An introduction to quantum spin liquids: fermions and gauge fields from bosons

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1. Mean-field approaches to spin liquids
   - Why standard mean-field approaches fail to describe spin liquids
   - Fermionic representation of a spin-1/2
   - Non-standard mean-field approaches for spin liquids
   - Beyond mean field: "low-energy" gauge fluctuations

2. The Kitaev compass model on the honeycomb lattice
   - Definition of the model
   - Majorana fermions
   - Representing the Kitaev model with Majorana fermions
   - Solving the Kitaev model

Standard mean-field approach

Consider the spin-1/2 Heisenberg model on a generic lattice

$$\mathcal{H} = \sum_{ij} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j$$

In a standard mean-field approach, each spin couples to an effective field generated by the surrounding spins:

$$\mathcal{H}_{\rm MF} = \sum_{ij} J_{ij} \left\{ \langle \mathbf{S}_i \rangle \cdot \mathbf{S}_j + \mathbf{S}_i \cdot \langle \mathbf{S}_j \rangle - \langle \mathbf{S}_i \rangle \cdot \langle \mathbf{S}_j \rangle \right\}$$

However, by definition, spin liquids have a zero magnetization:

$$\langle \mathbf{S}_i \rangle = 0$$

How can we construct a mean-field approach for such disordered states?

We need to construct a theory in which all classical order parameters are vanishing

Halving the spin operator

- The first step is to decompose the spin operator in terms of spin-1/2 quasi-particles creation and annihilation operators.
- One possibility is to write:

$$\boxed{S_i^\mu = \frac{1}{2} c_{i,\alpha}^\dagger \sigma_{\alpha,\beta}^\mu c_{i,\beta}}$$

$\sigma_{\alpha,\beta}^\mu$ are the Pauli matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$c_{i,\alpha}^\dagger$ ($c_{i,\beta}$) creates (destroys) a quasi-particle with spin-1/2
These may have various statistics, e.g., bosonic or fermionic

At this stage, splitting the original spin operator in two pieces is just a formal trick.
Whether or not these quasi-particles are true elementary excitations is THE question

Fermionic representation of a spin-1/2

- A faithful representation of spin-1/2 is given by:

$$
\begin{aligned}
S_i^z &= \frac{1}{2}\left(c^\dagger_{i,\uparrow}c_{i,\uparrow} - c^\dagger_{i,\downarrow}c_{i,\downarrow}\right)\\
S_i^+ &= c^\dagger_{i,\uparrow}c_{i,\downarrow}\\
S_i^- &= c^\dagger_{i,\downarrow}c_{i,\uparrow}
\end{aligned}
$$

$\{c_{i,\alpha}, c^\dagger_{j,\beta}\} = \delta_{ij}\delta_{\alpha\beta}$
$\{c_{i,\alpha}, c_{j,\beta}\} = 0$
$c^\dagger_{i,\uparrow}$ (or $c^\dagger_{i,\downarrow}$) changes S_i^z by 1/2 (or −1/2)
and creates a "spinon"

- For a model with one spin per site, we must impose the constraints:

$$\boxed{c^\dagger_{i,\uparrow}c_{i,\uparrow} + c^\dagger_{i,\downarrow}c_{i,\downarrow} = 1} \qquad \boxed{c_{i,\uparrow}c_{i,\downarrow} = 0}$$

- Compact notation by using a 2×2 matrix:

$$\Psi_i = \begin{bmatrix} c_{i,\uparrow} & c^\dagger_{i,\downarrow} \\ c_{i,\downarrow} & -c^\dagger_{i,\uparrow} \end{bmatrix} \qquad S_i^\mu = -\frac{1}{4}\mathrm{Tr}\left[\sigma^\mu \Psi_i \Psi_i^\dagger\right] \qquad G_i^\mu = \frac{1}{4}\mathrm{Tr}\left[\sigma^\mu \Psi_i^\dagger \Psi_i\right] = 0$$

Local redundancy and "gauge" transformations

$$S_i^\mu = -\frac{1}{4}\mathrm{Tr}\left[\sigma^\mu \Psi_i \Psi_i^\dagger\right]$$

$$\mathbf{S}_i \cdot \mathbf{S}_j = \frac{1}{16}\sum_\mu \mathrm{Tr}\left[\sigma^\mu \Psi_i \Psi_i^\dagger\right] \mathrm{Tr}\left[\sigma^\mu \Psi_j \Psi_j^\dagger\right] = \frac{1}{8}\mathrm{Tr}\left[\Psi_i \Psi_i^\dagger \Psi_j \Psi_j^\dagger\right]$$

• Spin rotations are left rotations:

$$\Psi_i \to R_i \Psi_i$$

The Heisenberg Hamiltonian is invariant under global rotations

• The spin operator is invariant upon local SU(2) "gauge" transformations, right rotations:

$$\Psi_i \to \Psi_i W_i$$

$$\mathbf{S}_i \to \mathbf{S}_i$$

There is a huge redundancy in this representation

Affleck, Zou, Hsu, and Anderson, Phys. Rev. B **38**, 745 (1988)

Mean-field approximation

- We transformed a spin model into a model of interacting fermions (subject to the constraint of one-fermion per site)
- The first approximation to treat this problem is to consider a mean-field decoupling:

$$\Psi_i^\dagger \Psi_j \Psi_j^\dagger \Psi_i \to \langle \Psi_i^\dagger \Psi_j \rangle \Psi_j^\dagger \Psi_i + \Psi_i^\dagger \Psi_j \langle \Psi_j^\dagger \Psi_i \rangle - \langle \Psi_i^\dagger \Psi_j \rangle \langle \Psi_j^\dagger \Psi_i \rangle$$

We define the mean-field 2×2 matrix

$$U_{ij}^0 = \frac{J_{ij}}{4} \langle \Psi_i^\dagger \Psi_j \rangle = \frac{J_{ij}}{4} \left[\begin{array}{cc} \langle c_{i,\uparrow}^\dagger c_{j,\uparrow} + c_{i,\downarrow}^\dagger c_{j,\downarrow} \rangle & \langle c_{i,\uparrow}^\dagger c_{j,\downarrow}^\dagger + c_{j,\uparrow}^\dagger c_{i,\downarrow}^\dagger \rangle \\ \langle c_{i,\downarrow} c_{j,\uparrow} + c_{j,\downarrow} c_{i,\uparrow} \rangle & -\langle c_{j,\downarrow}^\dagger c_{i,\downarrow} + c_{j,\uparrow}^\dagger c_{i,\downarrow} \rangle \end{array} \right] = \left[\begin{array}{cc} \chi_{ij} & \eta_{ij}^* \\ \eta_{ij} & -\chi_{ij}^* \end{array} \right]$$

- $\chi_{ij} = \chi_{ji}^*$ is the spinon hopping
- $\eta_{ij} = \eta_{ji}$ is the spinon pairing

Mean-field approximation

The mean-field Hamiltonian has a BCS-like form:

$$\mathcal{H}_{MF} = \sum_{ij} \chi_{ij}(c^\dagger_{j,\uparrow} c_{i,\uparrow} + c^\dagger_{j,\downarrow} c_{i,\downarrow}) + \eta_{ij}(c^\dagger_{j,\uparrow} c^\dagger_{i,\downarrow} + c^\dagger_{i,\uparrow} c^\dagger_{j,\downarrow}) + h.c.$$

$$+ \sum_i \mu_i \, (c^\dagger_{i,\uparrow} c_{i,\uparrow} + c^\dagger_{i,\downarrow} c_{i,\downarrow} - 1) + \sum_i \zeta_i \, c^\dagger_{i,\uparrow} c^\dagger_{i,\downarrow} + h.c.$$

- $\{\chi_{ij}, \eta_{ij}, \mu_i\,, \zeta_i\,\}$ define the mean-field Ansatz
- At the mean-field level:
  - χ_{ij} and η_{ij} are fixed numbers
  - Constraints are satisfied only in average

At the mean-field level, spinons are free.
Beyond this approximation, they will interact with each other
Do they remain asymptotically free (at low energies)?

Redundancy of the mean-field approximation

- Let $|\Phi_{MF}(U^0_{ij})\rangle$ be the ground state of the mean-field Hamiltonian (with a given Ansatz for the mean-field U^0_{ij})
- $|\Phi_{MF}(U^0_{ij})\rangle$ cannot be a valid wave function for the spin model (the Hilbert space is wrong, it has not one fermion per site!)
- A valid wave function for the spin model is obtained by projecting $|\Phi_{MF}(U^0_{ij})\rangle$ onto the sub-space with one fermion per site

$$|\Psi_{\rm spin}(U^0_{ij}\rangle = \mathcal{P}|\Phi_{MF}(U^0_{ij})\rangle$$

- Let us consider an arbitrary *site-dependent* SU(2) matrix $W_i \implies \Psi_i \to \Psi_i W_i$
  It leaves the spin unchanged $\mathbf{S}_i \to \mathbf{S}_i \implies U^0_{ij} \to W_i^\dagger U^0_{ij} W_j$
  Therefore, U^0_{ij} and $W_i^\dagger U^0_{ij} W_j$ define the same physical state
  (the same physical state can be represented by many different Ansätze U^0_{ij})

$$\langle 0| \prod_i c_{i,\alpha_i} |\Phi_{MF}(U^0_{ij})\rangle = \langle 0| \prod_i c_{i,\alpha_i} |\Phi_{MF}(W_i^\dagger U^0_{ij} W_j)\rangle$$

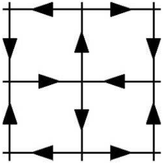
An example of the redundancy on the square lattice

- The staggered flux state is defined by

Affleck and Marston, Phys. Rev. B **37**, 3774 (1988)

$$j \in A \begin{cases} \chi_{j,j+x} = e^{i\Phi_0/4} \\ \chi_{j,j+y} = e^{-i\Phi_0/4} \end{cases}$$

$$j \in B \begin{cases} \chi_{j,j+x} = e^{-i\Phi_0/4} \\ \chi_{j,j+y} = e^{i\Phi_0/4} \end{cases}$$



- The d-wave "superconductor" state is defined by

Baskaran, Zou, and Anderson, Solid State Commun. **63**, 973 (1987)

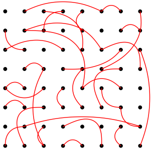
$$\begin{cases} \chi_{j,j+x} = 1 \\ \chi_{j,j+y} = 1 \\ \eta_{j,j+x} = \Delta \\ \eta_{j,j+y} = -\Delta \end{cases}$$

- For $\Delta = \tan(\Phi_0/4)$, these two mean-field states become the same state after projection
- The mean-field spectrum is the same for the two states

- The mean-field wave function has a BCS-like form

$$|\Phi_{MF}\rangle = \exp\left\{ \frac{1}{2} \sum_{i,j} f_{i,j} c^{\dagger}_{i,\uparrow} c^{\dagger}_{j,\downarrow} \right\} |0\rangle$$

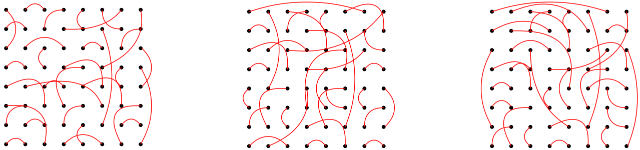
It is a linear superposition of all singlet configurations (that may overlap)



$+ \ldots$

- After projection, only non-overlapping singlets survive:
  the resonating valence-bond (RVB) wave function

Anderson, Science **235**, 1196 (1987)



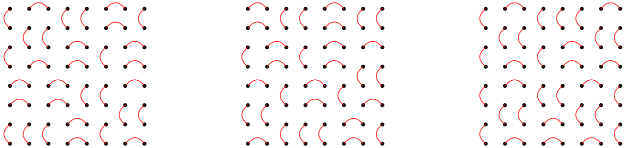
$+ \ldots$

The projected wave function

- The mean-field wave function has a BCS-like form

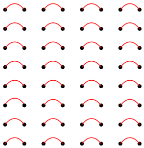
$$|\Phi_{MF}\rangle = \exp\left\{\frac{1}{2}\sum_{i,j} f_{i,j} c^{\dagger}_{i,\uparrow} c^{\dagger}_{j,\downarrow}\right\} |0\rangle$$

- Depending on the pairing function $f_{i,j}$, different RVB states may be obtained:



$+ \ldots$

- ...even with valence-bond order (valence-bond crystals)



Beyond mean field: "low-energy" gauge fluctuations

- Beyond mean field we can consider fluctuations of U^0_{ij}

$$U^0_{ij} = \frac{J_{ij}}{4} \langle \Psi^\dagger_i \Psi_j \rangle \Longrightarrow U^0_{ij} + \delta U_{ij}$$

- Wen's conjecture:

Amplitude fluctuations have a finite energy gap and are not essential

Phase fluctuations instead are important: $U^0_{ij} \Longrightarrow U^0_{ij} e^{iA_{ij}}$

In particular, all A_{ij} that leave U^0_{ij} invariant: $\mathcal{G}^\dagger_i U^0_{ij} \mathcal{G}_j = U^0_{ij}$

A_{ij} plays the role of a gauge field coupled to spinons

Wen, Phys. Rev. B **65**, 165113 (2002)

By adding "low-energy" fluctuations on top of the mean field Ansatz, we obtain a theory of matter (spinons) coupled to gauge fields

The structure of the "low-energy" gauge fluctuations may be different from the original "high-energy" one, we can have Z_2, U(1), SU(2)... spin liquids

Fluctuations above the mean field and gauge fields

- Some results about lattice gauge theory (coupled to matter, i.e., spinons) may be used to discuss the stability/instability of a given mean-field Ansatz

- What is known about U(1) gauge theories?
  Monopoles proliferate $\rightarrow$ confinement

  Polyakov, Nucl. Phys. B **120**, 429 (1977)

  Spinons are glued in pairs by strong gauge fluctuations and are not physical excitations

- Deconfinement may be possible in presence of gapless matter field
  The so-called U(1) spin liquid

  Hermele et al., Phys. Rev. B **70**, 214437 (2004)

- In presence of a charge-2 field (i.e., spinon pairing) the U(1) symmetry can be lowered to Z_2 $\rightarrow$ deconfinement

  Fradkin and Shenker, Phys. Rev. D **19**, 3682 (1979)

- For example in D=2:
  - Z_2 gauge field (gapped) + gapped spinons may be a stable deconfined phase short-range RVB physics Read and Sachdev, Phys. Rev. Lett. **66**, 1773 (1991)
  - U(1) gauge field (gapless) + gapped spinons should lead to an instability towards confinement and valence-bond order Read and Sachdev, Phys. Rev. Lett. **62**, 1694 (1989)

Summary of "low-energy" gauge theories

- The spin operator is written in terms of "more fundamental" objects: spinons
- The Hilbert space is artificially enlarged
- A constraint must be introduced to go back to the original Hilbert space of spins

  $\Longrightarrow$ A gauge redundancy appears
- At the mean-field level, there are free particles (spinons)
- Beyond mean field, spinons interact with gauge fluctuations
- Is the "low-energy" picture stable and valid to describe the original spin model?

  Arguments suggest that a (gapped) Z_2 gauge field may preserve the mean-field results

  Here, gauge excitations are called visons

  A vison is a quantized (magnetic) flux threading an elementary plaquette

  Senthil and Fisher, Phys. Rev. B **62**, 7850 (2000)

“To believe or not to believe”

How can a purely bosonic model have an effective theory described by gauge fields and fermions? This is incredible

Wen, *Quantum Field Theory of Many-Body Systems* (Oxford University Press 2004)

- There are many attempts to define *ad hoc* bosonic models having fermions and gauge fields as elementary excitations
- One class of these models are based upon string-net theories

Wen, Phys. Rev. Lett. **90**, 016803 (2003)

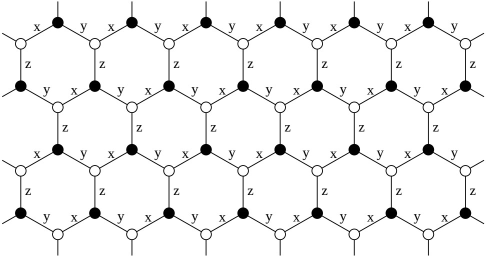
Kitaev, Ann. Phys. **303**, 2 (2003)

In the following, I will consider a spin model that is exactly described by fermions and gauge fields

The Kitaev compass model on the honeycomb lattice

- Rather artificial spin model breaking SU(2) symmetry
- Possible physical realization in Iridates with strong spin-orbit coupling

  Jackeli and Khaliullin, Phys. Rev. Lett. **102**, 017205 (2009)



$$\mathcal{H} = -J_x \sum_{x\text{-links}} \sigma_j^x \sigma_k^x - J_y \sum_{y\text{-links}} \sigma_j^y \sigma_k^y - J_z \sum_{z\text{-links}} \sigma_j^z \sigma_k^z$$

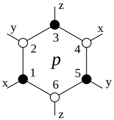
J_x, J_y, and J_z are model parameters

σ_j^x, σ_j^y, and σ_j^z are Pauli matrices on site j

Kitaev, Ann. Phys. **321**, 2 (2006)

Properties of the Kitaev model

- Take a cluster with $2N$ sites $\Longrightarrow$ N plaquettes
- There are $N-1$ integrals of motion W_p:



$$K_{jk} = \begin{cases} \sigma_j^x \sigma_k^x, & \text{if } (j,k) \text{ is an } x\text{-link;} \\ \sigma_j^x \sigma_k^y, & \text{if } (j,k) \text{ is an } y\text{-link;} \\ \sigma_j^x \sigma_k^z, & \text{if } (j,k) \text{ is an } z\text{-link.} \end{cases}$$

- All operators K_{jk} commute with

$$W_p = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z = K_{12} K_{23} K_{34} K_{45} K_{56} K_{61}.$$

- Different operators W_p commute with each other
- "Only" $N-1$ independent W_p because $\prod_p W_p = 1$
- Each operator W_p has eigenvalues $+1$ and -1

Properties of the Kitaev model

- The existence of $N-1$ operators commuting with $\mathcal{H}$ simplifies the problem
- $\Longrightarrow$ The Hamiltonian can be diagonalized in each sector separately
- The total Hilbert space is 2^{2N}
- $\Longrightarrow$ The dimension of each sector is $2^{2N}/2^{N-1} = 2^{N+1}$
- The problem is still exponentially hard
- However, the degrees of freedom in each sectors can be described by free Majorana fermions
- Solution in terms of free particles in presence of Z_2 magnetic fluxes, i.e., visons (values of W_p for each plaquette)

What is a Majorana fermion?

Let us consider a system with L fermionic modes

- This is usually described by annihilation and creation operators a_k and $a_k^\dagger$ with $k = 1, \dots, L$

  $\{a_k, a_p\} = \{a_k^\dagger, a_p^\dagger\} = 0$ and $\{a_k, a_p^\dagger\} = \delta_{k,p}$

- Instead, one can use linear combinations

$$c_{2k-1} = a_k^\dagger + a_k$$
$$c_{2k} = i(a_k^\dagger - a_k)$$

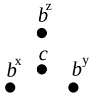
- They are called Majorana operators

  The operators c_j ($j = 1, \dots, 2L$) are Hermitian and obey the following relations:

$$c_j^2 = 1$$
$$c_i c_j = -c_j c_i \qquad i \neq j$$

- Let us represent the spin operator by 4 Majorana fermions

$$\boxed{\sigma^x = ib^x c \qquad \sigma^y = ib^y c \qquad \sigma^z = ib^z c}$$



- $\Longrightarrow$ We enlarge the Hilbert space

  2 physical spin states versus 4 unphysical fermionic states

$$\boxed{\sigma^x \sigma^y \sigma^z = ib^x b^y b^z c = iD}$$

- The physical Hilbert space is defined by states $|\xi\rangle$ that satisfy

$$\boxed{D|\xi\rangle = |\xi\rangle}$$

- The operator D may be thought of as a gauge transformation for the group Z_2

Representing the Kitaev model with Majorana fermions

$$\mathcal{H} = -J_x \sum_{x\text{-links}} \sigma_j^x \sigma_k^x - J_y \sum_{y\text{-links}} \sigma_j^y \sigma_k^y - J_z \sum_{z\text{-links}} \sigma_j^z \sigma_k^z$$

$$K_{jk} = \begin{cases} \sigma_j^x \sigma_k^x, & \text{if } (j,k) \text{ is an } x\text{-link;} \\ \sigma_j^x \sigma_k^y, & \text{if } (j,k) \text{ is an } y\text{-link;} \\ \sigma_j^x \sigma_k^z, & \text{if } (j,k) \text{ is an } z\text{-link.} \end{cases}$$

- By using the Majorana fermions

$$K_{jk} = (ib_j^\alpha c_j)(ib_k^\alpha c_k) = -i\,(ib_j^\alpha b_k^\alpha)\,c_j c_k$$

- We define the Hermitian operator $u_{jk} = ib_j^\alpha b_k^\alpha$, associated to each link (j,k)
  The index α takes values x, y or z depending on the direction of the link

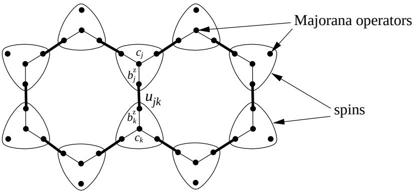
- The Hamiltonian becomes:

$$\mathcal{H} = \frac{i}{4} \sum_{j,k} A_{jk} c_j c_k, \qquad A_{jk} = \begin{cases} 2J_{\alpha_{jk}} u_{jk} & \text{if } j \text{ and } k \text{ are connected} \\ 0 & \text{otherwise} \end{cases}$$

$$u_{jk} = -u_{kj}$$

Representing the Kitaev model with Majorana fermions

$$\mathcal{H} = \frac{i}{4} \sum_{j,k} A_{jk} c_j c_k , \qquad A_{jk} = \begin{cases} 2J_{\alpha_{jk}} u_{jk} & \text{if } j \text{ and } k \text{ are connected} \\ 0 & \text{otherwise} \end{cases}$$

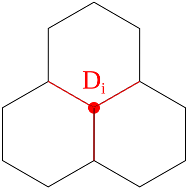


Now, the great simplification!

- All operators u_{jk} commute with the Hamiltonian and with each other
- $\Longrightarrow$ The Hilbert space splits into eigenspaces with fixed u_{jk} labeled by the eigenvalues $u_{jk} = \pm 1$
- $\Longrightarrow$ The Hamiltonian is quadratic in the c operators The set $\{u\}$ determine static magnetic fluxes through the plaquettes
- $\Longrightarrow$ All eigenfunctions $|\Psi_u\rangle$ with a fixed set $\{u\}$ can be found exactly

Remarks on the new representation

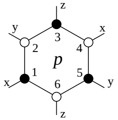
- The Hamiltonian commutes with all operators u_{jk}: $[\mathcal{H}, u_{jk}] = 0$
- The Hamiltonian commutes with all constraints D_i: $[\mathcal{H}, D_i] = 0$
- However, the link operators u_{jk} do not commute with the constraints D_i
  In particular, $D_j u_{jk} = -u_{jk} D_j$
  Applying D_j changes the values of u_{jk} on the links connecting j with the neighbors



- $\Longrightarrow$ The subspace with fixed u_{jk} is not gauge invariant

Remarks on the new representation

- The gauge-invariant objects are the fluxes through each plaquette $W_p = -u_{12} u_{23} u_{34} u_{45} u_{56} u_{61}$



D_j acts as a gauge transformation:
it changes u_{jk} but not the fluxes W_p (every plaquette changes 2 links)

- The eigenfunctions $|\Psi_u\rangle$ with a fixed set of $\{u\}$ do not belong to the physical subspace

- To obtain a physical wave function, we must symmetrize over all gauge transformations

$$|\Phi_w\rangle = \mathcal{P}|\Psi_u\rangle = \prod_j \left(\frac{1 + D_j}{2}\right)|\Psi_u\rangle$$

w denotes the equivalence class of u under the gauge transformations

Since $[\mathcal{P}, \mathcal{H}] = 0$, $|\Phi_w\rangle$ has the same eigenvalue as $|\Psi_u\rangle$

Diagonalizing the Kitaev model

$$\mathcal{H} = \frac{i}{4} \sum_{j,k} A_{jk} c_j c_k , \qquad A \text{ is a skew-symmetric matrix of size } 2N$$

- Diagonalize the Hamiltonian by considering the canonical form

$$\mathcal{H}_{\text{canonical}} = \frac{i}{2} \sum_{k=1}^{N} \epsilon_k b'_k b''_k = \sum_{k=1}^{N} \epsilon_k \left(a_k^\dagger a_k - \frac{1}{2} \right) \qquad \epsilon_k \geq 0$$

where b'_k, b''_k are normal modes

$$(b'_1, b''_1, \ldots, b'_N, b''_N) = (c_1, c_2, \ldots, c_{2N-1}, c_{2N}) Q$$

$$A = Q \begin{pmatrix} 0 & \epsilon_1 & & & \\ -\epsilon_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \epsilon_N \\ & & & -\epsilon_N & 0 \end{pmatrix} Q^T$$

$a_k^\dagger$ and a_k are the corresponding creation and annihilation operators

$$a_k^\dagger = \frac{1}{2}(b'_k - i b''_k) \qquad a_k = \frac{1}{2}(b'_k + i b''_k)$$

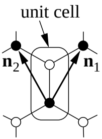
- The energy minimum is obtained by the vortex-free configuration (no visons)

  $W_p = 1$ for all plaquettes

- $\Longrightarrow$ We may assume $u_{jk} = 1$ for all links (j, k)

- $\Longrightarrow$ Translational symmetry $\Longrightarrow$ the spectrum can be found by the Fourier transform

  We take $\mathbf{n}_1 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $\mathbf{n}_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$



$$iA(\mathbf{q}) = \begin{pmatrix} 0 & if(\mathbf{q}) \\ -if(\mathbf{q})^* & 0 \end{pmatrix} \qquad \epsilon(\mathbf{q}) = \pm|f(\mathbf{q})|$$

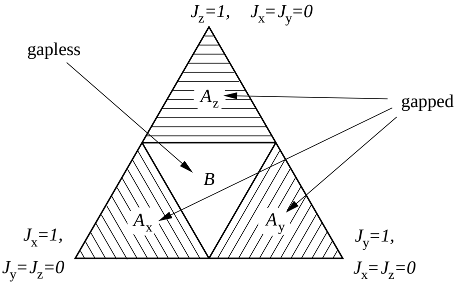
$$f(\mathbf{q}) = 2(J_x e^{i\mathbf{q}\cdot\mathbf{n}_1} + J_y e^{i\mathbf{q}\cdot\mathbf{n}_2} + J_z)$$

The phase diagram

The spectrum may be gapless or gapped

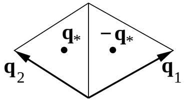
$$f(\mathbf{q}) = 2(J_x e^{i\mathbf{q}\cdot\mathbf{n}_1} + J_y e^{i\mathbf{q}\cdot\mathbf{n}_2} + J_z) = 0$$

has solutions only if $|J_x| \leq |J_y| + |J_z| \quad |J_y| \leq |J_x| + |J_z| \quad |J_z| \leq |J_x| + |J_y|$

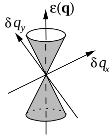


- In the gapless phase B, there are 2 gapless points at $\mathbf{q} = \pm\mathbf{q}_*$
- The gapped phases A_x, A_y, and A_z are distinct (but related by rotational symmetry)

- In the symmetric case $J_x = J_y = J_z$ the zeros of the spectrum are given by



$$
\begin{aligned}
+\mathbf{q}_* &= \tfrac{1}{3}\mathbf{q}_1 + \tfrac{2}{3}\mathbf{q}_2 \\
-\mathbf{q}_* &= \tfrac{2}{3}\mathbf{q}_1 + \tfrac{1}{3}\mathbf{q}_2
\end{aligned}
$$



- Gapless excitations with relativistic dispersion (Dirac cones)

- If $|J_x|$ and $|J_y|$ decrease (with constant $|J_z|$), $\pm\mathbf{q}_*$ move toward each other until they fuse and disappear

Phase diagram: discussion

Gapless B phase

- In presence of a finite number of vortices (visons) the problem is still easy (diagonalization of a $2N \times 2N$ matrix)
- States with a finite number of visons are gapped

  Remark: In this model visons are static
- A full gap opens when adding perturbations that break time reversal symmetry

Gapped A phase

- The A phases are gapped but show non-trivial structure
- By using perturbation theory for $|J_x|, |J_y| \ll |J_z| \Longrightarrow$ The Toric Code

  Kitaev, Ann. Phys. **303**, 2 (2003)

  Topological order (four-fold degeneracy of the ground state)

  Abelian anyons (non-trivial braiding rules between e and m excitations)

Conclusions

A purely bosonic model can have an effective theory described by gauge fields and fermions. This is incredible, but it is true