (b) indicate that the Mott-insulator and the Bose-glass phase are characterized by an exponential and algebraic decay of  $\overline{G}_>(\tau)$ , respectively. These analytical results agree qualitatively with Monte Carlo simulations [47]–[49], [52, 54].

Since even at finite temperature  $\overline{G}_>(\tau)$  decays only like  $1/\tau$ , the superfluid susceptibility still diverges logarithmically. Vice versa, the divergent superfluid susceptibility  $\bar{\chi}$  has the consequence that the density of states for the single-particle excitations at zero energy does not vanish in contrast to the case of finite  $\bar{\chi}$  [38]. In the next section, we will see that the density of states is easier to analyse at finite temperature than the full Green's function.

#### 4.2. Density of states

The density of states for the single-particle excitations can be determined in terms of the Fourier transformed single-particle Green's function  $\tilde{G}(\omega) = \int_{-\infty}^{+\infty} dt \exp(i\omega t) G(t)$  as  $\rho(\omega, \mu) = -\frac{1}{\pi} \text{Im} \tilde{G}(\omega)$  [28]. The Fourier transformation of equations (11) and (12) gives the density of states for the pure case

$$\rho(\omega, \mu) = \frac{1}{Z_0(\mu)} \sum_{m=0}^{\infty} e^{-\beta E_m(\mu)} [m\delta(\omega + \mu - U(m-1)) + (m+1)\delta(\omega + \mu - Um)]. \quad (14)$$



**Figure 4.** (a) Density of states in the Mott-insulator phase for  $\Delta/U = 0.5$ ,  $\mu/U = 0.5$ ,  $kT/U = 0.01$  (solid line),  $0.2$  (dashed line). (b) Density of states in the Bose-glass phase for  $\Delta/U = 0.5$ ,  $\mu/U = 1$ ,  $kT/U = 0.01$  (solid line),  $0.2$  (dashed line).

The two  $\delta$ -functions correspond to the hole and particle excitations, respectively. After the disorder averaging we obtain

$$\overline{\rho(\omega, \mu)} = \sum_{m=0}^{\infty} \frac{(m+1)p(Um - \mu - \omega)}{Z_0(Um - \omega)} [e^{-\beta E_{m+1}(Um - \omega)} + e^{-\beta E_m(Um - \omega)}]. \quad (15)$$

This disorder averaged density of states is plotted in figure 4(a) for the Mott-insulator phase with the energy gap  $\omega_g = Un - \Delta/2 - \mu$  and in figure 4(b) for the Bose-glass phase. Since  $E_{m+1}(Um) = E_m(Um)$ , we get finally

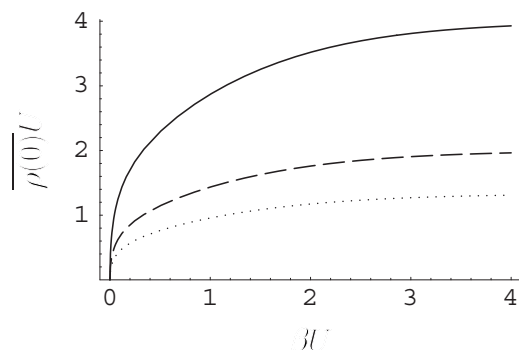
$$\overline{\rho(0, \mu)} = 2 \sum_{m=0}^{\infty} \frac{(m+1)p(Um - \mu)}{Z_0(Um)} e^{-\beta E_m(Um)}. \quad (16)$$

For the homogeneous disorder distribution (8), the summation in equation (16) is restricted by  $m = n_-, \dots, n'_+$ , where  $n_-$  is the smallest integer greater than or equal to  $(\mu - \Delta/2)/U$  and  $n'_+$  is the greatest integer less than or equal to  $(\mu + \Delta/2)/U$ . If  $\Delta < U$ , equation (16) takes the form

$$\overline{\rho(0, \mu)} = \begin{cases} \frac{2n}{\Delta Z_0(U(n-1))} e^{-\beta E_{n-1}(U(n-1))} & \mu \in \mathcal{G}_1, \\ 0 & \mu \in \mathcal{M}, \\ \frac{2(n+1)}{\Delta Z_0(Un)} e^{-\beta E_n(Un)} & \mu \in \mathcal{G}_2. \end{cases} \quad (17)$$

where we defined  $n = n(\mu)$  as the smallest integer larger than or equal  $\mu/U$  and where  $\mathcal{G}_1 = [U(n-1), U(n-1) + \Delta/2]$ ,  $\mathcal{G}_2 = [Un - \Delta/2, Un]$ ,  $\mathcal{M} = [U(n-1) + \Delta/2, Un - \Delta/2]$ . The temperature dependence of  $\overline{\rho(0, \mu)}$  is plotted in figure 5. In the limit  $T \rightarrow 0$ , equation (17) reduces to

$$\overline{\rho(0, \mu)} = \begin{cases} n/\Delta & \mu \in \mathcal{G}_1, \\ 0 & \mu \in \mathcal{M}, \\ (n+1)/\Delta & \mu \in \mathcal{G}_2. \end{cases} \quad (18)$$



**Figure 5.** Density of states at zero energy in the Bose-glass phase for  $\mu/U = 1$ ,  $\Delta/U = 0.5$  (solid line), 1.0 (dashed line), 1.5 (dotted line).

The simple analytical expressions (17) and (18) show that the lines  $\mu = U(n - 1) + \Delta/2$  and  $\mu = Un - \Delta/2$  determine the boundaries between the Mott-insulator and Bose-glass phases at an arbitrary temperature (see figure 1). In the case  $\Delta > U$  of strong disorder,  $\rho(0, \mu)$  does not vanish for finite temperature and the Mott-insulator phase does not exist.

## 5. Compressibility

The compressibility of the system is defined as  $\overline{\kappa(\mu)} = -\partial^2 \overline{F(\mu)} / \partial \mu^2$ , where  $\overline{F(\mu)} = -\ln Z(\mu) / \beta$ . In a non-superfluid phase  $Z(\mu) = Z_0(\mu)$  is given by equation (5). Partial integration over  $\epsilon$  gives

$$\overline{\kappa(\mu)} = -\frac{1}{\beta} \frac{\partial}{\partial \mu} \int_{-\infty}^{+\infty} \ln Z_0(\mu + \epsilon) \frac{dp(\epsilon)}{d\epsilon} d\epsilon. \quad (19)$$

For the homogeneous distribution (8), we get

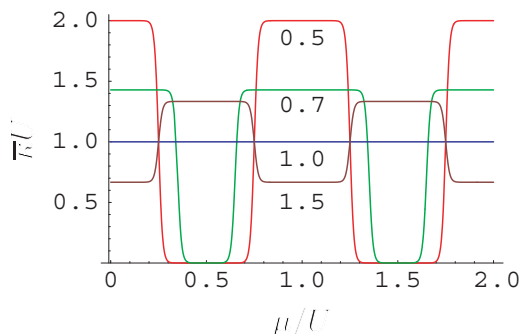
$$\overline{\kappa(\mu)} = \frac{1}{\Delta} [N(\mu + \Delta/2) - N(\mu - \Delta/2)], \quad (20)$$

where

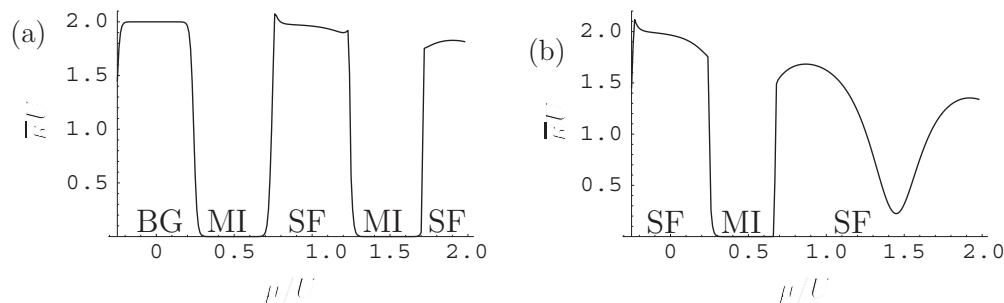
$$N(\mu) = \langle a^\dagger a \rangle = \frac{1}{Z_0(\mu)} \sum_{m=0}^{\infty} m e^{-\beta E_m(\mu)}, \quad (21)$$

is the mean particle number per lattice site in the pure case. For the Bose-glass phase, the compressibility (20) does not vanish. One can easily show that

$$\lim_{\Delta \rightarrow 0} \overline{\kappa(\mu)} = \kappa(\mu) = \beta [\langle (a^\dagger a)^2 \rangle - \langle a^\dagger a \rangle^2] = \beta \left[ \frac{1}{Z_0(\mu)} \sum_{m=0}^{\infty} m^2 e^{-\beta E_m(\mu)} - N^2(\mu) \right]. \quad (22)$$



**Figure 6.** Compressibility for  $kT/U = 0.01$ ,  $J = 0$  and the values of  $\Delta/U$  given below the lines worked out according to equation (20).



**Figure 7.** Compressibility for  $kT/U = 0.01$ ,  $\Delta/U = 0.5$ ,  $2dJ/U = 0.05$  (a),  $2dJ/U = 0.1$  (b) (along the dotted lines in figure 1(b)). The results are obtained by numerically diagonalizing the Hamiltonian (2) and calculating the second derivative of the free energy with respect to  $\mu$  at  $\psi = \psi_m$ .

If  $\beta U \gg 1$ , the compressibility (20) expanded for small temperatures has the form

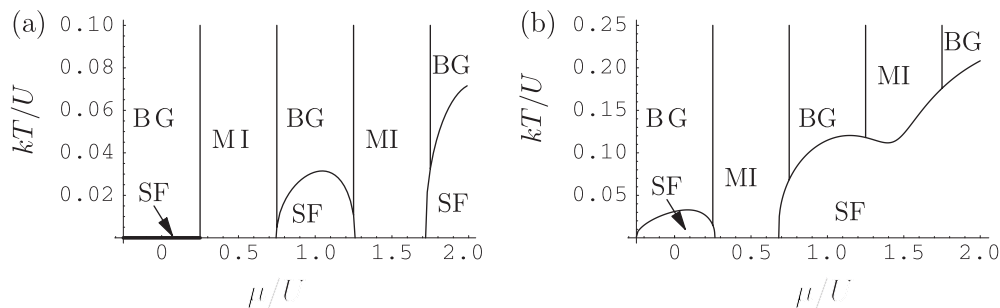
$$\overline{\kappa(\mu)} \approx \frac{n_+ - n_-}{\Delta} + \alpha e^{-\beta\delta}, \quad (23)$$

where  $\delta(\mu) = E_n - \min(E_{n-1}, E_{n+1})$  is the energy difference between the first excited state and the ground state in the pure case (cf (5)), and  $\alpha$  is some finite constant. This equation shows that the Mott-insulator phase, which occurs for  $n_+ = n_-$ , has an exponentially small compressibility at nonzero temperature, in contrast to the Bose-glass phase.

The dependence of the compressibility on  $\mu$  for small temperatures is shown in figures 6 and 7. Since the compressibility does not vanish at nonzero temperatures and is a continuous function of the system parameters, it cannot be used as a criterion to distinguish between different phases. Thus, we deduce that the transitions are better defined in terms of the superfluid order parameter  $\psi$  and the density of states.

## 6. Conclusions

The Mott-insulator phase and the Bose-glass phase at vanishing temperatures can be defined either by their thermodynamic properties or by the spectral properties of their low-lying excitations.



**Figure 8.** Phase diagram in  $(\mu, T)$ -plane corresponding to the two dotted lines in figure 1(b), with the tunnelling amplitude  $2dJ/U = 0.05$  (a),  $0.1$  (b).

Both characterizations are, of course, closely related. Both phases are non-superfluid, i.e., the corresponding Goldstone modes, the phonons, are absent. In the case of the Mott-insulator phase, the spectral characterization by an energy gap implies a vanishing compressibility and vanishing particle-number fluctuations at  $T = 0$ . In the Bose-glass phase, the non-vanishing density of states at zero energy implies a non-vanishing compressibility. These features allow a sharp distinction between the two phases at zero temperature.

However, at non-vanishing temperatures, the characterization of the Mott-insulator and Bose-glass phases by their thermodynamic properties is no longer sharp—the Mott-insulator phase has an exponentially small but finite compressibility which corresponds to non-vanishing fluctuations of the particle number density. Still, as we have pointed out in this paper, the characteristic spectral features remain present also at  $T > 0$  and can therefore be used for a sharp definition and distinction between these low temperature phases. We employed this possibility to calculate finite-temperature phase diagrams within a Bose–Hubbard model with on-site disorder within the mean-field approximation. For experiments with optical lattices it is usually easiest to change system parameters like the tunnelling amplitude  $J$  at fixed temperature, i.e., phase diagrams of the format of figure 1, where system parameters are used as variables, are most natural from this point of view. However, from a thermodynamic point of view, it may be more natural to give the phase diagram in the  $(\mu, T)$ -plane. This is done in figure 8 for the two values of  $J$  marked in figure 1(b) as dotted lines. On the high temperature side, the phase diagram of figure 8 is incomplete, because there the transition to the normal gas phase must occur, which we have not considered in the present study. For both the Mott-insulator and the Bose-glass phase this transition would be sharp, if the energy gap or the finite density of states would start to appear suddenly at a critical temperature. Alternatively, the transition could also take the form of a smooth crossover. For the Mott-insulator phase the crossover could occur at the temperature  $kT \approx \omega_g$ , where  $\omega_g$  is the energy-gap for thermal excitations [57, 62]. For the Bose-glass phase, this would happen at the temperature where the density of states starts to be dominated by the normal Bose gas.

In our calculations, we find direct transitions from the superfluid phase either to the Mott-insulator or to the Bose-glass phase under variation of  $J$ , depending on the values of  $\mu$  and the disorder strength  $\Delta$  (figure 1). The character of the transition from the superfluid to the Bose-glass phase is in agreement with Monte Carlo calculations. The direct transition from the superfluid to the Mott-insulator phase in the case of weak disorder  $\Delta < U$  is also in agreement with the results obtained by the path-integral Monte Carlo techniques [45] and by the

Monte Carlo simulations based on the  $J$ -current model, which neglects amplitude fluctuations of the bosonic quantum fields, for  $\mu/U = 0.5$  [48, 49, 51, 52] and for some finite interval of  $\mu$  near the tip of the Mott-insulator lobe [50]. However, more recent Monte Carlo investigations of the  $J$ -current model [53], still neglecting the amplitude fluctuations, show that the superfluid and the Mott-insulator regions on the phase diagram are separated by a narrow region occupied by the Bose-glass phase if the number of lattice sites is large enough. Thus, different Monte Carlo techniques give different results on the character of the superfluid-Mott-insulator transition in the presence of disorder. Other methods lead also to mutually conflicting results concerning this point [64]. Here, we have presented the results of the mean-field approach.

Experimentally, the Bose-glass phase may not be easy to identify with ultracold atoms in a suitably disordered lattice. In fact, it might best be identifiable indirectly by the absence of properties which are present in the competing phases for  $T \rightarrow 0$ , like the absence of a macroscopic wavefunction and the absence of an energy gap or of incompressibility [19]. The finite density of states at  $\omega \rightarrow 0$  would show up in a specific heat proportional to  $T$  for  $T \rightarrow 0$  and in a logarithmically diverging susceptibility  $\bar{\chi}$ . It would certainly be of great interest if a way could be found to measure  $\rho(0)$  directly.

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