ELSEVIER

Physica D 125 (1999) 123–141

PHYSICA D

Travelling wave dynamics in a nonlinear interferometer with spatial field transformer in feedback

E.V. Grigorieva [a,*], H. Haken [a], S.A. Kashchenko [b], A. Pelster [c]

[a] *Institute for Theoretical Physics and Synergetics, University of Stuttgart, Pfaffenwaldring 57, 70550 Stuttgart, Germany*
[b] *Department of Mathematics, Yaroslavl State University, 150000 Yaroslavl, Russia*
[c] *Institute for Theoretical Physics, Free University of Berlin, Arnimallee 14, 14195 Berlin, Germany*

Received 2 September 1997; accepted 22 June 1998
Communicated by Y. Kuramoto

Abstract

The properties of different spatial structures which emerge in a nonlinear resonator with a spatial transformer in a feedback are investigated. In detail, we determine the parameter regions, the amplitudes and the rotation frequency of various optical reverberators. Special emphasis is devoted to critical cases of infinite dimension which occur in a system with large geometrical size. Evolution equations obtained describe multistable travelling waves, slowly rotating step structures, or modulated waves in dependence on the spatial shift in the feedback. Sensitive dependence of the phase space construction on geometrical size of the system is demonstrated even in the case of a large-scale system. © 1999 Elsevier Science B.V.

PACS: 42.65.Sf; 02.30.Jr
Keywords: Nonlinear optics; Feedback; Pattern formation; Bifurcation analysis

1. Introduction

Recently, different types of instability of a light field were observed in a nonlinear interferometer with spatial field transformer in two-dimensional feedback (FB) [1,2]. In particular, various rotating patterns and a transition to optical turbulence were demonstrated experimentally. From a theoretical point of view, this system has the advantage that a single evolution equation with a shifted spatial argument describes a broad variety of nonlinear wave structures.

Usually, such self-organizing phenomena are studied in the vicinity of a bifurcation point by approximately constructing the simplified models which describe the important properties of the original problem in terms of few order parameters [3]. This problem can be solved accurately in the critical case of a finite dimension by applying either the adiabatic elimination procedure of synergetics [3,4] or the normal form theory [5,6]. In this way, the

* Corresponding author. Permanent address: Department of Physics, Belarus State University, 220050 Minsk, Belarus. E-mail: grigore@phys.bsu.unibel.by

PII: S0167-2789(98)00196-1

existence of rotating multi-petal structures, the optical reverberators, has been shown for a Hopf bifurcation on the basis of linear analysis [2] and of normal form theory [7,8].

The aim of this paper is to consider a more complex situation of a system with a "large" geometrical size which exhibits the critical cases of infinite dimension. In this case, the known results concerning the existence of integral manifolds and the method of normal forms are inapplicable immediately. One possible way around this problem is to use so-called multiple-scaling technique [9,10] to derive amplitude equations in the limit of an infinite geometrical size. However, optical systems involve a very fast wave processes and, hence, the influence of the boundary conditions may prove to be important. Here we apply the scaling technique [11,12] which uses and develops the ideas of [4,13], and allows us, in addition, to investigate the effects of finite geometry in a large-scale system.

The mathematical description of optical reverberators [1,2] is based on the parabolic equation with a shifted spatial argument

$$\frac{\partial u}{\partial t} = -u + D\frac{\partial^2 u}{\partial \theta^2} + K\sin[u(t, \theta - \Delta) + \phi] \tag{1}$$

and periodic boundary conditions

$$u(t, \theta) = u(t, \theta + 2\pi). \tag{2}$$

Eqs. (1) and (2) describe the behaviour of the light beam which has the form of a thin circle. Here $u(t, \theta)$ is the phase shift of the field in the medium depending on the normalized time t and on the angle coordinate θ, K is proportional to the input intensity of light, Δ denotes the rotation angle of the field in the feedback circuit, and D is an effective diffusion coefficient inversely proportional to the radius R of the circle mask. Note that such a normalization results in a variation of the diffusion coefficient with the system size R in accordance with $D \sim R^{-2}$.

Let u_0 be a homogeneous stationary state of the parabolic equation (1)

$$u_0 = K\sin(u_0 + \phi). \tag{3}$$

The stability of the every such state is determined by the roots λ_n of the characteristic equation for the corresponding linearized problem given as

$$\lambda_n = -Dn^2 - 1 + p\mathrm{e}^{-\mathrm{i}\Delta n}. \tag{4}$$

Here $n = 0, \pm 1, \ldots$ characterizes a spatial mode proportional to $\exp(\mathrm{i}n\theta)$. We have also used the notation $p = K\cos(u_0 + \phi)$ because the properties of stationary states corresponding to positive and negative values of p turn out to be different. In the following we will consider the parameter p, instead of K, as an independent parameter.

As is well-known, the stability spectrum of a scalar parabolic equation without shifted space variable is real. It means, in particular, that all bifurcation phenomena are connected only to a change of the quantity of stationary states. The more general result is that all limited solutions of such a boundary problem tend to homogeneous stationary states at $t \to \infty$. Eq. (4) shows that the spectrum of the single parabolic equation (1) becomes complex due to the shift Δ of the spatial variable. An opportunity of the bifurcation analysis is thus opened in cases when the characteristic roots cross the imaginary axis. For equations without spatial shift such an opportunity is opened only for a system consisting of not less than two parabolic equations, for example, a reaction–diffusion system. Moreover, the specific high-order spatial mode bifurcations which take place in Eqs. (1) and (2) may appear for a system of not less than three parabolic equations. Hence, the local dynamic properties of a model change essentially by the spatial transformer in the feedback.

In the following, we present a complete bifurcation analysis for the stationary states of the system (1), (2). Various types of critical cases can be distinguished depending on peculiarities of stability destruction for finite, i.e. asymptotically not small, or asymptotically small, values of the diffusion coefficient D and the rotation angle Δ.

First, in Section 2 of the paper, order parameter equations will be constructed in the critical cases of one, two and four dimensions. These cases occur if the diffusion coefficient D is not too small, i.e. local transverse interactions are rather strong. It turns out that the stationary states which are associated with positive p lose stability via transcritical bifurcation, irrespective of the rotation angle Δ in the feedback. The other stationary states become unstable via Andronov–Hopf bifurcation of different inhomogeneous spatial modes in dependence on rotation shift Δ. It will be shown that this bifurcation is always supercritical. Moreover, the emerging optical reverberators are also stable in the case of a bifurcation of codimension two due to the action of mode competition. The normal forms for critical cases of finite dimension are derived by applying multiple scaling technique. This formalism is shown in Sections 3 and 4 to expand correctly to critical cases of infinite dimension which appear if the diffusion coefficient D becomes small as the geometrical size of the system increases. An analysis of the constructed quasi-normal forms reveals that the high dimension of the critical cases can result in complex dynamics as well as in simple periodical structures. In Section 3, slowly oscillating (in space) patterns are described which are caused by a set of lower-order spatial modes for a finite rotation angle Δ. It is shown that the stationary states, corresponding to positive p, conserve a homogeneous structure. In contrast, the stationary states with negative p bifurcate to multistable travelling waves or to slowly rotating step structures bepending on the spatial shift Δ in the feedback. In Section 4, we deduce quasi-normal forms which describe the structure formation in the vicinity of higher-order spatial mode bifurcations. Such a situation can be observed in the large system with an asymptotically small spatial shift in the feedback or in the limits of experimental variation “without” rotation of the field in feedback. It is shown that in the neighbourhood of the stationary state the dynamics of the travelling waves is sensitive to a “natural” wavelength of the system even in the case of a large-scale system.

2. Normal forms for problem (1), (2)

Let both the diffusion coefficient D and rotation angle Δ not be too small. The respective bifurcation diagram is given in Fig. 1, showing three situations when the critical conditions are valid for the different numbers (one, two, and four) of spatial modes. It is known that such critical conditions imply the existence of a local invariant integral manifold of finite dimension in the neighbourhood of the stationary state. All solutions with initial conditions from this vicinity approach this manifold in the limit $t \to \infty$ [5,6]. Therefore, the problem to find solutions of Eqs. (1) and (2) is reduced to the investigation of solutions that belong to the manifold. Also, the dimension of the last one is equal to the dimension of corresponding critical case. In the following, we deduce these normal forms by applying the multiple-scaling technique. With this aim, we introduce a small parameter $\epsilon \ll 1$ and consider a sufficiently small vicinity of the bifurcation point

$$p = p_0 + \epsilon p_1, \qquad \Delta = \Delta_0 + \epsilon \Delta_1, \qquad D = D_0 + \epsilon D_1. \tag{5}$$

Here, p_0, Δ_0, D_0 act as bifurcation parameters which have to be specified for every type of bifurcation.

2.1. One-dimensional critical case

It follows from the characteristic equation (4) that the critical conditions for the single spatial mode with wave number $n = n_0 = 0$ occur at $p = p_0 = 1$, Fig. 1. In this situation, the characteristic root $\lambda_{n_0} = 0$ and the inequality $\operatorname{Re} \lambda_n < 0$ is valid for any other $n \neq 0$. The type of bifurcation remains the same for any rotation angle Δ. Note that

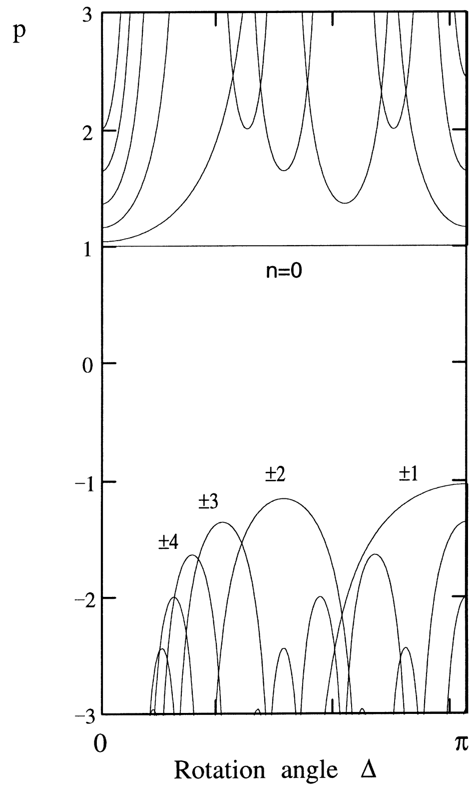

Fig. 1. Bifurcation diagram at the coefficient of diffusion $D = 0.04$. Numbers near curves correspond to wave numbers of the excited modes.

in this case the stationary state u_0 is determined as a root of the equation $u_0 = \tan(u_0 + \phi)$ and the input intensity of light is determined by $K = \cos^{-1}(u_0 + \phi)$.

In the case of $u_0 \neq 0$ transcritical bifurcation occurs and the corresponding normal form on the one-dimensional local invariant manifold reads

$$\frac{d\xi}{d\tau} = a\xi - b\xi^2, \tag{6}$$

with slow time variable $\tau = \epsilon t$. The function ξ is connected to the phase of the field by the series

$$u = u_0 + \epsilon\xi(\tau) + \epsilon^2 g(\tau) + \cdots \tag{7}$$

Inserting (5) and (7) into (1) and collecting the terms of the same order of ϵ, we find the coefficients of the normal form as

$$a = p_1, \qquad b = \tfrac{1}{2}u_0. \tag{8}$$

It follows from (6)–(8) that the steady solution of Eq. (1) which is homogeneous in space and stationary in time becomes $u = u_0 + \epsilon 2p_1/u_0$ for $p_1 > 0$ or $u = u_0$ for $p_1 < 0$.

The second-order term in the nonlinear part of Eq. (1) disappears for $u_0 = 0$. This degenerate case represents a pitchfork bifurcation and the normal form is

$$\frac{d\xi}{d\tau} = a\xi - b\xi^3. \tag{9}$$

Here, the function ξ is connected to the phase of the field by the series

$$u = \epsilon^{1/2}\xi(\tau) + \epsilon g(\tau) + \cdots \tag{10}$$

and we obtain for the coefficient of the normal form

$$a = p_1, \qquad b = \tfrac{1}{6}. \tag{11}$$

From (9)–(11) one concludes that the homogeneous stationary solution of the amplitude $u = \sqrt{6\epsilon p_1}$ is stable in a vicinity of the bifurcation point $p = 1, u_0 = 0$ for $p_1 > 0$.

Physically, the homogeneous stationary states which correspond to positive p do not change essentially when they lose their stability.

2.2. Two-dimensional critical case (Andronov–Hopf bifurcation)

Now, we turn to bifurcations of stationary states which are associated with negative p. The respective bifurcation diagram is shown in Fig. 1. Contrary to the previous one-dimensional critical case, different inhomogeneous spatial modes become unstable at different angles Δ in the feedback.

Two spatial modes with the wave numbers $n = \pm n_0, n_0 \neq 0$, fulfill the critical conditions if $\mathrm{Re}\,\lambda_{\pm n_0} = 0$ with $\mathrm{Im}\,\lambda_{\pm n_0} = \pm\omega(n_0)$, and the inequality $\mathrm{Re}\,\lambda_n < 0$ is valid for any other $n \neq \pm n_0$. For given $D = D_0$, $\Delta = \Delta_0$, the characteristic equation (4) determines the critical value of the parameter $p = p_0$, $p_0 < 0$ according to

$$p_0 = \frac{1 + D_0 n_0^2}{\cos(\Delta_0 n_0)}, \tag{12}$$

and the number n_0 of exited mode is obtained at the minimum of $|p_0|$. Furthermore, the corresponding frequencies are given by

$$\omega(n_0) = -p_0 \sin(\Delta_0 n_0), \qquad \omega(-n_0) = -\omega(n_0). \tag{13}$$

In the vicinity (5) of the bifurcation point Eqs. (1) and (2) are reduced to the normal form in a two-dimensional local invariant integral manifold

$$\frac{\mathrm{d}\xi}{\mathrm{d}\tau} = a(n_0)\xi + b(n_0)\xi|\xi|^2. \tag{14}$$

Here, the function ξ of the slow time variable $\tau = \epsilon t$ is connected to the phase of the field u by a power series of the excited harmonics

$$u = u_0 + \epsilon^{1/2}\xi(\tau)\mathrm{e}^{\mathrm{i}\eta} + \epsilon[u_{20}(\tau) + u_{21}(\tau)\mathrm{e}^{\mathrm{i}2\eta}] + \epsilon^{3/2} + \cdots + \text{c.c.} \tag{15}$$

with $\eta = \omega(n_0)t + n_0\theta$. The above critical factor $a(n_0)$ and the Lyapunov value $b(n_0)$ are determined by

$$a(n_0) = -D_1 n_0^2 + \mathrm{e}^{-\mathrm{i}n_0\Delta_0}(p_1 - \mathrm{i}p_0\Delta_1 n_0), \tag{16}$$

$$b(n_0) = \frac{\mathrm{e}^{-\mathrm{i}\Delta_0 n_0}}{2}\left[-p_0 + u_0^2\left(\frac{2}{1-p_0} + G(n_0, n_0)\right)\right], \tag{17}$$

with

$$G(x, y) = \frac{\mathrm{e}^{-\mathrm{i}\Delta_0(x+y)}}{\mathrm{i}[\omega(x) + \omega(y)] + 1 + D_0(x+y)^2 - p_0\mathrm{e}^{\mathrm{i}\Delta_0(x+y)}}. \tag{18}$$

Note that the function $G(x, y)$ is always determinable for $x + y \neq n_0$ due to the critical conditions

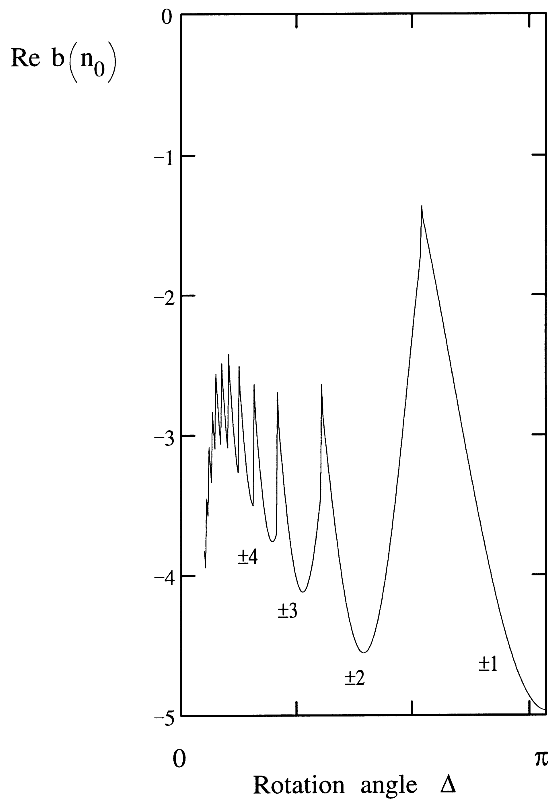

Fig. 2. Lyapunov value Re b corresponding to the bifurcation diagram in Fig. 1.

Eq. (14) has a stable periodic solution given as $\xi = \xi_0(n_0)\exp[\mathrm{i}r(n_0)\tau]$, where

$$\xi_0(n_0) = \sqrt{\frac{\mathrm{Re}\,a(n_0)}{-\mathrm{Re}\,b(n_0)}},$$

$$r(n_0) = \mathrm{Im}\,a(n_0) - \frac{\mathrm{Re}\,a(n_0)}{\mathrm{Re}\,b(n_0)}\mathrm{Im}\,b(n_0),$$

if the conditions $\mathrm{Re}\,a(n_0) > 0$ and $\mathrm{Re}\,b(n_0) < 0$ are fulfilled.

In accordance with (15), this limit cycle corresponds to the optical reverberator in the original system (1), (2)

$$u(\epsilon) = u_0 + \epsilon^{1/2} 2\xi_0(n_0)\cos(\Omega(n_0)t + n_0\theta) + \mathrm{O}(\epsilon),$$

with the frequency $\Omega(n_0) = \omega(n_0) + \epsilon r(n_0) + \mathrm{O}(\epsilon^2)$.

Fig. 2 shows that the condition $\mathrm{Re}\,b(n_0) < 0$ is valid for any n_0, i.e. for all inhomogeneous spatial modes which lose stability. Therefore, optical reverberators will appear along the whole boundary of stability. Experimental and numerical results [1,2] confirm this conclusion.

2.3. Four-dimensional critical case

In this section, we consider the critical conditions to be fulfilled for four spatial modes with the wave numbers $n = \pm n_1, \pm n_2$ excited simultaneously. The respective bifurcation parameters $D = D_0$, $\Delta = \Delta_0$, $p = p_0$, are determined from Eq. (12) and

$$D_0 = \frac{\cos(n_2 \Delta_0) - \cos(n_1 \Delta_0)}{n_2^2 \cos(n_1 \Delta_0) - n_1^2 \cos(\Delta_0 n_2)}. \tag{19}$$

In such a codimension two bifurcation we have $\mathrm{Re}\,\lambda_{\pm n_1, \pm n_2} = 0$ and $\mathrm{Re}\,\lambda_n < 0$ for any other $n \neq \pm n_1, \pm n_2$. For instance, this situation may be realized by the set $\pm(n_1, n_2)$ of exited modes: $\pm(1, 2)$; $\pm(2, 3)$; ..., or by $\pm(1, 4)$; $\pm(2, 5)$ ...

The corresponding frequencies are given by Eqs. (13) and it is important that low-order resonances do not occur due to the opposite direction of pattern rotation at the bifurcation point. In the vicinity (5) of the bifurcation point, we introduce a series:

$$u = u_0 + \epsilon^{1/2}[\xi(\tau)\mathrm{e}^{\mathrm{i}\eta_1} + \chi(\tau)\mathrm{e}^{\mathrm{i}\eta_2}] + \epsilon g(\tau, \eta_1, \eta_2) + \epsilon^{3/2} + \cdots + \text{c.c.}, \tag{20}$$

where, as before, $\eta_1 = \omega(n_1)t + n_1\theta$ and $\eta_2 = \omega(n_2)t + n_2\theta$. Now we have to insert (5) and (20) in (1) and to collect all terms of the same order of ϵ and a similar class of functions. Then we obtain the normal form in a four-dimensional local invariant manifold:

$$\frac{\mathrm{d}\xi}{\mathrm{d}\tau} = a(n_1)\xi + \xi[b(n_1)|\xi|^2 + \gamma_1(|\chi|^2],$$
$$\frac{\mathrm{d}\chi}{\mathrm{d}\tau} = a(n_2)\chi + \chi[b(n_2)|\xi|^2 + \gamma_2(|\xi|^2]. \tag{21}$$

Here, $a(n_1), a(n_2)$, and $b(n_1), b(n_2)$ are given by (16) and (17); the coefficients γ_1 and γ_2 which are responsible for the interaction of modes are determined from:

$$\gamma_1 = \mathrm{e}^{-\mathrm{i}\Delta_0 n_1}\left[-p_0 + u_0^2\left(\frac{1}{1-p_0} + G(n_1, n_2) + G(n_1, -n_2)\right)\right],$$
$$\gamma_2 = \mathrm{e}^{-\mathrm{i}\Delta_0 n_2}\left[-p_0 + u_0^2\left(\frac{1}{1-p_0} + G(n_2, n_1) + G(n_2, -n_1)\right)\right], \tag{22}$$

The normal form (21) has two periodic solutions

$$\xi = \sqrt{\frac{\mathrm{Re}\,a(n_1)}{-\mathrm{Re}\,b(n_1)}}\mathrm{e}^{\mathrm{i}r(n_1)\tau}, \quad \chi = 0;$$

and

$$\xi = 0, \quad \chi = \sqrt{\frac{\mathrm{Re}\,a(n_2)}{-\mathrm{Re}\,b(n_2)}}\mathrm{e}^{\mathrm{i}r(n_2)\tau}.$$

These cycles correspond to the same ones which appear via simple Hopf bifurcation. They are stable and coexist under conditions $\mathrm{Re}\,a(n_1), a(n_2) > 0$ as the other necessary conditions $\mathrm{Re}\,b(n_1), b(n_2) < 0$ are always valid for the whole boundary of stability, including the bifurcation points of codimension two. Therefore, hysteresis phenomena may be observed in a region of coexistence of both optical reverberators, which is in good agreement with the experimental results [2].

In addition, the normal form (21) has the solution $\xi \neq 0, \chi \neq 0$ which would exist in the case of a "weak" coupling between modes. Comparing (17) and (22) we obtain that this coupling is quite strong for our system

$$\frac{\mathrm{Re}\,\gamma_1 \mathrm{Re}\,\gamma_2}{\mathrm{Re}\,b(n_1)\mathrm{Re}\,b(n_2)} > 1.$$

Hence, the mode beating regime is unstable, at least, near the bifurcation point.

As an important result, the mechanism of the mode competition predominates in the case of the finite diffusion coefficient. It provides the existence of simple optical reverberators. The conditions for cooperative mode interaction can appear when the diffusion coefficient becomes small. They will be considered in the following.

3. Slow oscillating spatial structures

With an increase of the radius of circle mask $R \to \infty$, we obtain $D \sim R^{-2}$, $D \to 0$. Indeed, the values of the diffusion coefficient D from experimental data were estimated to be 10^{-5} [2]. In this situation, critical cases of infinite dimension arise, i.e. an asymptotically infinite number of spatial modes become unstable simultaneously near the boundary of stability. Fig. 3 illustrates this tendency for $D = 10^{-3}$. As can be seen, there exist different critical cases of infinite dimension corresponding to different critical values of p. Slowly oscillating spatial structures are formed if the angle of field rotation Δ in the feedback is close to rational number multiple π and if $p = 1$, $p = -1$ or $p < -1$. In addition, higher-order spatial modes bifurcate if both the rotation angle Δ and the diffusion coefficient D tend to zero at $p < -1$.

Applying the normal forms method we would have to analyse the infinite system of coupled equations. Instead of pursuing this approach, we will derive as a quasi-normal form a partial differential equation for the order parameter. By doing so the existence of a small parameter in form of the diffusion coefficient D becomes essential. To emphasize this point we explicitly introduce the diffusion coefficient $D = d \ll 1$ as a small parameter instead of arbitrary

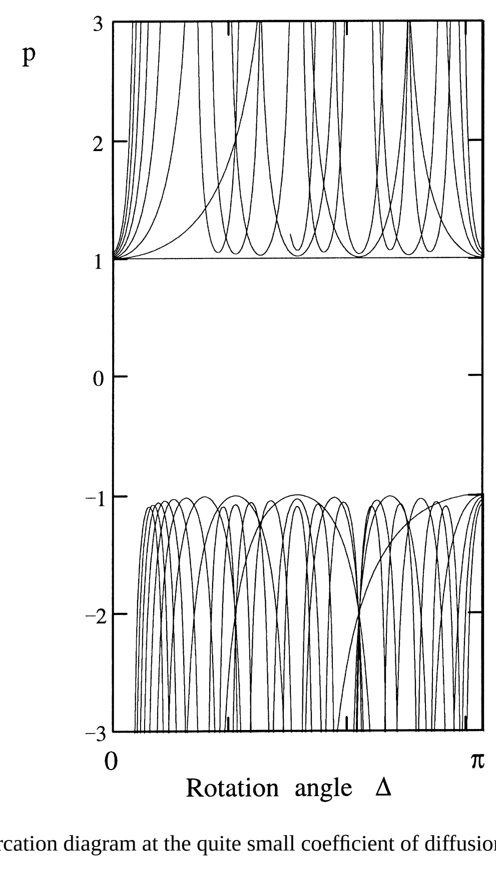

Fig. 3. Bifurcation diagram at the quite small coefficient of diffusion $D = 0.001$.

ϵ which was used previously. Now the time scaling and the scaling of the above critical parameters should be connected with the small diffusion parameter. Also, we have not to apply any scaling for spatial variable and, hence, we do not change the boundary conditions.

The equations obtained have the same meaning as the normal forms in the finite-dimensional critical cases and they have following advantages:
- they possess a universal nonlinearity, and only their coefficients depend on peculiarities of the original problem;
- they take into account the boundary effects even in the case of large-scale system;
- they do not possess a small parameter that allows us to compute them accurately.

3.1. Finite rotation angle Δ and $p = 1$

Let us first consider the bifurcation of the stationary solution which is associated with positive p in the case of $D = d, d \to 0$. An infinite number of roots λ_{n_k} of the characteristic equation (4) tends to the simple zero at $p = p_0 = 1$ and at the rotation angle Δ close to the rational number multiple $\pi : \Delta_0 = \pi(m_1/m_2)$, where m_1, m_2 are integers. The numbers of excited modes are given by $n_k = mk, k = 0, \pm 1, \pm 2, \ldots$ and $m = m_2$ if m_1 is even but $m = 2m_2$ if m is odd. Therefore, the critical case of infinite dimension is realized.

In the neighbourhood of the bifurcation point

$$\Delta = \Delta_0 + d^{1/2}\Delta_1 + d\Delta_2, \qquad p = 1 + dp_1, \tag{23}$$

we introduce, in analogy to (7), the formal series

$$u = u_0 + d\sum_{k=-\infty}^{\infty} \xi_k(\tau)\mathrm{e}^{\mathrm{i}mk\theta} + d^2\sum_{k=-\infty}^{\infty} z_k(\tau)\mathrm{e}^{\mathrm{i}mk\theta} + \cdots \tag{24}$$

Here, all "critical" harmonics are collected in the first sum; $\xi_k(\tau)$, and $z_k(\tau)$ are functions of the slow time variable $\tau = dt$. Inserting (23) and (24) into (1), we obtained an infinite system of equations

$$d^2\dot{\xi}_k = [-m^2k^2d(1 + \tfrac{1}{2}\Delta_1^2) + p_1 d - d^{1/2}\mathrm{i}mk(\Delta_1 + d^{1/2}\Delta_2)]d\xi_k + d^2\Phi_k(\xi) + \mathrm{O}(d^3), \tag{25}$$

where $\Phi_k(\xi)$ is a coefficient of $\exp(\mathrm{i}mk\theta)$ in the Fourier expansion of the function

$$\frac{u_0}{2}\left[\sum_{k=-\infty}^{\infty} \xi_k(t)\mathrm{e}^{\mathrm{i}mk\theta}\right]^2$$

Neglecting terms of high order of d, we can reformulate system (25) as the parabolic boundary problem

$$\frac{\partial v}{\partial \tau} = m^2\left(1 + \frac{\Delta_1^2}{2}\right)\frac{\partial^2 v}{\partial x^2} + p_1 v - \frac{u_0}{2}v^2, \qquad v(\tau, x) = v(\tau, x + 2\pi), \tag{26}$$

for the function

$$v(\tau, x) = \sum_{k=-\infty}^{\infty} \xi_k(\tau)\mathrm{e}^{\mathrm{i}kx}, \quad x = m[\theta - d^{1/2}(\Delta_1 + d^{1/2}\Delta_2)t], \quad \tau = dt.$$

The diffusion coefficient in quasi-normal form (26) is determined by the characteristics of the large-scale transverse interaction due to the field rotation in the feedback. Its minimum value at $m = 2$ is equal to 4. Such a strong diffusion supports the stability properties of the homogeneous solution. In fact, at $p_1 > 0$, the only stable solution of the problem (26) is given by

$$u = u_0 + d\frac{2p_1}{u_0} + \mathrm{O}(d^2),$$

which describes the homogeneous stationary state.

Thus, the “high” dimension of the critical case does not lead to complex dynamics in the local vicinity of the stationary states corresponding to positive p.

3.2. Finite rotation angle Δ and $p = -1$

Let us consider the critical conditions which occur at $p = p_0 = -1$. In this case, as follows from Eq. (4), at $\Delta = \Delta_0 = (m_1/m_2)\pi$, where m_1, m_2 are integer numbers and, in addition, m_1 is odd, an infinite number of characteristic roots λ_{n_k} tends to the simple zero. Here $n_k = m_2(2k+1)$, $k = 0, \pm 1, \pm 2, \ldots$

Contrary to the previous case, m_2-odd spatial modes become unstable simultaneously in the vicinity of the bifurcation point. Because the product of odd harmonics gives only even harmonics, we have to take into account at least the terms of the third order in the series of a small diffusion parameter d.

$$u = u_0 + d^{1/2}\sum_{k=-\infty}^{\infty}\xi_k(\tau)\mathrm{e}^{\mathrm{i}(2k+1)x} + du_2(\tau, x) + d^{3/2}u_3(\tau, x) + \cdots, \tag{27}$$

and to consider a small vicinity of the bifurcation point

$$\Delta = \Delta_0 + d^{1/2}\Delta_1 + d\Delta_2, \qquad p = -1 + dp_1. \tag{28}$$

As before, $\tau = dt$, $x = m_2[\theta - d^{1/2}(\Delta_1 + d^{1/2}\Delta_2)t]$. Combining (27), (28) and (1), one finds at the second step

$$u_2(\tau, x) = -\frac{u_0}{4}\left[\sum_{k=-\infty}^{\infty}\xi_k(\tau)\mathrm{e}^{\mathrm{i}(2k+1)x}\right]^2.$$

Neglecting the terms of the higher-order of d and introducing the function $z(\tau, x)$

$$z(\tau, x) = \sum_{k=-\infty}^{\infty}\xi_k(\tau)\mathrm{e}^{\mathrm{i}(2k+1)x},$$

we obtain the parabolic equation with antiperiodic boundary conditions

$$\frac{\partial z}{\partial \tau} = m_2^2\left(1 + \frac{\Delta_1^2}{2}\right)\frac{\partial^2 z}{\partial x^2} - p_1 z - \left(\frac{1}{6} + \frac{u_0^2}{4}\right)z^3, \qquad z(\tau, x) = -z(\tau, x+\pi). \tag{29}$$

Quasi-normal form (29) determines the local dynamics of system (1), (2) near the state u_0 which is the root of equation: $u_0 = -\tan(u_0 + \phi)$. The bifurcation value of the input light intensity in this case is proportional to $K = (1 + u_0^2)^{1/2}$.

As in previous case, the effective diffusion coefficient in (29) depends on characteristics of the rotation shift in the feedback and is quite large. However, specific antiperiodic boundary conditions may provide the existence of inhomogeneous structures. Indeed, for:

(1) $p_1 > -m_2^2(1 + \Delta_1^2/2)$, the homogeneous stationary solution $z = 0$ is stable (the solution $u = u_0$ is stable and there are no other stable solutions in some vicinity of the last one);

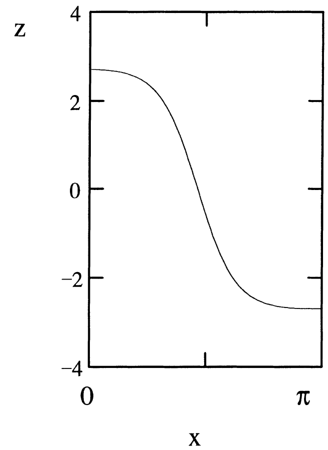

Fig. 4. The final state of the amplitude z, given by Eqs. (29), resulting from the different initial distributions. $K = 7$, $m_2 = 1$, $p_1 = -15.2$, and $\Delta_1 = 0$.

(2) $p_1 < -m_2^2(1 + \Delta_1^2/2)$, the spatial inhomogeneous solution is stable, which corresponds to a stable inhomogeneous slowly rotating pattern

$$u(t, \theta, d) = u_0 + d^{1/2} z_0(m_2\theta - d^{1/2} m_2[\Delta_1 + d^{1/2}\Delta_2]t) + \mathrm{O}(d).$$

A typical form of the solution is shown in Fig. 4. Evidently, the function $z_0(x)$ tends to a step-function with the amplitude $\pm 2\sqrt{-p_1/(u_0^2 + 2/3)}$ as the parameter p_1 decreases. That corresponds to the stationary contrast structures observed experimentally [1,2]. Such a pattern is formed to cooperative interaction of all excited modes. In this next section, we derive another type of dynamical behaviour – mode competition – leading to multistability of the reverberators.

3.3. Finite rotation angle Δ and $p < -1$

Let us determine the critical conditions at $p = p_0 <= -1$. In this case, as can be seen from Eq. (4), with

$$\Delta = \Delta_0 = \frac{\pi m_1}{m_2},$$

where m_1, m_2 are integer and, in addition, m_1 is even, an infinite number of the characteristic roots λ_{n_k} have asymptotically small real parts but, contrary to the previous case, all excited modes have nonzero frequency of rotation.

For simplicity, we consider here only the particular case $m_1 = 2, m_2 = 3$, i.e. $\Delta_0 = 2\pi/3$, in which an infinite set of spatial modes with wave numbers $n_k = (3k \pm 1)$, $k = 0, \pm 1, \pm 2, \ldots$ and with frequencies $\omega_3 \sim \mp\sqrt{3}$ become unstable simultaneously at $p = p_0 - 2$.

As before, to obtain the quasi-normal form, we introduce the power series of the small diffusion parameter d

$$u = u_0 + d^{1/2} \mathrm{e}^{\mathrm{i}\eta} \sum_{k=-\infty}^{\infty} \xi_k(\tau) \mathrm{e}^{\mathrm{i}3k\theta} + d[u_{20}(\tau, \theta) + u_{21}(\tau, \theta)\mathrm{e}^{\mathrm{i}2\eta}] + +d^{3/2} + \cdots + \text{c.c.}, \tag{30}$$

with $\eta = \theta - t\sqrt{3}$ in the neighbourhood of the bifurcation point

$$\Delta = \Delta_0 + d\Delta_1, \qquad p = -2 + dp_1. \tag{31}$$

Acting analogously, we find at the second step:

$$u_{20}(\tau,x) = -\frac{u_0}{3}\left[\sum_{k=-\infty}^{\infty} \xi_k(\tau)\mathrm{e}^{\mathrm{i}3k\theta}\right]\left[\sum_{k=-\infty}^{\infty} \xi_k^*(\tau)\mathrm{e}^{-\mathrm{i}3k\theta}\right],$$

$$u_{21}(\tau,x) = -\frac{u_0(1+\mathrm{i}\sqrt{3})}{12\mathrm{i}\sqrt{3}}\left[\sum_{k=-\infty}^{\infty} \xi_k(\tau)\mathrm{e}^{\mathrm{i}3k\theta}\right]^2.$$

Neglecting terms of higher-order of d, putting $x = m_2\theta$, and

$$y(\tau,x) = \sum_{k=-\infty}^{\infty} \xi_k(\tau)\mathrm{e}^{\mathrm{i}kx},$$

it becomes possible to pack back the infinite system into the parabolic equation with periodic boundary conditions

$$\frac{\partial y}{\partial \tau} = 9\frac{\partial^2 y}{\partial x^2} + 3c\frac{\partial y}{\partial x} + ay + by|y|^2, \qquad y(\tau,x) = y(\tau, x+2\pi). \tag{32}$$

The coefficients of this quasi-normal form are given by the expressions

$$a = -\frac{1}{2}(1+\mathrm{i}\sqrt{3})(p_1 + 2\mathrm{i}\Delta_1) - 1,$$

$$b = -\frac{1}{2}\left(1+\frac{u_0^2}{3}\right)\left[1+\mathrm{i}\sqrt{3}\left(1-\frac{u_0^2}{9+3u_0^2}\right)\right],$$

$$c = -\Delta_1(1+\mathrm{i}\sqrt{3}) + 2\mathrm{i}.$$

Typical solutions of Eqs. (32) are travelling waves (T W) which may be stable and coexist at the same parameters. This phenomenon of multistability of travelling waves corresponds to a set of optical reverberators in the original system:

$$u = u_0 + d^{1/2} y_{0k}\cos[\Omega(k)t + (3k+1)\theta] + \mathrm{O}(d), \quad k = 0, \pm 1, \ldots,$$

where the amplitude of the reverberator is determined by

$$y_{0k} = \sqrt{\frac{-(3k^2 + 1 - \Delta_1\sqrt{3}/2)^2 - p_1 + 3\Delta_1^2/2}{1 + u_0^2/3}},$$

and the rotation frequency is given as

$$\Omega(k) = -\sqrt{3} - d\frac{\sqrt{3}}{2}\left[p_1 + \frac{8}{\sqrt{3}}\Delta_1 + y_{0k}^2\left(1+\frac{u_0^2}{3}\right)\left(1-\frac{u_0^2}{9+3u_0^2}\right)\right] + \mathrm{O}(d^2).$$

The existence of such a k-travelling wave is determined by the condition $u_{0k}^2 > 0$ and the stability condition. The last one shows that such simple pattern may be only observed in a sufficiently small vicinity $d \cdot \Delta_1$ of bifurcation point Δ_0. As the detuning increases up to the order of $d^{1/2}\Delta_1$, all travelling waves lose stability abruptly. In fact, in this case the quasi-normal form is transformed to:

$$\frac{\partial y}{\partial \tau'} = m_2 c\frac{\partial y}{\partial x} + ay + by|y|^2, \tag{33}$$

and for every T W there is an infinite number of eigenvalues which have positive real parts.

Thus, the considered high-dimensional critical case results in a set of coexisting simple optical reverberators which may be observed in a close vicinity of the bifurcation point.

4. Fast oscillating spatial patterns

In the previous cases steady solutions are formed by a set of spatial modes with not too high wave numbers. Now we consider patterns, fast oscillating in space, which are formed at a small diffusion and at a small rotation shift in the feedback circuit.

The set of high-order spatial modes with wave numbers close to $d^{-1/2}$ may be excited simultaneously if the rotation shift is separated from zero at least by the value $\Delta = d^{1/2}\delta_0$. It turns out that, as the geometrical size of the system increases, the wave numbers increase too, whereas the corresponding rotation angle decreases due to $d \to 0$.

To obtain such critical conditions, let us introduce the parameter $S^2 = d \cdot n^2$ which has the property $s \not\to 0$ as $d \to 0$. Furthermore, we consider this parameter as independent and uninterrupted. Then the bifurcation values of the parameters $S = S_0$, $p = p_0 < -1$ can be determined from Eq. (4). The conditions $\mathrm{Re}\,\lambda(S_0) = 0$ and for any other $S \neq S_0$ the real part of the corresponding characteristic root is negative, lead to the system:

$$-1 - S_0^2 + p_0 \cos(S_0\delta_0) = 0, \qquad -2S_0 - \delta_0 p_0 \sin(S_0\delta_0) = 0. \tag{34}$$

All solutions of Eqs. (1) and (2) from a small (independent on d) vicinity of u_0 tend to u_0 at $\Delta < d^{1/2}\delta_0$ as $t \to \infty$. In the opposite case, there is an infinite number of characteristic roots having asymptotically small real parts. Therefore, an infinite number of high-order spatial modes with wave numbers around $(S_0 d^{-1/2})$ become unstable and their frequencies coverge to

$$\omega(S_0) = -p_0 \sin(S_0\delta_0). \tag{35}$$

Because of the large wave numbers ($\sim d^{-1/2}$) of excited harmonics, the rate of the changing of the harmonic amplitude due to spatial interaction has the order of $d^{1/2}$. That is why it is convenient to choose the above critical parameters of the same order and to consider the small vicinity of the bifurcation point

$$\Delta = d^{1/2}(\delta_0 + d^{1/2}\delta_1), \qquad p = p_0 + d^{1/2} p_1. \tag{36}$$

We then seek solutions of Eqs. (1) and (2) in the form

$$u = u_0 + d^{1/4}\xi(s,\theta)\mathrm{e}^{\mathrm{i}\eta} + d^{1/2}[u_{20}(s,\theta) + u_{21}(s,\theta)\mathrm{e}^{\mathrm{i}2\eta}] + d^{3/4} + \cdots + \text{c.c.}, \tag{37}$$

where the functions $\xi(s,\theta)$, $u_{20}(s,\theta,\eta), \ldots$ are 2π-periodic. Slow temporal and travelling variables are connected to the small parameter of the diffusion d by

$$s = d^{1/2}t, \qquad \eta = n(d)\theta + \omega(S_0)t.$$

Here, the wave number

$$n(d) = d^{-1/2}S_0 + \sigma$$

is a central one for excited spatial modes. We have introduced the “internal” parameter σ to make $n(d)$ an integer, so that $\sigma = \{\epsilon^{-1/2}S_0\} - \epsilon^{-1/2}S_0$, $\sigma \in (-1, 0)$. This parameter allows us to fulfill the periodic boundary conditions. Note that in particular, it depends on the geometrical size of the system.

Let us substitute Eqs. (36) and (37) into Eq. (1), and collect the coefficients of the same powers of d. At the second step, we obtain the expressions for functions $u_{20}(s,\theta)$, $u_{21}(s,\theta)$, and at the next step, we find that ξ satisfies

$$\frac{\partial\xi}{\partial s} = \omega'(S_0)\left(\frac{\partial\xi}{\partial\theta} + \mathrm{i}\sigma\xi\right) + A\xi + B\xi|\xi|^2, \tag{38}$$

with periodic boundary conditions

$$\xi(s,\theta) \equiv \xi(s,\theta+2\pi), \tag{39}$$

where

$$A = (p_1 - \mathrm{i}S_0 p_0 \delta_1)\mathrm{e}^{-\mathrm{i}S_0\delta_0},$$
$$B = \mathrm{e}^{-\mathrm{i}\delta_0 S_0}\left[-\frac{p_0}{2} + \frac{u_0^2}{2}\left(\frac{2}{1-p_0} + \frac{\mathrm{e}^{-2\mathrm{i}\delta_0 S_0}}{2\mathrm{i}\omega(S_0) + 1 + 4S_0^2 + p_0\mathrm{e}^{-\mathrm{i}2S_0\delta_0}}\right)\right].$$

The quasi-normal form given by (38) and (39) describes the dynamics of TW:

$$\xi_k(s,\theta) = \xi_{0k}\exp(\mathrm{i}\rho_k s + \mathrm{i}k\theta), \qquad k = 0, \pm1, \pm2, \dots,$$

with the amplitude

$$\xi_{0k} = \sqrt{\frac{\mathrm{Re}\,A}{-\mathrm{Re}\,B}}$$

and the frequency

$$\rho_k = (k+\sigma)\omega'(S_0) + \mathrm{Im}\,A + \xi_{0k}^2\,\mathrm{Im}\,B.$$

Linear analysis shows that characteristic roots for every such solution are

$$\lambda_m^+ = \mathrm{i}m\omega(S_0), \qquad \lambda_m^- = \mathrm{i}m\omega(S_0) + 2_{0k}^2\,\mathrm{Re}\,B, \quad m = 0, \pm1, \pm2, \dots$$

The expression for the coefficient B is analogous to Eq. (17) for b and therefore, as before, Re $B < 0$. Thus, the obtained TW are neutrally stable and the second normalization is necessary to find the direction of their evolution. However, it is possible to demonstrate the existence of the stable travelling waves in the "super" small vicinity of the bifurcation point

$$\Delta = d^{1/2}(\delta_0 + d\delta_1), \qquad p = p_0 + dp_1.$$

In this case, we have to introduce the series

$$u = u_0 + d^{1/2}\xi(s_1,\Theta)\mathrm{e}^{\mathrm{i}\zeta} + d[u_{20}(s_1,\Theta) + u_{21}(s_1,\Theta)\mathrm{e}^{\mathrm{i}2\zeta}] + d^{3/2} + \cdots + \mathrm{c.c.}, \tag{40}$$

with the amplitudes depending slowly on time $s_1 = dt$ and with moving variables

$$\zeta = \eta + d^{1/2}\sigma\omega'(S_0)t, \qquad \Theta = \theta + d^{1/2}\omega'(S_0)t.$$

With these notations, the quasi-normal form becomes

$$\frac{\partial\xi}{\partial s_1} = C\left(\frac{\partial^2\xi}{\partial\Theta^2} + \frac{\partial\xi}{\partial\Theta}2\mathrm{i}\sigma - \sigma^2\xi\right) + A\xi + B\xi|\xi|^2, \qquad \xi(s_1,\Theta) = \xi(s_1,\Theta+2\pi). \tag{41}$$

Here

$$C = 1 + \tfrac{1}{2}\delta_0^2 p_0 \mathrm{e}^{-\mathrm{i}S_0\delta_0}.$$

The quasi-normal form (41) is a complex Ginzburg–Landau type equation. As before, the diffusion coefficient C is determined by the characteristic of the rotation shift in the feedback. However, contrary to previous situations it is a complex coefficient. That is why Eq. (41) is more complicated in comparison with the analogous one for a two-component reaction–diffusion system under high-order spatial mode bifurcation [15]. On the other hand, the quasi-normal form (41) is more complex than standard Ginzburg–Landau equation [10] due to the coefficient σ taking into account the geometrical size of the system. It is important that this coefficient does not disappear asymptotically as the size of the system becomes larger. Below we investigate the effects of finite geometry in the dynamics of the most simple solutions, the travelling waves, of the system.

In fact, Eq. (41) describes the phenomenon of multistability of travelling waves

$$\xi_k = \xi_{0k} \exp(\mathrm{i}\omega_2 s_1 + \mathrm{i}k\Theta), \quad k = 0, \pm 1, \pm 2, \ldots$$

In the original system, they correspond to optical reverberators with a “quite large” number of light spot: $n(d) + k$, and with the amplitude $d^{1/2}\xi_{0k}$, where:

$$\xi_{0k} = \sqrt{\frac{\mathrm{Re}\, C}{-\mathrm{Re}\, B}}\sqrt{a - (\sigma + k)^2},$$

and with the rotation frequency:

$$\Omega(k) = \omega(S_0) + d^{1/2}\omega_1(k) + d\omega_2(k),$$
$$\omega_1(k) = \omega'(S_0)(\sigma + k),\ \omega_2(k) = -(\sigma + k)^2 \mathrm{Im}\, C + \mathrm{Im}\, A + \xi_{0k}^2\, \mathrm{Im}\, B.$$

Hereafter, we use the notations

$$a = \frac{\mathrm{Re}\, A}{\mathrm{Re}\, C}, \quad b = -\frac{\mathrm{Im}\, B}{\mathrm{Re}\, B}, \quad c = \frac{\mathrm{Im}\, C}{\mathrm{Re}\, C}.$$

The expression for the rotation frequency includes small corrections to $\omega(S_0)$, which, in particular, depend on geometrical size of the system via the parameter σ. That is the evident effect of the boundaries. More essentially, the existence and stability of such $k-$TW depends also on the internal parameter σ. Indeed, the TW-solution with wave number k exists if

$$a > (\sigma + k)^2. \tag{42}$$

According to this condition, the typical domains of the k-TW existence are given in Fig. 5 for different values of σ. At a sufficiently large value of a, several different travelling waves may exist, arising from different initial data. Investigating their stability, one finds the characteristic equation

$$\lambda_m^2 - 2\lambda_m h_1 + h_2 = 0, \quad m = 0, \pm 1, \pm 2, \ldots, \tag{43}$$

$$h_1 = -\xi_{0k}^2 - m^2 - \mathrm{i}2mc(\sigma + k),$$
$$h_2 = 2\xi_{0k}^2[m^2(1 + bc) + 2\mathrm{i}m(\sigma + k)(c - b)] + [m^2 - 4m^2(\sigma + k)^2](1 + c^2),$$

which gives the stability boundaries of the k-TW shown in Fig. 5 by the dotted curves. We can estimate approximately these boundaries as

$$a > (\sigma + k)^2\left(3 - \frac{2bc}{1 + bc}\right) - \frac{1}{2}\frac{1 + c^2}{1 + bc}. \tag{44}$$

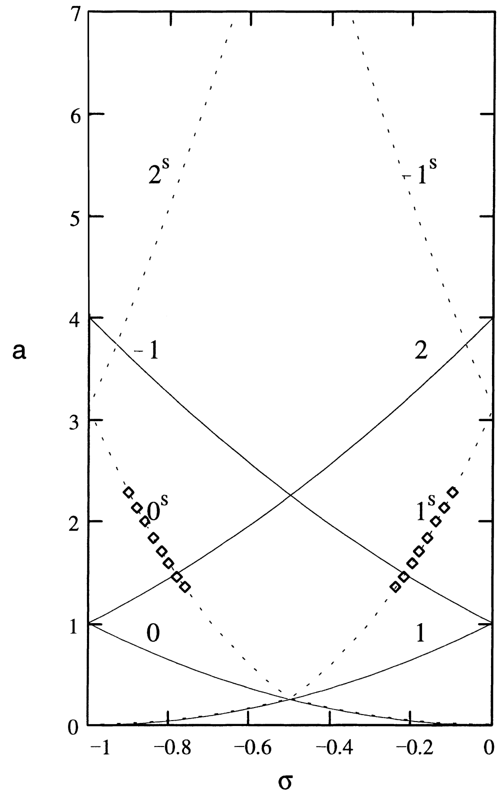

Fig. 5. The domains of the existence (solid line) and stability (dotted line) of the travelling wave solutions of Eqs. (41). The numbers near curves denote the wave numbers k of travelling waves. The part corresponding negative Re β is marked. The intensity of input field $K = 5$, the other parameters are found from Eqs. (3) and (34): $p_0 = -4.273$, $\delta_0 = 1.593$, $S_0 = 1.653$, $c = 0.459$, and $b = -0.594$.

One can see from conditions (42) and (44) that homogeneous solution (TW with $k = 0$) becomes unstable if $1 + bc < 0$. It agrees with the well-known result for the Ginzburg–Landau equation [10] and in this case the possibility of complex irregular dynamics occurs [16]. The parameters of the initial problem described by Eqs. (1) and (2) give us the value bc in the limits $(-0.5, 0)$, therefore such a possibility is not realized in the system.

In the opposite case $(1 + bc > 0)$, there are some intervals of a, σ for which the condition of stability follows the condition of existence of the homogeneous solution. Apart from these intervals, there is a region of the Eckhaus instability [17], where the TW-solution exits, but is unstable. In Fig. 5, this region is situated between the solid and dotted curves. The detailed analysis of the Eckhaus instability has been presented, for example, in [18] for the pitchfork bifurcation which was found to be always subcritical.

The characteristic equation (43) for the normal equation (41) presents another critical case: a pair of the roots has pure imaginary values (at $m = \pm 1$) at $a = a_0, \sigma = \sigma_0$ so that $\lambda_1(a_0, \sigma_0) = \mathrm{i}l$, $\lambda_{-1}(a_0, \sigma_0) = -\mathrm{i}l$. In addition, one simple zero eigenvalue (at $m = 0$) is associated with the symmetry of Eq. (41), namely, $\xi_k \exp(\mathrm{i}\phi)$ is also the solution of this equation. In the local vicinity of the bifurcation parameters

$$a = a_0 + \epsilon a_1, \qquad \sigma = \sigma_0 + \epsilon \sigma_1$$

the corresponding normal form on three-dimensional local invariant manifold reads

$$\frac{\mathrm{d}v}{\mathrm{d}\tau} = \alpha v + \beta v|v|^2, \qquad \frac{\mathrm{d}w}{\mathrm{d}\tau} = 0. \tag{45}$$

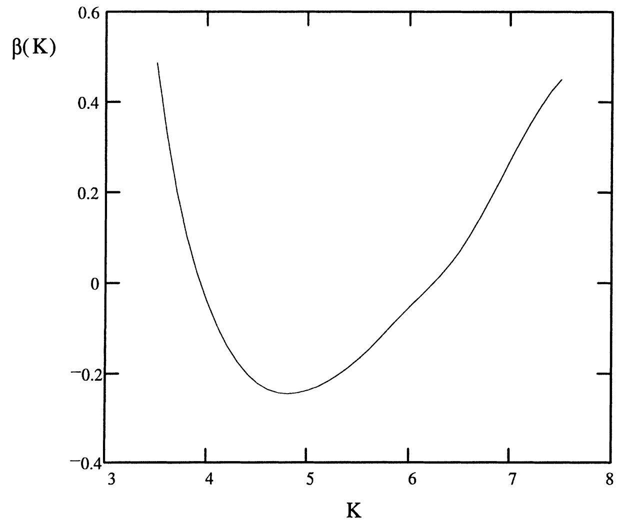

Fig. 6. The dependence of the Lyapunov value $\mathrm{Re}\,\beta$ on the intensity K of input field. $\sigma = -0.9$.

Here $\tau = \epsilon s_1$ denotes a slow time and the function $v(\tau)$ is connected to the amplitude of the travelling wave by the series

$$\xi(s_1, \Theta) = \xi_{0k}[1 + 2\epsilon^{1/2}|v|((1 + \mathrm{i}\,\mathrm{Re}(q))\cos(ls_1 + \Theta) + \mathrm{i}\,\mathrm{Im}(q)\sin(ls_1 + \Theta)) + \epsilon\cdots]\mathrm{e}^{\mathrm{i}\omega_2(1+\epsilon\cdots)s_1 + \mathrm{i}k\Theta}. \tag{46}$$

The parameter q is determined by

$$q = \frac{2\mathrm{i}(\sigma_0 + k) - 2b|\xi_{0k}|^2 - c}{1 + \mathrm{i}l + 2\mathrm{i}c(\sigma_0 + k)},$$

and, in addition, it is convenient to choose the constant value of the function w to be zero, $w = 0$. The expressions for the coefficients α, β of the normal form (45) are too awkward hence we present here only the final result. Fig. 6 shows the dependence of the real part of the Lyapunov value β on the intensity of the input light K. One can see that $\mathrm{Re}\,\beta < 0$ at $K \in (4, 6.3)$. It means that the normal form (45) has a stable periodic solution

$$v(\tau) = \left(-\frac{\mathrm{Re}\,\alpha}{\mathrm{Re}\,\beta}\right)^{1/2} \exp\left(\mathrm{i}\tau\,\mathrm{Re}\,\alpha\left(\frac{\mathrm{Im}\,\alpha}{\mathrm{Re}\,\alpha} - \frac{\mathrm{Im}\,\beta}{\mathrm{Re}\,\beta}\right)\right)$$

in the region of TW-instability. This limit cycle results in a two-frequency attractor (torus) in the original system. For the Lyapunov value $\mathrm{Re}\,\beta > 0$, the unstable cycle exists in the local vicinity of the stable TW-solution.

Thus, complicated pattern dynamics becomes possible at asymptotically small angle of the field rotation in the feedback. Generally, it is interesting to note that, while the diffusion coefficient $d \to 0$, the specific σ value cannot be fixed. A small change of d causes the displacement of the system along line $a =$ const. Fig. 5 shows how the type of solution can change in this case. The part of the bifurcation boundary corresponding to the negative Lyapunov value is specially marked. When σ (or equivalently d) is varied one rotating structure changes into another one, in particular, a complicated beating of spatial modes can be observed. It means that despite of "large" geometrical size R, the dynamics of the system depends unexpectedly on the quantity $R \bmod \sqrt{d}/S_0$.

In conclusion, the complete bifurcation analysis has been done for optical system described by a single parabolic equation with a retarded argument. It provides an excellent agreement with experimental results even on the level of the linear analysis. For instance, the quantitative comparison between theoretical and experimental rotation frequencies of patterns was given in [2] for the parameters closed to critical. Our results explain the stability of such patterns and the existence of the hysteresis which has been also observed experimentally. The last one is a consequence of the mode competition in the vicinity of the bifurcation point of codimension two. Hence, such an accurate experiment would be useful to test more refined conclusions of the theory in the case of the large scale system, in particular, the transformation of the pattern shape to the step-form, the multistability, and, at last, mode-beating regimes depending sensitively on the geometrical size. Since the obtained quasi-normal forms are universal ones, we may also expect analogous self-organization phenomena in different systems. Indeed, a parabolic equation with special antiperiodic boundary conditions appears as a quasi-normal form for some large-time-delay equations [14] to be responsible for step-like temporal structures. Note also a sensitive dependence of the so-called "blinking" state on the aspect ratio which has been experimentally and numerically observed in a large hydrodynamical system [19,20].

Various types of critical cases of finite as well as infinite dimensions arise in the system due to specific "delay" in space. That is why it is interesting to compare the equation with a spatial shift (1) to a parabolic equation with a time delay:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial \theta^2} - u + f[u(t-h,\theta)].$$

On the one hand, the same phenomenon is observed: a complex spectrum and, as a consequence, an occurrence of nonstationary structures. On the other hand, the equation with a time delay should attribute to more difficult problems, as far as its phase space $C_{[-h,0]} \times C_{[0,2\pi]}$ is essentially more difficult than the phase space $C_{[0,2\pi]}$ of Eq. (1). Besides, the spatial shift Δ is limited by the interval $(0, 2\pi)$ (i.e. by the size of the system), while unlimited change of the time delay h is possible that can result in extremely complicated dynamics, even at $D = 0$ [21].

However, there is an important factor which distinguishes between the role of the delay h and of the shift Δ. It is connected with a study of the equations with a small coefficient of diffusion D. Such a peculiarity arises naturally in many problems as due to some physical aspects of wave propagation, as to the spatial size of an object under investigation. If the value of the time delay h is less than some threshold value, there are no local bifurcations at any value of the diffusion coefficient $D \in (0, \infty)$. For Eq. (1) with spatial shift, that is not valid. In fact, the parameters D and Δ are coupled so that at their simultaneous decreasing the bifurcations occur and lead to complex spatial structures even at the level of local dynamics.

Acknowledgements

This work was supported by the Alexander von Humboldt Foundation.

References

[1] S.A. Akhmanov, M.A. Vorontsov, in: Nonlinear waves. Dynamics and evolution, Moscow, 1989, 228–238.
[2] S.A. Akhmanov, M.A. Vorontsov, V. Yu. Ivanov, A.V. Larichev, N.I. Zheleznykh, JOSA B 9 (1992) 78.
[3] H. Haken, Synergetics. An Introduction, Springer, Berlin, 1977.
[4] H. Haken, Advanced Synergetics, Springer, Berlin, 1983.
[5] A.D. Bruno, Local Method of Nonlinear Analysis of Differential Equations, Moscow, 1979.
[6] J. Guckenheimer, P. Holmes, Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields, Springer, Berlin, 1983.
[7] S.A. Kashchenko, Sov. J. Num. Math. and Math. Phys. 31 (1991) 467.
[8] N.I. Zheleznykh, M. Le Berre, E. Ressayre, A. Tallet, Chaos, Solitons, and Fractals 4 (1994) 1717.
[9] A.C. Newell, J.A. Whitehead, J. Fluid. Mech. 38 (1969) 279.

[10] Y. Kuramoto, T. Tsuzuki, Prog. Theor. Phys. 54 (1975) 687.
[11] S.A. Kashchenko, Differen. Uravn. 25 (1989) 262.
[12] S.A. Kashchenko, Sov. Math. Dokl. 37 (1988) 510.
[13] H. Haken, Z. Phys. B 21 (1975) 105.
[14] S.A. Kashchenko, Int. J. Bifurc. and Chaos 6 (1996) 1093.
[15] I.B. Bokolishvily, S.A. Kaschenko, G.G. Malinetskii, A.B. Potapov, Nonlinear Sci. 4 (1994) 545.
[16] T.S. Akhromeyeva, S.P. Kurdyumov, G.G. Malinetskii, A.A. Samarskii, Phys. Rep. 176 (1989) 189.
[17] W. Eckhaus. Studies in Nonlinear Stability Theory, Springer, Berlin, 1965.
[18] L.S. Tuckerman, D. Barkley, Physica D 46 (1990) 57.
[19] M. Bestehorn, R. Friederich, H. Haken, Z. Phys. B 77 (1989) 151.
[20] P. Kolodner, C.M. Surko, H. Williams, Physica D 37 (1989) 319.
[21] K. Ikeda, H. Daido, O. Akimoto, Phys. Rev. Lett. 45 (1980) 709.