An introduction to quantum spin liquids: general definitions and physical properties

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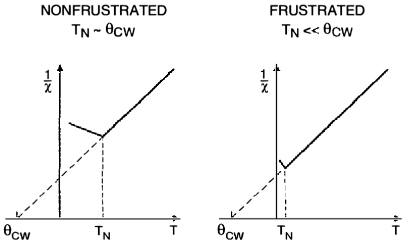
Searching for non-magnetic ground states

- In a spin model, magnetic order is expected at (mean field):

$$k_B T_N \propto zS(S+1)|J|$$

z is the coordination number, S is the spin and J is the super-exchange coupling



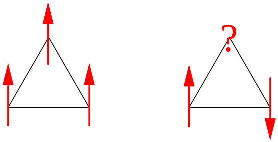
$$\chi = \frac{C}{T - \theta_{cw}} \qquad T \gg T_N$$

θ_{CW} is the Curie-Weiss temperature

$$f = \frac{|\theta_{cw}|}{T_N}$$

- Can quantum fluctuations prevent magnetic order down to $T = 0$?

$\Longrightarrow$ Look for low spin S, low coordination z, competing interactions:



Ultimate frustration?

Looking for a magnetically disordered ground state

- Many theoretical suggestions since P.W. Anderson (1973)

  Anderson, Mater. Res. Bull. **8**, 153 (1973)

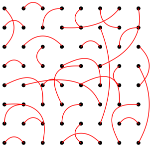
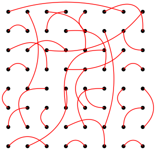
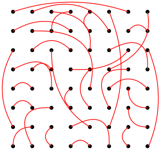
  Fazekas and Anderson, Phil. Mag. **30**, 423 (1974)

  "Resonating valence-bond" (quantum spin liquid) states

  Idea: the best state for two spin-1/2 spins is a valence bond (a spin singlet):

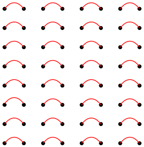
$$|VB\rangle_{\mathbf{R},\mathbf{R}'} = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_{\mathbf{R}} |\downarrow\rangle_{\mathbf{R}'} - |\downarrow\rangle_{\mathbf{R}} |\uparrow\rangle_{\mathbf{R}'} \right)$$

  Every spin of the lattice is coupled to a partner
  Then, take a superposition of different valence bond configurations

$\Psi =$  $+$  $+$  $+ \dots$

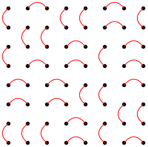
Valence-bond states: liquids and solids

Valence-bond solid



breaks translational/rotational symmetries

Short-range RVB



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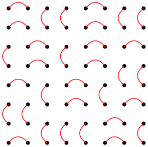

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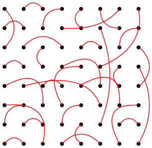
Long-range RVB



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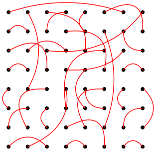


$+ \ldots$

General properties of valence-bond states

- The formation of a valence bond implies a gap to excite those two spins
- Long-range valence bonds are more weakly bound: a gapless spectrum is possible
- The number of resonating valence-bond states is vast
  (according to different linear superpositions)
- It is now clear that the number of distinct quantum spin liquids is also huge
  hundreds of different quantum spin liquids have been classified
  (all with the same symmetry $\Longrightarrow$ topological order)

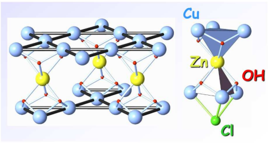
  Wen, Phys. Rev. B **65**, 165113 (2002)
- It is usually believed that such states may be described by gauge theories
  (at least at low energies/temperatures)
  $\Longrightarrow$ Gauge excitations should be visible in the spectrum!

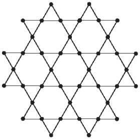
Candidate materials for $S = 1/2$ spin liquids

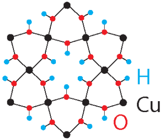
- Many experimental efforts to synthetize new materials

Two-dimensional Kagome lattice: Herbertsmithite and Volborthite

$ZnCu_3(OH)_6Cl_2$ and $Cu_3V_2O_7(OH)_2$ $2H_2O$

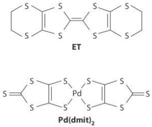


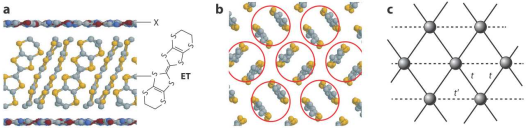




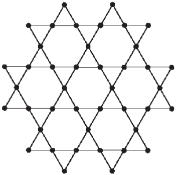
Two-dimensional anisotropic lattice: organic materials

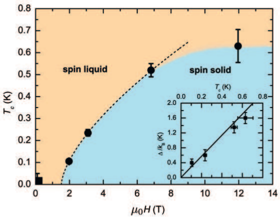
κ-$(BEDT\text{-}TTF)_2Cu_2(CN)_3$ and $EtMe_3Sb[Pd(dmit)_2]_2$



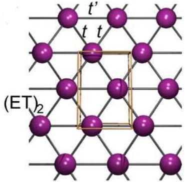


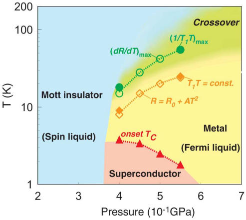
Candidate materials for $S = 1/2$ spin liquids

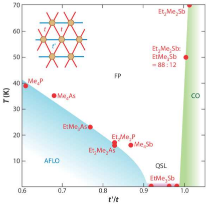




Jeong *et al.*, Phys. Rev. Lett. **107**, 237201 (2011)







Kanoda and Kato, Annu. Rev. Condens. Matter Phys. **2**, 167 (2011)

Shimizu *et al.*, Phys. Rev. Lett. **91**, 107001 (2003)

Candidate materials for $S = 1/2$ spin liquids

Material	Lattice	$\lvert\theta_{cw}\rvert$	f
κ-(BEDT-TTF)$_2$Cu$_2$(CN)$_3$	$\approx$ triangular	375K	$> 10^3$
EtMe$_3$Sb[Pd(dmit)$_2$]$_2$	$\approx$ triangular	350K	$> 10^3$
$ZnCu_3(OH)_6Cl_2$	kagome	240K	$> 10^3$
$Cu_3V_2O_7(OH)_2 \cdot 2H_2O$	$\approx$ kagome	120K	≈ 100
$BaCu_3V_2O_8(OH)_2$	$\approx$ kagome	80K	$> 10^2$
Cs_2CuCl_4	quasi one-dimensional	4K	≈ 10

From Hubbard to Heisenberg

- Zero temperature $T = 0$
- Correlated electrons on the lattice

The starting point is the Hubbard model:

$$\mathcal{H} = -\sum_{i,j,\sigma} t_{i,j} c^{\dagger}_{i,\sigma} c_{j,\sigma} + h.c. + U \sum_{i} n_{i,\uparrow} n_{i,\downarrow}$$

At half-filling (i.e., $N_e = N_s$) for $U \gg t$, an insulating state exists

For $U/t \to \infty$, by perturbation theory, we obtain the Heisenberg model:

$$\mathcal{H} = \sum_{i,j} J_{i,j} \mathbf{S}_i \cdot \mathbf{S}_j + \sum_{i,j,k,l} (P_{i,j,k,l} + h.c.) + \dots$$

- Spin SU(2) symmetric models

Here, I will discuss **spin models** (frozen charge degrees of freedom)

Simple considerations for classical spins

We want to find the lowest-energy spin configuration for classical spins
Consider the case of Bravais lattices (i.e., one site per unit cell)

$$E[\{\mathbf{S}_i\}] = \frac{1}{2} \sum_i \sum_r J(r) \mathbf{S}_i \cdot \mathbf{S}_{i+r}$$

with the *local* constraint $\mathbf{S}_i^2 = 1$
By Fourier transform:

$$E = \frac{1}{2} \sum_k J(k) \mathbf{S}_k \cdot \mathbf{S}_{-k}$$

Look for solutions with the *global* constraint: $\sum_i \mathbf{S}_i^2 = N \longrightarrow \sum_k \mathbf{S}_k \cdot \mathbf{S}_{-k} = N$

Assume $J(k)$ minimized for $k = k_0$

Take $\mathbf{S}_k = 0$ for all k's except for $k = \pm k_0$

$$\mathbf{S}_{k_0} = \frac{\sqrt{N}}{2} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \qquad \mathbf{S}_{-k_0} = \mathbf{S}_{k_0}^* = \frac{\sqrt{N}}{2} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

$$\mathbf{S}_i = \frac{1}{\sqrt{N}} \left(\mathbf{S}_{k_0} e^{i k_0 r_i} + h.c. \right) = \{\cos(k_0 r_i), \sin(k_0 r_i), 0\}$$

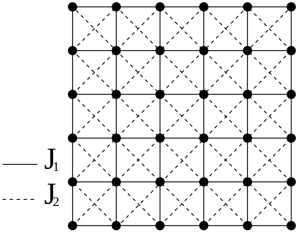
The *local* constraint is automatically satisfied!

Spiral configuration (in general non-collinear – coplanar)

Example: Classical $J_1 - J_2$ model on the square lattice

$J(k) = 2J_1 (\cos k_x + \cos k_y) + 4J_2 \cos k_x \cos k_y$

- For $J_2/J_1 < 1/2$, $k_0 = (\pi, \pi)$
- For $J_2/J_1 > 1/2$, $k_0 = (\pi, 0)$ or $(0, \pi)$
  The two sublattices are decoupled
  (free angle between spins in A and B sublattices)
- For $J_2/J_1 = 1/2$, $k_0 = (\pi, k_y)$ or (k_x, π)
  highly-degenerate ground state:
  $\mathcal{H} = \text{const.} + \sum_{\text{plaquettes}} (\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4)^2$



Quantum fluctuations

In order to include the quantum fluctuations, perform a $1/S$ expansion

$$\mathcal{H} = \sum_{i,j} J_{i,j} \mathbf{S}_i \cdot \mathbf{S}_j$$

- Let us denote by $\theta_j = k_0 \cdot r_j$
- Make a rotation around the z axis

$$\begin{cases} \tilde{S}_j^x = \cos\theta_j S_j^x + \sin\theta_j S_j^y \\ \tilde{S}_j^y = -\sin\theta_j S_j^x + \cos\theta_j S_j^y \\ \tilde{S}_j^z = S_j^z \end{cases}$$

- Perform the Holstein-Primakoff transformations:

$$\begin{cases} \tilde{S}_j^x = S - a_j^\dagger a_j \\ \tilde{S}_j^y \simeq \sqrt{\frac{S}{2}} \left(a_j^\dagger + a_j \right) \\ \tilde{S}_j^z \simeq i\sqrt{\frac{S}{2}} \left(a_j^\dagger - a_j \right) \end{cases}$$

Quantum fluctuations

At the leading order in $1/S$, we obtain:

$$\mathcal{H}_{\rm sw} = \mathrm{E}_{\rm cl} + \frac{S}{2}\sum_k \left\{ A_k a_k^\dagger a_k + \frac{B_k}{2}\left(a_k^\dagger a_{-k}^\dagger + a_{-k} a_k \right) \right\}$$

Where:

$$\mathrm{E}_{\rm cl} = \frac{1}{2} N S^2 J_{k_0}$$

$$\begin{cases} A_k = J_k + \frac{1}{2}(J_{k+k_0} + J_{k-k_0}) - 2J_{k_0} \\ B_k = \frac{1}{2}(J_{k+k_0} + J_{k-k_0}) - J_k \end{cases}$$

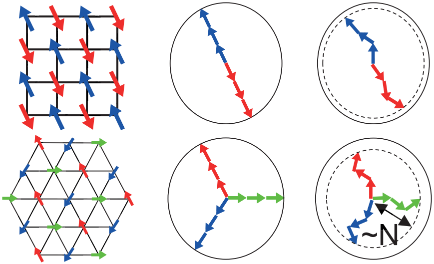
By performing a Bogoliubov transformation:

$$\boxed{\mathcal{H}_{sw} = \mathrm{E}_{\rm cl} + \sum_k \omega_k (\alpha_k^\dagger \alpha_k + \tfrac{1}{2})}$$

- Leading-order corrections to the magnetization $\langle \tilde{S}_j^x \rangle = S - \langle a_j^\dagger a_j \rangle$
- Excitations are called magnons (analog of phonons for lattice waves)
- Presence of gapless excitations for broken SU(2) systems (Goldstone mode)

Renormalization of the classical state

The classical ground state is "dressed" by quantum fluctuations



- The lattice breaks up into sublattices
- Each sublattice keeps an extensive magnetization

$$S(q) = \frac{1}{N}\langle\Psi_0|\left|\sum_j \mathbf{S}_j e^{iqr_j}\right|^2|\Psi_0\rangle = \frac{1}{N}\sum_{j,k}\langle\Psi_0|\mathbf{S}_j\cdot\mathbf{S}_k|\Psi_0\rangle e^{iq(r_j-r_k)}$$

$$S(q) = \begin{cases} O(1) & \text{for all q's} \quad \rightarrow \text{short-range correlations} \\ S(q_0) \propto N & \text{for} q = q_0 \quad \rightarrow \text{long-range order} \end{cases}$$

Fingerprints in finite clusters

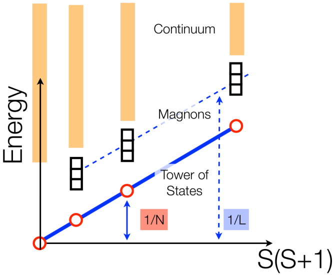
- Spontaneous symmetry breaking is only possible in the thermodynamic limit
  Spontaneously broken SU(2) symmetry $\Longrightarrow$ Gapless spin waves

- How can we detect it on finite lattices (e.g., by exact diagonalizations)?
  $\Longrightarrow$ Tower of states

Anderson, Phys. Rev. **86**, 694 (1952)

Bernu, Lhuillier, and Pierre, Phys. Rev. Lett. **69**, 2590 (1992)

Bernu, Lecheminant, Lhuillier, and Pierre, Phys. Rev. B **50**, 10048 (1994)



A family of states with S up to $O(\sqrt{N})$ collapse to the ground state with $\Delta E_S \propto S(S+1)/N$

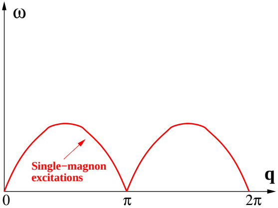
In the thermodynamic limit $\Delta E_S \to 0$
Linear combinations of states with different S $\Longrightarrow$ broken SU(2) symmetry

Inelastic Neutron scattering: magnon excitations and continuum

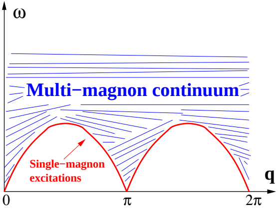
The inelastic Neutron scattering is a probe for the dynamical structure factor

$$S(q,\omega) = \int dt \langle \Psi_0 | S^{\alpha}_{-q}(t) S^{\alpha}_{q}(0) | \Psi_0 \rangle e^{i\omega t} = \sum_{n \neq 0} |\langle \Psi_n | S^{\alpha}_{q} | \Psi_0 \rangle|^2 \delta(\omega - \Delta\omega_{n0})$$

Within the harmonic approximation there is only a single branch of excitations (magnons)

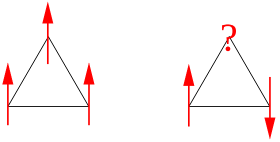


In reality, a continuum of multi-magnon excitations exists above the threshold.
Single magnon excitations are well defined
$S(q,\omega) = Z_q \delta(\omega - \omega_q) +$ incoherent part



We have to stay away from the classical limit

- Small value of the spin S, e.g., $S = 1/2$ or $S = 1$
- Frustration of the super-exchange interactions
  (not all terms of the energy can be optimized simultaneously)



- Low spatial dimensionality: $D = 2$ is the "best" choice
  In $D = 1$ there is no magnetic order, given the Mermin-Wagner theorem
  (not possible to break a continuous symmetry in D=1, even at $T = 0$)

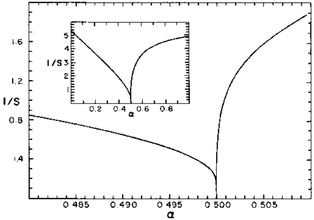
  Pitaevskii and Stringari, J. Low Temp. Phys. **85**, 377 (1991)

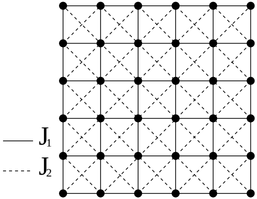
- [Large continuous rotation symmetry group, e.g., SU(2), SU(N) or Sp(2N)]

  Arovas and Auerbach, Phys. Rev. B **38**, 316 (1988); Arovas and Auerbach, Phys. Rev. Lett. **61**, 617 (1988)

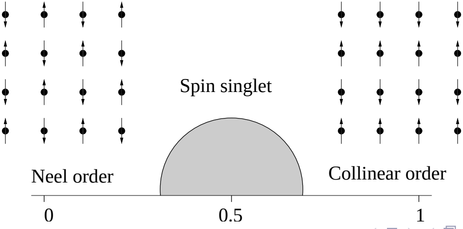
  Read and Sachdev, Phys. Rev. Lett. **66**, 1773 (1991); Read and Sachdev, Nucl. Phys. **B316**, 609 (1989)

$$\mathcal{H} = \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j + \alpha \sum_{\langle\langle i,j \rangle\rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$





Chandra and Doucot, Phys. Rev. B **38**, 9335 (1988)



In D=1 many exactly solvable models (e.g., Heisenberg and Haldane-Shastry)

Bethe, Z. Phys. **71**, 205 (1931).

Haldane, Phys. Rev. Lett. **60**, 635 (1988); Shastry, Phys. Rev. Lett. **60**, 639 (1988).

Simple example: the one-dimensional XY model:

$$\mathcal{H} = J \sum_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) = \frac{J}{2} \sum_i (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+)$$

- Representing spin operators via hard-core bosons

$$S_i^+ = b_i^\dagger \qquad S_i^- = b_i \qquad S_i^z = b_i^\dagger b_i - \frac{1}{2}$$

- Perform a Jordan-Wigner transformation

  Jordan and Wigner, Z. Phys. **47**, 631 (1928).

$$b_j = c_j e^{i\pi \sum_{n<j} c_n^\dagger c_n} \quad \Longleftarrow \text{String}$$

c_i are (spinless) fermionic operators

$$\mathcal{H} = \frac{J}{2} \sum_i (c_i^\dagger c_{i+1} + h.c.)$$

Free fermions with gapless excitations

Ground state and excitations

$$\mathcal{H} = \frac{J}{2} \sum_i (c_i^\dagger c_{i+1} + h.c.)$$

Boundary conditions depend upon the number N of fermions (or bosons):
N odd $\Longrightarrow$ periodic boundary conditions
N even $\Longrightarrow$ anti-periodic boundary conditions

- Ground state (always unique because of the boundary conditions)

$$|\Psi_0\rangle = \prod_{|k|>k_F} c_k^\dagger |0\rangle$$

- Single-particle excitation

$$|\Psi_k\rangle = c_k |\Psi_0\rangle \qquad |k| > k_F$$

does not live in the correct (bosonic) Hilbert space:
One must also change boundary conditions!
$\Longrightarrow S_k^+$ or S_k^- do not create elementary excitations

- Particle-hole excitations

$$|\Psi_{k,q}\rangle = c_{k+q}^\dagger c_k |\Psi_0\rangle \qquad |k| > k_F \;\; \text{and} \;\; |k+q| < k_F$$

They are terribly complicated in terms of bosons (because of the string)!

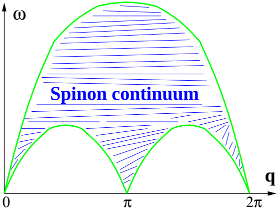
Absence of magnon excitations

- In $D = 1$ systems, elementary excitations are spinons carrying $S = 1/2$

Faddeev and Takhtajan, Phys. Lett. **85A**, 375 (1981)

$$S(q,\omega) = \int dt \langle \Psi_0 | S^z_{-q}(t) S^z_q(0) | \Psi_0 \rangle e^{i\omega t} = \sum_{n \neq 0} |\langle \Psi_n | S^z_q | \Psi_0 \rangle|^2 \delta(\omega - \Delta\omega_{n0})$$

$S(q,\omega)$ has only the incoherent part
No delta function
Singularity at the bottom of the spectrum



$S(q,\omega)$ can be computed exactly also in the Haldane-Shastry model:

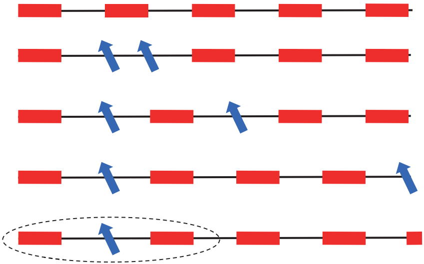
$$\mathcal{H} = J \sum_{m<n} [d(m-n)]^2 \mathbf{S}_m \cdot \mathbf{S}_n \qquad d(n) = \frac{N}{\pi} \sin(\frac{\pi n}{N})$$

Here, the $S = 1$ state $S^\alpha_n | \Psi_0 \rangle$ is completely expressible in terms of two spinons

Haldane and Zirnbauer, Phys. Rev. Lett. **71**, 4055 (1993)

Fractionalization

- Majumdar-Gosh chain (1D): $\mathcal{H} = J \sum_i \mathbf{S}_i \cdot \mathbf{S}_{i+1} + \frac{J}{2} \sum_i \mathbf{S}_i \cdot \mathbf{S}_{i+2}$
- The exact ground state is known (two-fold degenerate), perfect dimerization



The "initial" $S = 1$ excitation can decay into two spatially separated spin-1/2 excitations (spinons)

Finite-energy state with an isolated spinon (the other is far apart) domain wall between two dimerization patterns

- A spinon is a neutral spin-1/2 excitation, "one-half" of a $S = 1$ spin flip. (it has the same spin as the electron, but no charge)
- Spinons can only be created by pairs in finite systems
  In one dimension, they can propagate at large distances, as two elementary particles

A spin liquid is a state without long-range magnetic order

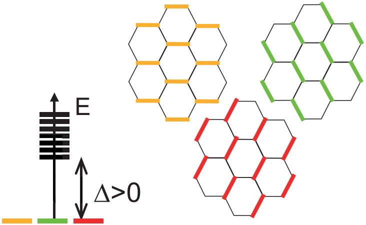
A spin liquid is a state without magnetic order
the structure factor $S(q)$ does not diverge, whatever the q is

$$S(q) = \frac{1}{N} \langle \Psi_0 | \left| \sum_j \mathbf{S}_j e^{iqr_j} \right|^2 | \Psi_0 \rangle = \frac{1}{N} \sum_{j,k} \langle \Psi_0 | \mathbf{S}_j \cdot \mathbf{S}_k | \Psi_0 \rangle e^{iq(r_j - r_k)}$$

$$S(q) = \begin{cases} O(1) & \text{for all q's} \quad \rightarrow \text{short-range correlations} \\ S(q_0) \propto N & \text{for} q = q_0 \quad \rightarrow \text{long-range order} \end{cases}$$

- Can be checked by using Neutron scattering
- Mermin-Wagner theorem implies that *any* 2D Heisenberg model at $T > 0$ is a spin liquid according to this definition

A spin liquid is a state without long-range magnetic order

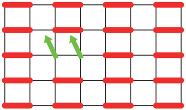


$J_1 - J_2$ Heisenberg model on the hexagonal lattice

Fouet, Sindzingre, and Lhuillier, Eur. Phys. J. B **20**, 241 (2001)

Properties:

- Short-range spin-spin correlations
- Spontaneous breakdown of some lattice symmetries $\rightarrow$ ground-state degeneracy
- Gapped $S = 1$ excitations ("magnons" or "triplons")



Spin liquid: a second definition

A spin liquid is a state without any spontaneously broken (local) symmetry

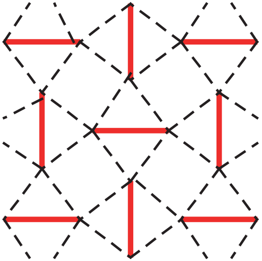
- It rules out magnetically ordered states that break spin SU(2) symmetry (also NEMATIC states)
- It rules out valence-bond crystals that break some lattice symmetries

Remark I: “local” means that there is a physical order parameter that can be measured by some local probe

Remark II: within this definition we also rule out chiral spin liquids that break time-reversal symmetries

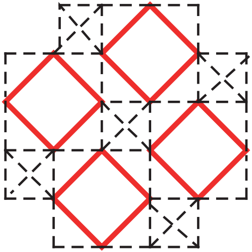
Wen, Wilczek, and Zee, Phys. Rev. B **39**, 11413 (1989)

There are few examples of magnetic insulators without any broken symmetry



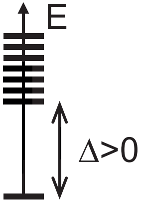
$SrCu_2(BO_3)_2$

Kageyama et al., Phys. Rev. Lett. **82**, 3168 (1999)



CaV_4O_9

Taniguchi et al., J. Phys. Soc. Jpn. **64**, 2758 (1995)

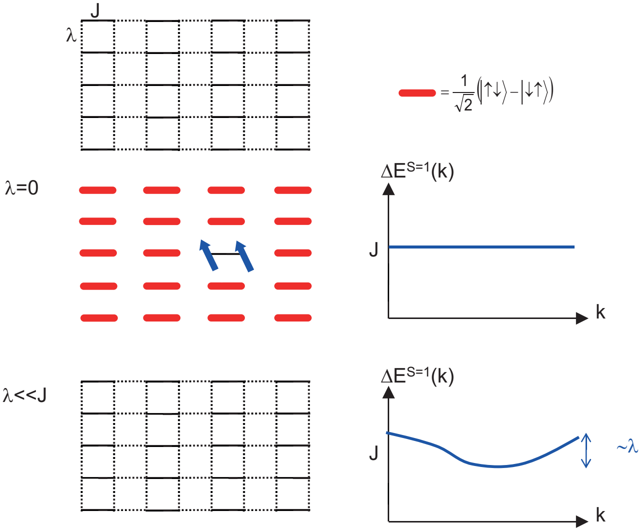


Non-degenerate ground state

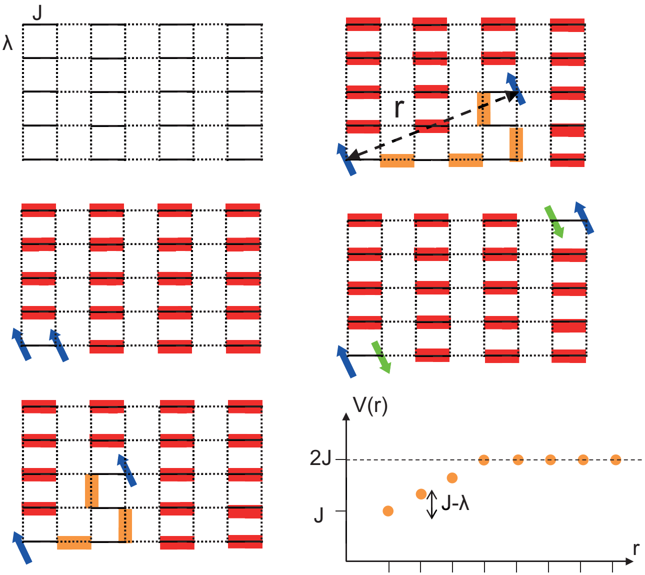
Properties:

- No broken symmetries
- Even number of spin-1/2 in the unit cell
- Adiabatically connected to the (trivial) limit of decoupled blocks
- No phase transition between $T = 0$ and $\infty \Longrightarrow$ "simple" quantum paramagnet

Quantum paramagnets:excitation spectrum



Quantum paramagnets and VBCs are not fractionalized



> A system with half-odd-integer spin in the unit cell cannot have a gap and a unique ground state

Valid in the thermodynamic limit for periodic boundary conditions and

$$L_1 \times L_2 \times \cdots L_D = odd$$

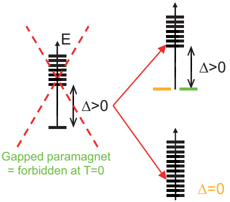
- The original theorem by Lieb, Schultz, and Mattis refers to 1D

Lieb, Schultz, Mattis, Ann. Phys. (N.Y.) **16**, 407 (1961); Affleck and Lieb, Lett. Math. Phys. **12**, 57 (1986)

- Since then, several attempts to generalize it in 2D

Affleck, Phys. Rev. B **37**, 5186 (1988); Bonesteel, Phys. Rev. B **40**, 8954 (1989);

Oshikawa, Phys. Rev. Lett. **84**, 1535 (2000); Hastings, Phys. Rev. B **69**, 104431 (2004)



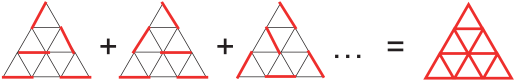
Case 1) Ground-state degeneracy
a) Valence-bond crystal
b) Resonating-valence bond spin liquid
(gapped but with a topological degeneracy)
Case 2) Gapless spectrum
a) Continuous broken symmetry (magnetic order)
b) Resonating-valence bond spin liquid
(gapless, i.e., critical state)

The short-range RVB picture

- Anderson's idea: the short-range resonating-valence bond (RVB) state:

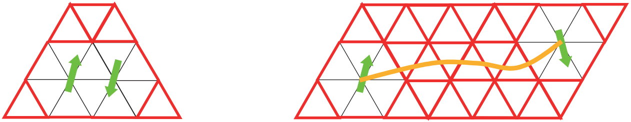
  Anderson, Mater. Res. Bull. **8**, 153 (1973)

  Linear superposition of many (an exponential number) of valence-bond configurations

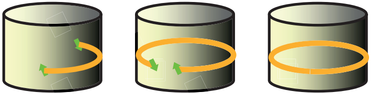


Spatially uniform state

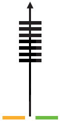
- Spin excitations? No dimer order → we may have deconfined spinons



- Spinon fractionalization and topological degeneracy



Distinct ground states that are not connected by any local operator



Wen, Phys. Rev. B **44**, 2664 (1991); Oshikawa and Senthil, Phys. Rev. Lett. **96**, 060601 (2006)

Spin liquid: a third definition

A spin liquid is a state without any spontaneously broken (local) symmetry, with a half-odd-integer spin in the unit cell

- It rules out magnetically ordered states that break spin SU(2) symmetry (also NEMATIC states)
- It rules out valence-bond crystals that break some lattice symmetries
- It rules out quantum paramagnets that have an even number of spin-half per unit cell

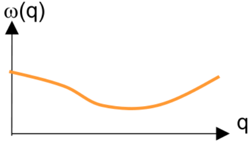
A spin liquid sustains fractional (spin-1/2) excitations

Inelastic Neutron scattering: spinon continuum

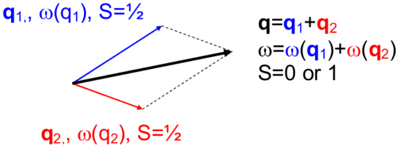
The inelastic Neutron scattering is a probe for the dynamical structure factor

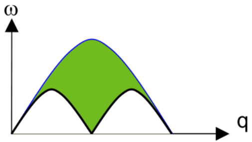
$$S(q,\omega) = \int dt \langle \Psi_0 | S^\alpha_{-q}(t) S^\alpha_q(0) | \Psi_0 \rangle e^{i\omega t}$$

- The elementary excitations are spin-1 magnons: $S(q,\omega)$ has a single-particle pole at $\omega = \omega(q)$



- The spin-flip decays into two spin-1/2 excitations $S(q,\omega)$ exhibits a two-particle continuum

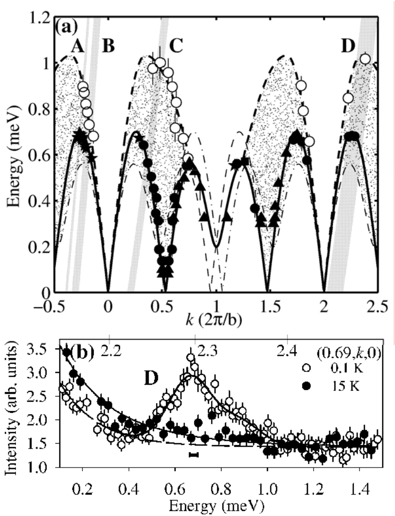




Inelastic Neutron scattering: spinon continuum

Neutron scattering on Cs_2CuCl_4

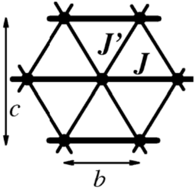
Coldea, Tennant, Tsvelik, and Tylczynski , Phys. Rev. Lett. **86**, 1335 (2001)



Almost decoupled layers

Strongly-anisotropic triangular lattice

$J' \simeq 0.33J$: **quasi-1D**

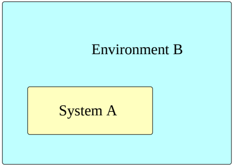


Entanglement entropy

- Given the ground-state wave function $|\Psi\rangle$, the density matrix of the whole lattice is

$$\rho = |\Psi\rangle\langle\Psi|$$

- Suppose to split the lattice in two regions (system A and environment B)



- Define the reduced density matrix of the system A:

$$\rho_A = \mathrm{Tr}_\mathrm{B}|\Psi\rangle\langle\Psi|$$

- The von Neumann entropy of the system A is

$$S_A = -\mathrm{Tr}_\mathrm{A}(\rho_A \log \rho_A)$$

Hard to compute (easy by density-matrix renormalization group)

Rényi entropy $\Longrightarrow$ $S_A = \frac{1}{1-n} \log \mathrm{Tr}_\mathrm{A}(\rho_A^n)$

- S_A quantifies the entanglement between A and B

  For example: given two spins

$$|\uparrow\rangle|\uparrow\rangle \quad \Longrightarrow \quad S_A = 0$$

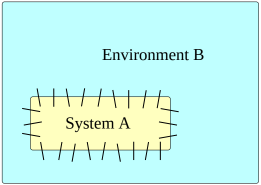
$$\frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle) \quad \Longrightarrow \quad S_A = \log 2$$

- "Standard" ground states have the area law

$$S_A = \alpha L^{D-1} + \cdots$$

The area law is due to the local entanglement across the boundary of A
The coefficient α is non-universal

In gapless 1D systems: $S_A = \frac{c}{3} \log L + \cdots$
where c is the central charge



- Free fermions have a deviation from the area law (due to the Fermi surface)
  $\Longrightarrow S_A = \alpha L^{D-1} \times \log L$

Wolf, Phys. Rev. Lett. **96**, 010404 (2006); Gioev and Klich, Phys. Rev. Lett. **96**, 100503 (2006)

A fourth definition for (gapped) spin liquids

- In two dimensions, topologically ordered states have an extra term:

$$S_A = \alpha L - \gamma + \cdots$$

- γ is the topological entanglement entropy (related to fractionalized excitations)

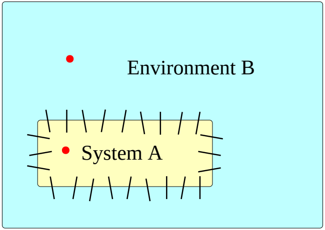
γ assumes universal values in gapped states:

$\gamma = \log \sqrt{\sum_a d_a^2}$

d_a are "quantum dimensions" of particles

For example $\gamma = log 2$ for the toric code

Kitaev, Ann. Phys. **303**, 2 (2003)



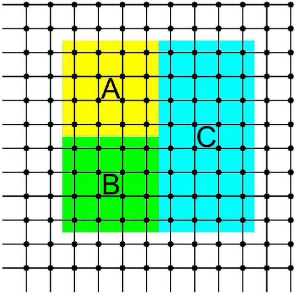
Kitaev and Preskill, Phys. Rev. Lett. **96**, 110404 (2006)

Levin and Wen, Phys. Rev. Lett. **96**, 110405 (2006)

A gapped spin liquid is a highly entangled state with a finite and universal topological entanglement entropy

A fourth definition for (gapped) spin liquids

A linear combination of different entropies may be considered



$$-\gamma = S_A + S_B + S_C - S_{AB} - S_{BC} - S_{AC} + S_{ABC}$$

- γ is a topological invariant
- γ is a universal quantity (unchanged by smooth deformations of the Hamiltonian, i.e., unless a quantum critical point is encountered)

Kitaev and Preskill, Phys. Rev. Lett. **96**, 110404 (2006)