Persistent Currents in a Two-component Bose Gas in the Ring Geometry

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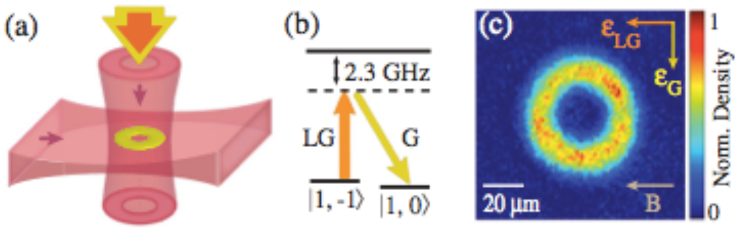
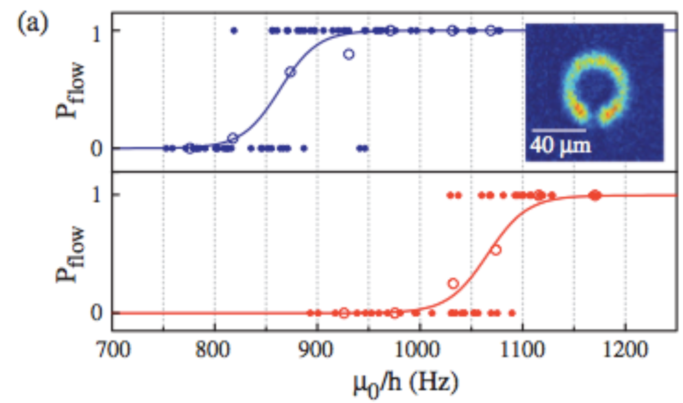
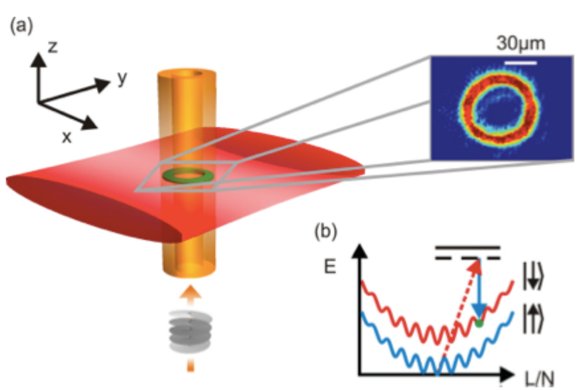
Work done in collaboration with Konstantin Anoshkin and Zhigang Wu; also Smyrnakis, Magiropoulos, Efremidis and Kavoulakis

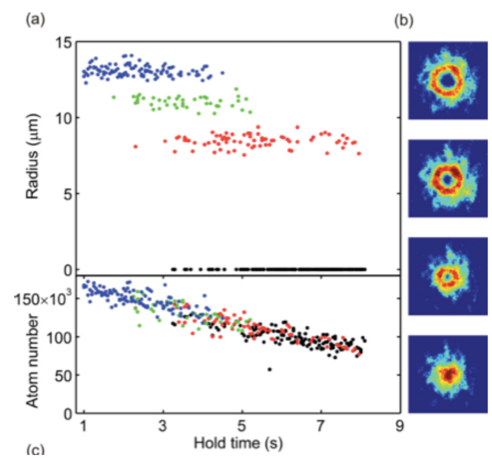
Experiments on Persistent Currents





A. Ramanathan *et al*., Phys. Rev. Lett. **106**, 130401 (2011)





S. Moulder *et al*., Phys. Rev. A **86**, 013629 (2012)

*Bloch's Criterion for Persistent Currents**

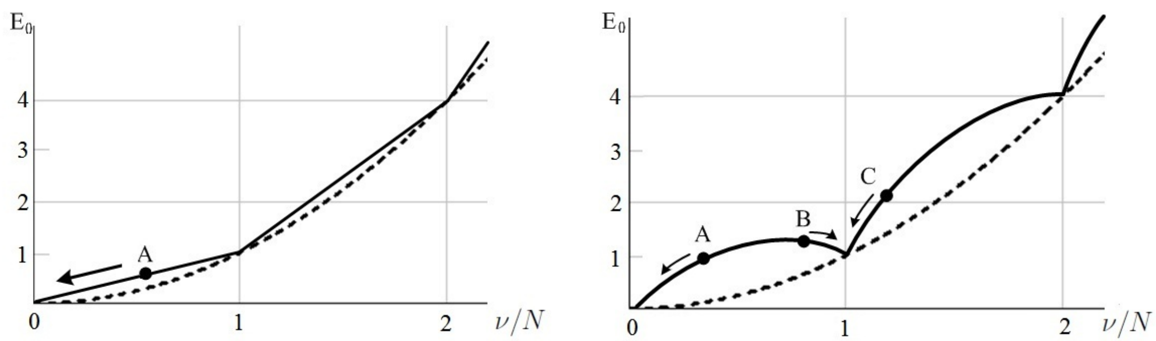
- for a single-species system in the one-dimensional ring geometry, Bloch showed that the ground state energy takes the form

$$E_0(L) = \frac{L^2}{2M_T R^2} + e_0(L) \qquad L = \nu\hbar$$

Yrast Spectrum

where $e_0(L)$ is even and periodic:

$$e_0(-L) = e_0(L), \quad e_0(L + N\hbar) = e_0(L)$$



(a) (b)

- Bloch argued that, if $E_0(L)$ exhibits local minima at $L_n = nN\hbar$, persistent currents are *stable*

*F. Bloch, Phys. Rev. A**7**, 2187 (1973)

Yrast spectrum of the Lieb-Liniger model and connection to the soliton solutions of the GP equation

- the Hamiltonian for 1D bosons interacting via a delta function potential is given by

$$\hat{H} = -\frac{\hbar^2}{2MR^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{2}U\sum_{ij}\delta(\theta_i - \theta_j)$$

- the many-body wavefunctions can be obtained using the Bethe ansatz (*Lieb and Liniger, 1963*)

- particle momentum shows no BEC in the thermodynamic limit

- the yrast spectrum corresponds to Lieb's type II excitations (*Lieb*, *1963*); it can be determined explicitly using the Lieb-Liniger solution (*Kaminishi et al., 2011*)

- the excitations corresponding to the yrast spectrum can be identified as *solitons* (*Ishikawa and Takayama, 1980*)

Mean-field Analysis for the Single-component Case

- the Gross-Pitaevskii energy functional for bosons on a ring is

$$\bar{E}[\psi] = \int_0^{2\pi} d\theta \left| \frac{d\psi}{d\theta} \right|^2 + \pi\gamma \int_0^{2\pi} d\theta |\psi(\theta)|^4$$

- the yrast spectrum is obtained by minimizing the GP energy with respect to ψ subject to the constraint that the average angular momentum has the value

$$\bar{L} = \frac{1}{i} \int_0^{2\pi} d\theta\, \psi^* \frac{d\psi}{d\theta} = l$$

- this can be achieved by minimizing the functional

$$\bar{F}[\psi] = \bar{E}[\psi] - \Omega\bar{L} - \mu \int_0^{2\pi} d\theta |\psi(\theta)|^2$$

where Ω and μ are Lagrange multipliers

- this leads to the GP equation

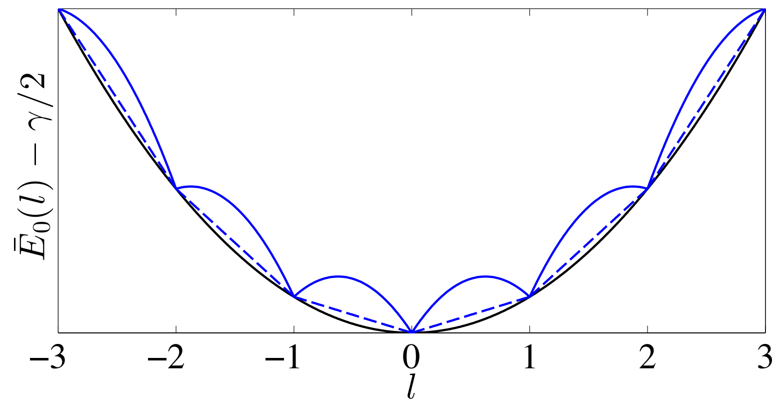
$$-\psi''(\theta) + i\Omega\psi'(\theta) + 2\pi\gamma\rho(\theta)\psi(\theta) = \mu\psi(\theta)$$

Mean-field Solutions and Yrast Spectrum

- the mean-field solutions for a general value of l are *solitons*
- this stationary state solution represents a travelling soliton as viewed in a rotating frame; in the lab frame, the soliton is the time-dependent state

$$\psi(\theta, t) = \psi(\theta - \Omega t) e^{-i\mu t}$$

- the energy of the mean-field soliton agrees with the exact many-body energy if the interactions are not too strong (*Kanamoto et al., 2010*)



Extension of Bloch's Argument to the Two-species System

- we consider an ideal 1D ring geometry with N_A particles of mass M_A and N_B particles of mass M_B; $N = N_A+N_B$, $M_T = N_AM_A+N_BM_B$

- the many-body wave function can be written as

$$\Psi_{L\alpha}(\theta_1, ..., \theta_N) = \exp(iNl\Theta_{\rm cm})\chi_{L\alpha}(\theta_1, ..., \theta_N)$$

where

$$\Theta_{\rm cm} = \frac{1}{M_T}\sum_{i=i}^{N} M_i\theta_i, \quad L = lN\hbar$$

- $\chi_{L\alpha}(\theta_1,\ldots,\theta_N)$ is a function of coordinate differences $\theta_i - \theta_j$ and is therefore a zero angular momentum wave function

Extension of Bloch's Argument, cont'd

- $\chi_{L\alpha}(\theta_1,\ldots,\theta_N)$ satisfies the Schrödinger equation

$$H\chi_{L\alpha} = e_\alpha(L)\chi_{L\alpha} \qquad e_\alpha(L) = E_\alpha(L) - \frac{L^2}{2M_T R^2}$$

  with the boundary conditions

$$\chi_{L\alpha}(\cdots,\theta_i + 2\pi,\cdots) = \exp\left(-i2\pi\frac{\nu M_i}{M_T}\right)\chi_{L\alpha}(\cdots,\theta_i,\cdots)$$

- if $M_A/M_B = p/q$, a rational number, $e_\alpha(L)$ is a periodic function with period $\tilde{N}\hbar$ where

$$\tilde{N} = pN_A + qN_B$$

- for $M_A = M_B = M$, $p = q = 1$ and the periodicity is N as for the single-species case

Connection with Landau's Criterion

- $M_A = M_B = M$; Bloch's argument allows for persistent currents at $L_n = nN\hbar$; for $L = L_n + \Delta L$

$$E_0(L_n + \Delta L) = \frac{1}{2} M_T R^2 \Omega_n^2 + \Omega_n \Delta L + E_0(\Delta L)$$

where we have defined the angular velocity

$$\Omega_n = \frac{L_n}{M_T R^2}$$

- assuming $E_0(\Delta L)$ to correspond to a single quasiparticle excitation with energy $\varepsilon(m)$ and angular momentum $\Delta L = m\hbar$, we have

$$E_0(L_n + \Delta L) = E_0(L_n) + \varepsilon(m) + m\hbar\Omega_n$$

- Bloch's criterion for persistent currents, $E_0(L_n + \Delta L) > E_0(L_n)$, then implies

$$\Omega_n < \left(\frac{\varepsilon(m)}{\hbar|m|}\right)_{\min}$$

- this is the Landau criterion for a ring

Bogoliubov Excitations in a Ring

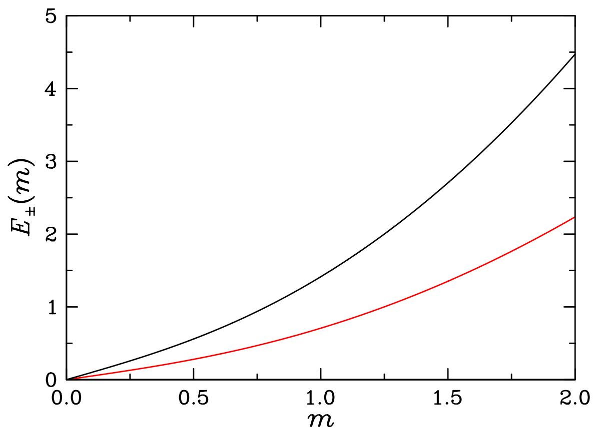
- the two-species system has Bogoliubov excitations with energies

$$E_{\pm}^{2} = \frac{1}{2}\left(E_A^2 + E_B^2\right) \pm \frac{1}{2}\sqrt{\left(E_A^2 + E_B^2\right)^2 + 4\left(4\epsilon_A\epsilon_B g_{AB}^2 - E_A^2 E_B^2\right)}$$

where

$$E_s = \sqrt{\epsilon_s^2 + 2\epsilon_s g_{ss}}, \quad \epsilon_s = \frac{\hbar^2 m^2}{2M_s R^2}$$

$$g_{ss} = \frac{U_{ss'}\sqrt{N_s N_{s'}}}{2\pi}$$



- the Landau criterion is satisfied for most choices of the parameters, implying the stability of superfluid flow at L_n

- however, if $M_A = M_B$ and $U_{AA}U_{BB} = U^2{}_{AB}$, the $E\_$ mode is *particle-like* and supercurrents are not stable at L_n

Mean-field Analysis

- the stability of persistent currents can be analyzed using mean-field theory; this was first done by Smyrnakis *et al*.*
- for the special case $M_A = M_B$ and $U_{AA} = U_{BB} = U_{AB} = U$, the so-called symmetric model, the Gross-Pitaevskii energy functional is

$$\bar{E}[\psi_A, \psi_B] = \int_0^{2\pi} d\theta \left(x_A \left| \frac{d\psi_A}{d\theta} \right|^2 + x_B \left| \frac{d\psi_B}{d\theta} \right|^2 \right) + \pi\gamma \int_0^{2\pi} d\theta \left(x_A |\psi_A|^2 + x_B |\psi_B|^2 \right)^2$$

with

$$x_A = N_A/N, \quad x_B = N_B/N, \quad \gamma = NMR^2U/\pi\hbar^2$$

- the objective is to minimize the GP energy with respect to ψ_A and ψ_B subject to the constraint that the average total angular momentum has the value $L = lN\hbar$
- this can be achieved by minimizing the functional

$$\bar{F}[\psi_A, \psi_B] = \bar{E}[\psi_A, \psi_B] - \Omega\bar{L} - \sum_s x_s \mu_s \int_0^{2\pi} d\theta |\psi_s(\theta)|^2$$

where Ω and μ_s are Lagrange multipliers

*J. Smyrnakis *et al*., Phys. Rev. Lett **103**, 100404 (2009)

Minimizing the GP Energy

- the condensate wave functions are expanded as

$$\psi_A(\theta) = \sum_m c_m \phi_m(\theta), \quad \psi_B(\theta) = \sum_m d_m \phi_m(\theta)$$

where

$$\phi_m(\theta) = \frac{e^{im\theta}}{\sqrt{2\pi}}$$

- the expansion coefficients must satisfy the normalization constraints

$$\sum_m |c_m|^2 = 1, \qquad \sum_m |d_m|^2 = 1$$

and the angular momentum constraint

$$l = x_A \sum_m m|c_m|^2 + x_B \sum_m m|d_m|^2$$

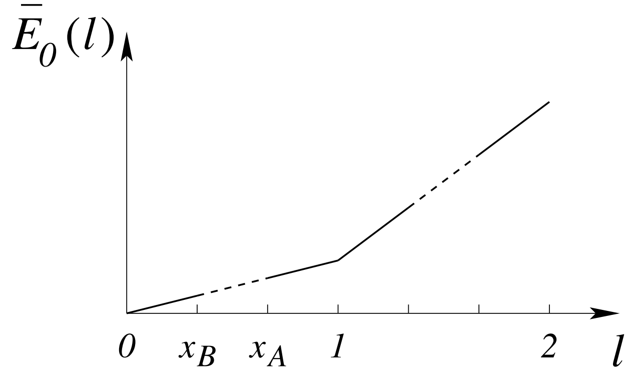
Two-component Analysis

- the simplest variational ansatz is

$$\psi_A = c_0\phi_0 + c_1\phi_1, \quad \psi_B = d_0\phi_0 + d_1\phi_1$$

- minimizing the energy with respect to c_0, c_1, d_0 and d_1, one finds

$$\bar{E}_0(l) = l + \gamma/2$$

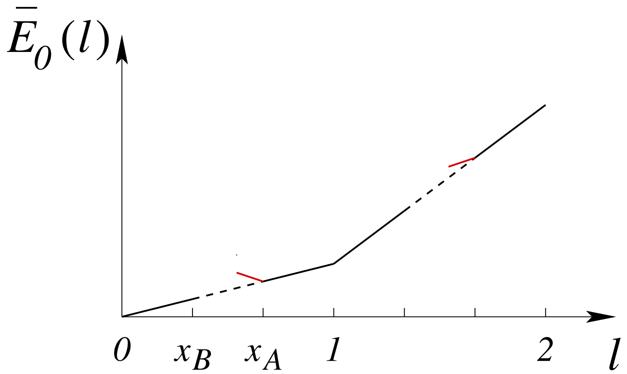


- this result is *exact* if

$$0 \leq l \leq x_B \quad \text{or} \quad x_A \leq l \leq 1$$

- as predicted by the Landau criterion, superfluid flow is unstable at L_n
- however, there is a possibility that persistent currents might be stable for l in the range $x_B < l < x_A$
- to examine this possibility, an improved variational ansatz is required

Persistent Currents at $l = x_A + n$



$$\bar{E}_0(l) = l^2 + \bar{e}_0(l)$$

- the stability of persistent currents at $l = x_A + n$ is determined by the slope

$$\left.\frac{d\bar{E}_0(l)}{dl}\right|_{l=(x_A+n-1)^-} = 2n - 1 - \lambda$$

- for $n = 1$, the critical value of the interaction parameter is

$$\gamma_{cr} = \frac{3}{2(4x_A - 3)}$$

- this gives the correct value of $\gamma_{\mathrm{cr}} = 3/2$ for $x_A = 1$; however, the above expression predicts that persistent currents are not possible for $n > 1$ (Smyrnakis *et al*., 2009)

*Analytic Soliton Solutions**

- the coupled GP equation for equal interactions strengths

$$-\psi_s''(\theta) + i\Omega\psi_s'(\theta) + 2\pi\gamma\rho(\theta)\psi_s(\theta) = \mu_s\psi_s(\theta)$$

- here, the angular velocity, Ω, is a Lagrange multiplier introduced to ensure the angular momentum takes a specific value l
- modulus-phase representation

$$\psi_s(\theta) = \sqrt{\rho_s(\theta)}e^{i\phi_s(\theta)}, \quad \int_0^{2\pi} d\theta\, \rho_s(\theta) = 1$$

- boundary conditions

$$\rho_s(\theta + 2\pi) - \rho_s(\theta) = 0$$

$$\phi_s(\theta + 2\pi) - \phi_s(\theta) = 2\pi J_s, \quad J_s = 0, \pm 1, \pm 2, \cdots$$

- J_s is the soliton winding number

*Z. Wu and E. Zaremba, Phys. Rev. A **88**, 063640 (2013)

Density Ansatz

- the ansatz (*Porubov and Parker, 1994; Smyrnakis et al., 2012*)

$$\rho_B = \frac{1-r}{2\pi} + r\rho_A$$

  reduces the coupled system to two uncoupled equations for the densities

$$\frac{1}{2}\rho_s\rho_s'' - \frac{1}{4}(\rho_s')^2 - 2\pi\gamma_s\rho_s^3 + \tilde{\mu}_s\rho_s^2 - \frac{W_s^2}{4} = 0$$

  with different interaction strengths

$$\gamma_A = (x_A + r x_B)\gamma$$

$$\gamma_B = (r^{-1}x_A + x_B)\gamma$$

- the density equation for the single-component system was solved by others; it has analytic solutions in terms of Jacobi elliptic functions

$$\rho_s(\theta) = \mathcal{N}(\eta_s)\left[1 + \eta_s dn^2\,(u|m)\right] \qquad u = \frac{jK(m)}{\pi}(\theta - \theta_0)$$

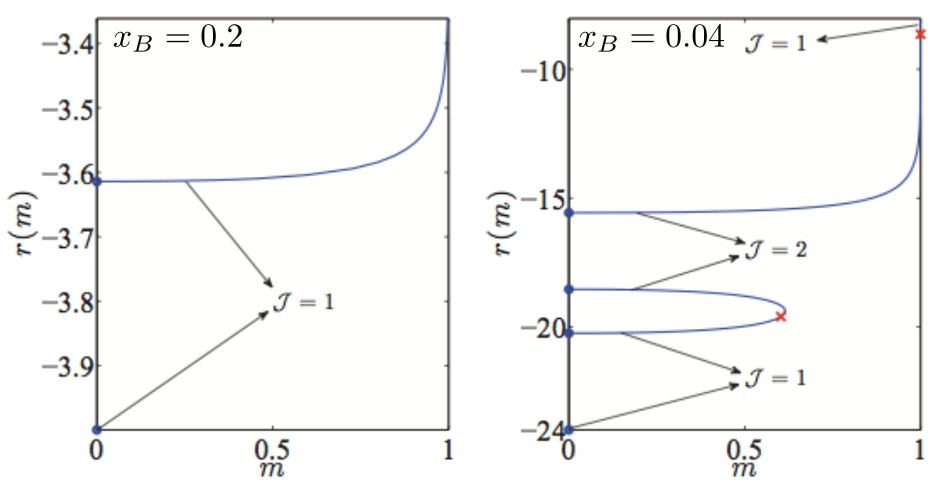
  where j is the soliton train index and η_s is a number which depends on γ_s (hence x_A, γ and r); m is the elliptic parameter defining the complete elliptic integral $K(m)$

Phase Boundary Condition

- with the density solutions in hand, the phase boundary condition gives

$$\mathcal{J} \equiv J_B - J_A = \frac{1}{4\pi}\int_0^{2\pi} d\theta \left[\frac{W_B}{\rho_B(\theta)} - \frac{W_A}{\rho_A(\theta)}\right]$$

- this equation provides a relation between the elliptic index m defining the density distributions and the parameter r appearing in the density ansatz

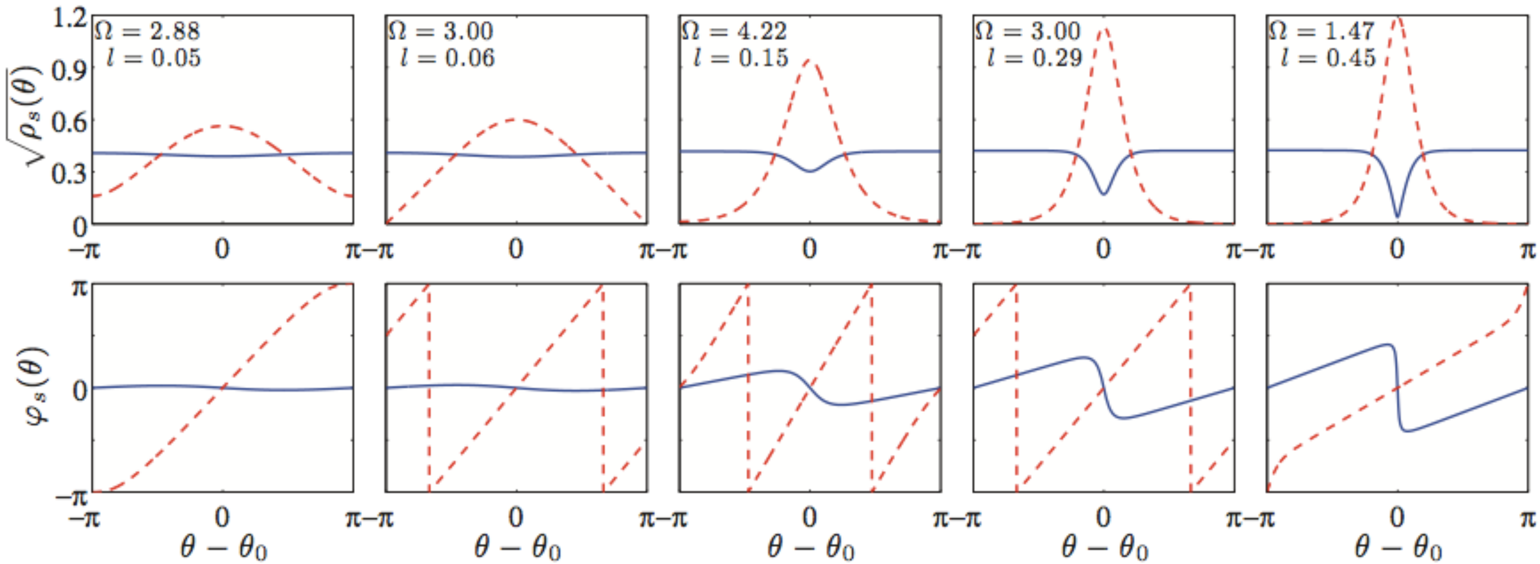


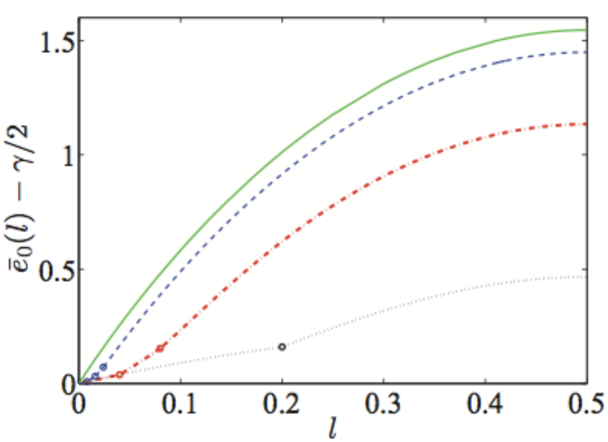
$\gamma = 23$

- the different branches correspond to different soliton states in different ranges of angular momentum given by $kx_B \leq l \leq (k+1)x_B$
- the winding numbers J_A and J_B take specific values along each of the branches

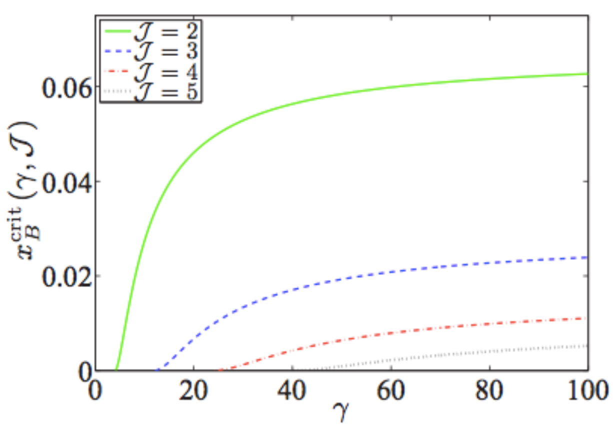
Soliton States and Yrast Spectrum

$x_B = 0.04 \quad \gamma = 23$

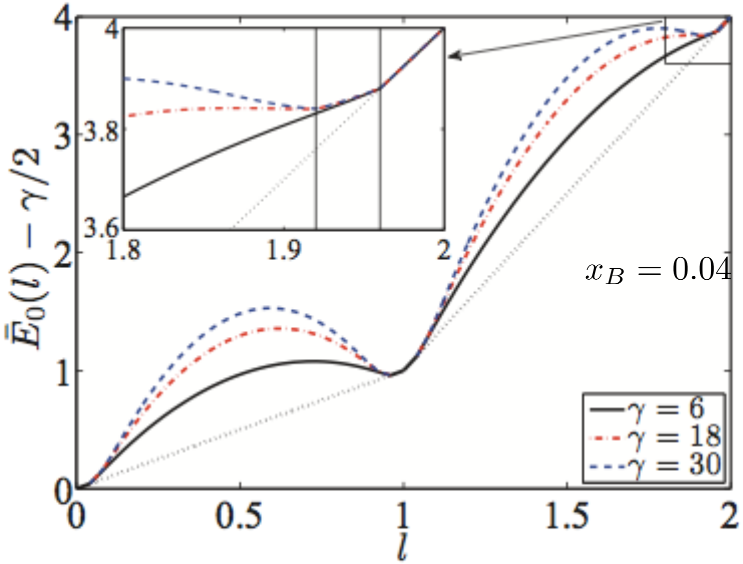




$x_B = 0.2,\ 0.04,\ 0.008,\ 0$



Persistent Currents at Higher Angular Momenta



- the soliton solutions explain how the $x_B \to 0$ limit is reached
- persistent currents are in fact possible at higher angular momenta for sufficiently small x_B and sufficiently large γ

Asymmetric Interactions: $\gamma_{AA}, \gamma_{BB}, \gamma_{AB}$

- there are no known analytic solutions to the coupled GP equations for asymmetric interactions – the density ansatz used for the symmetric model does not work
- the analysis of the symmetric model showed that certain plane wave states are special in that they can be yrast states and can sustain persistent currents
- we expect certain plane wave states to continue playing an important role in the yrast spectrum of the asymmetric model

Can one obtain a criterion for determining whether a plane wave state is an yrast state and secondly, whether this state supports persistent currents?

*Local Minima of the GP Energy Functional**

- we suppose (ϕ_μ, ϕ_ν) is a candidate plane-wave yrast state
- for an arbitrary deviation $\psi_A = \phi_\mu + \delta\psi_A,\ \psi_B = \phi_\nu + \delta\psi_B$, the change in GP energy is

$$\begin{aligned} \delta\bar{E} &= \bar{E}[\psi_A, \psi_B] - \bar{E}[\phi_\mu, \phi_\nu] \\ &\simeq \sum_{m>0} \mathbf{v}_m^\dagger \mathcal{H}_m \mathbf{v}_m, \quad \mathbf{v}_m = (\delta c_{\mu-m}\, \delta c^*_{\mu+m}\, \delta d_{\nu-m}\, \delta d^*_{\nu+m})^T \end{aligned}$$

- the energy has a local minimum if $\mathcal{H}_m$ is positive-definite; this yields the following conditions:

$$2x_A\gamma_{AA} + 1 - 4\mu^2 > 0$$

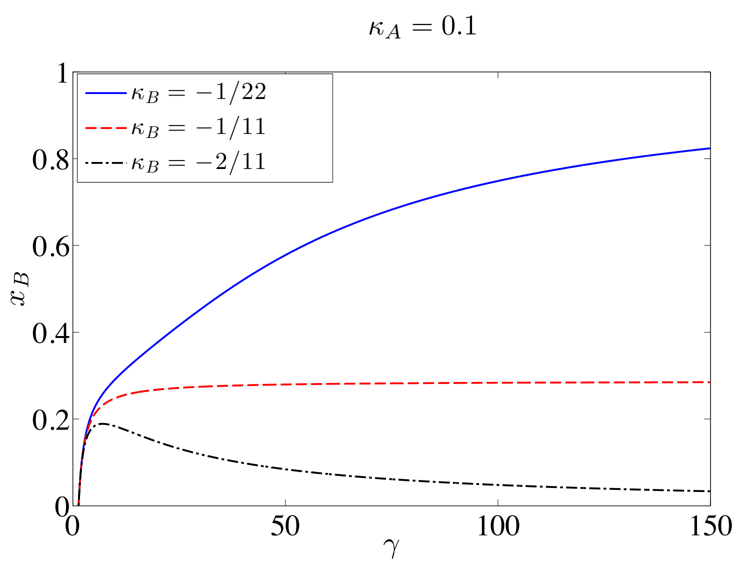
$$(2x_A\gamma_{AA} + 1 - 4\mu^2)(2x_B\gamma_{BB} + 1 - 4\nu^2) - 4x_Ax_B\gamma_{AB}^2 > 0$$

- if these conditions are not satisfied, persistent currents are *not* possible

*Z. Wu *et al*., Phys. Rev. A **92**, 033630 (2015)

Effect of Interaction Asymmetry on the Stability of Persistent Currents

- the inequalities allow one to determine the range of parameters for which persistent currents at a particular plane-wave state are possible
- example of (ϕ_1, ϕ_0)



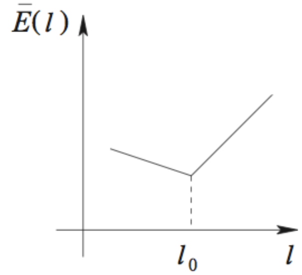
$$\gamma = \gamma_{AB}$$

$$\gamma_{AA} = (1+\kappa_A)\gamma,\ \gamma_{BB} = (1+\kappa_B)\gamma$$

- for κ = *-1/11*, $\gamma_{AA}\gamma_{BB} - \gamma_{AB}^2 = 0$; the x_B-γ boundary is similar to that found in the symmetric model

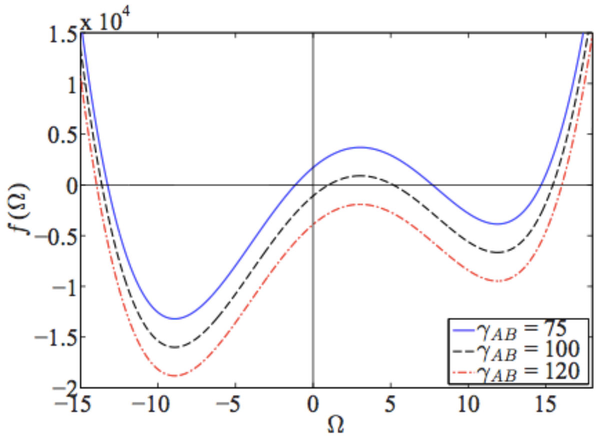
On the Cusp

- for a plane-wave yrast state supporting persistent currents at $l_0 = \mu x_A + \nu x_B$, the yrast spectrum looks locally like:



- the slopes of the yrast spectrum can be obtained by minimizing the GP energy functional subject to the constraint $\bar{L} = l_0 + \delta l$

$$\Omega(l) = \frac{\partial \bar{E}_0(l)}{\partial l}$$



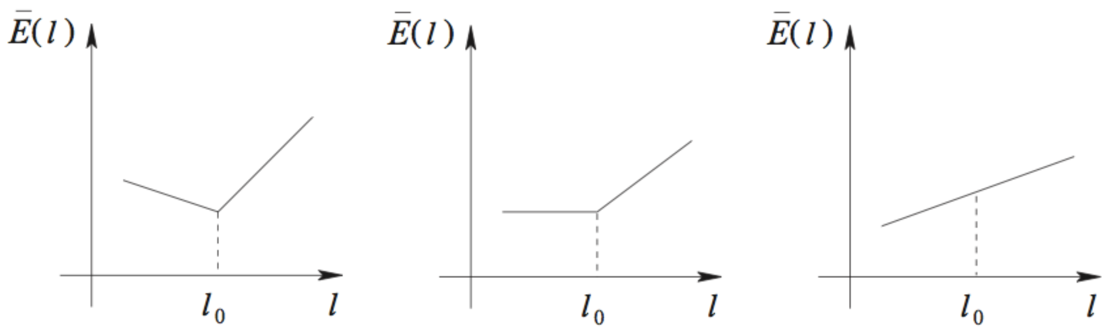
- one obtains a quartic with roots

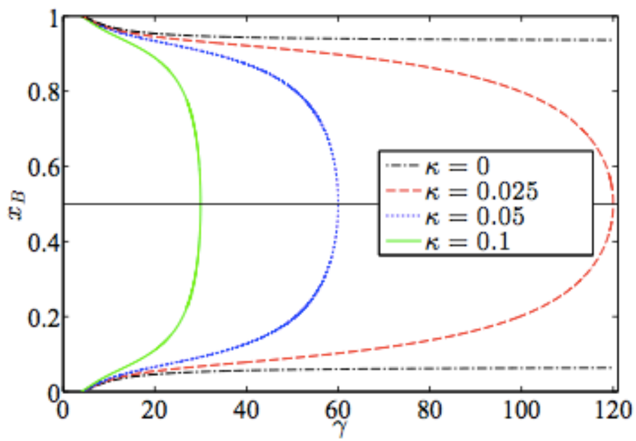
$$\Omega_1 < \Omega_2 < \Omega_3 < \Omega_4$$

- the middle two give the slopes of the yrast spectrum at the plane-wave state
- if one of these roots goes to zero, persistent currents are no longer possible; this happens precisely when the inequality is violated

Possible Criterion for Plane-wave Yrast States

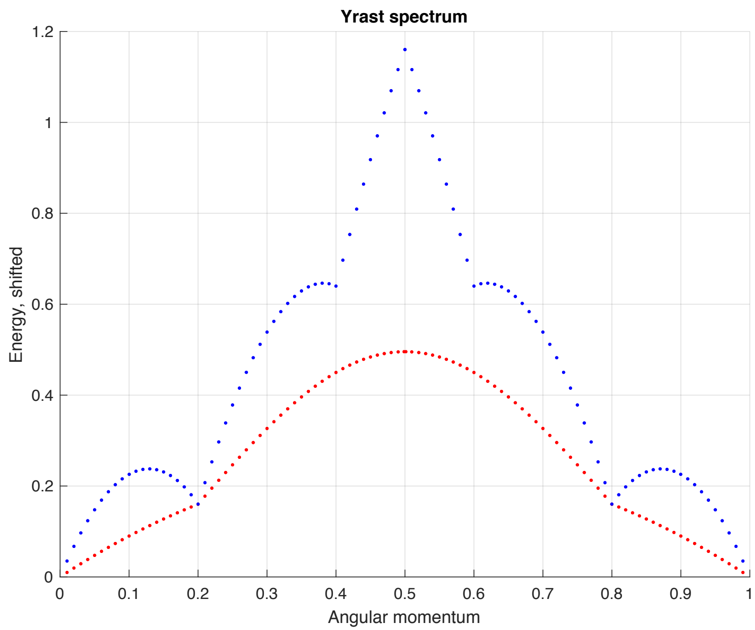
- a double root of the quartic signals when the plane-wave state ceases to be an yrast state





- whether this scenario is correct must be checked by numerical calculations

Numerical Solutions; Preliminary Results



- one can obtain solutions to the coupled GP equations by imaginary time propagation
- the results support the criterion for the stability of persistent currents at plane-wave states
- but there is evidence that the criterion for a plane-wave state being an yrast state is not generally valid
- there is no difficulty solving the GP equations for arbitrary masses; results for mass ratios equal to rational numbers are consistent with our general predictions