Aalto University
School of Science

Superfluidity and interactions in topologically and geometrically nontrivial bands

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ACADEMY OF FINLAND

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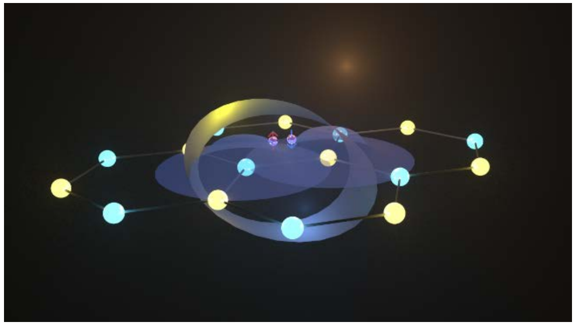

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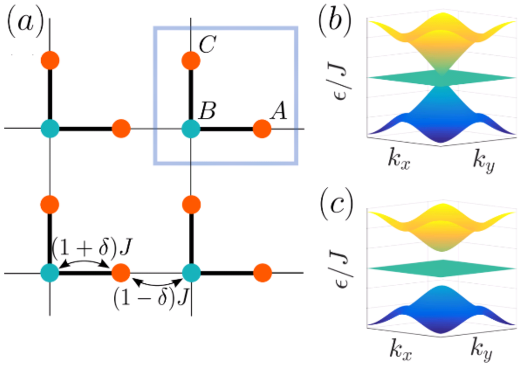

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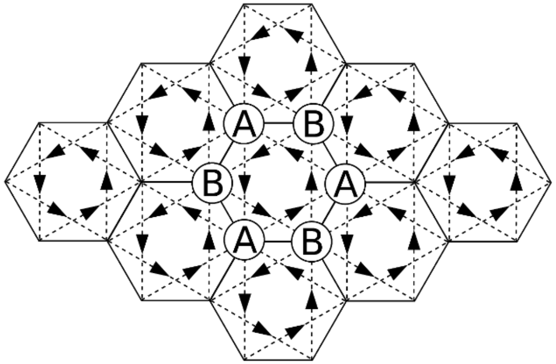

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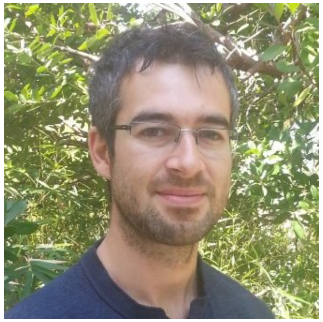
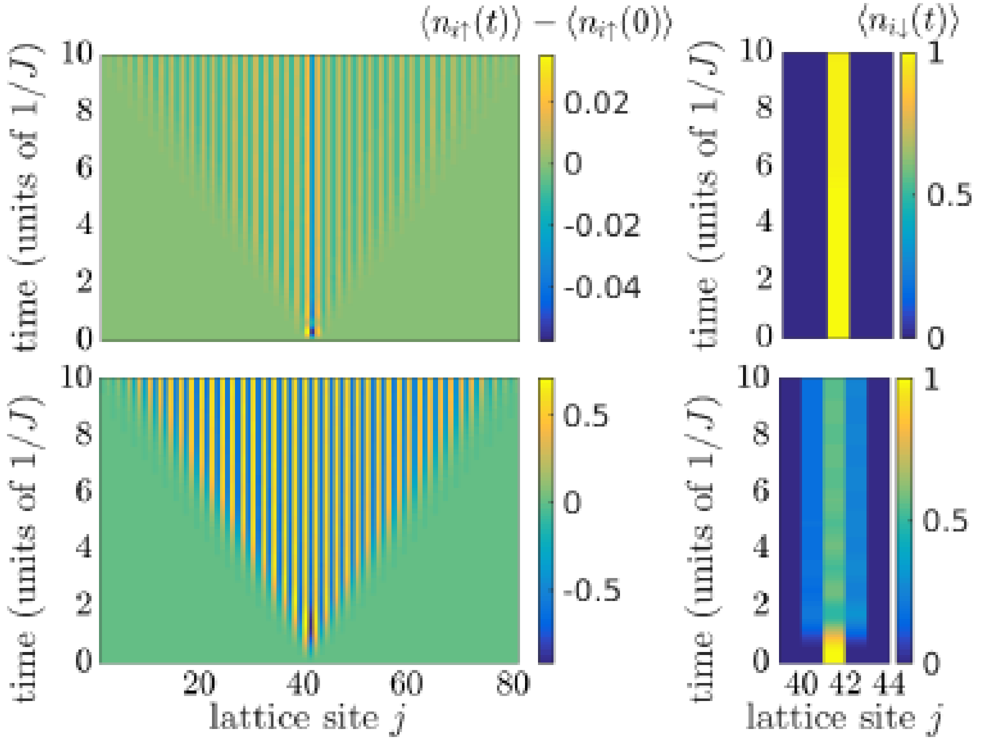


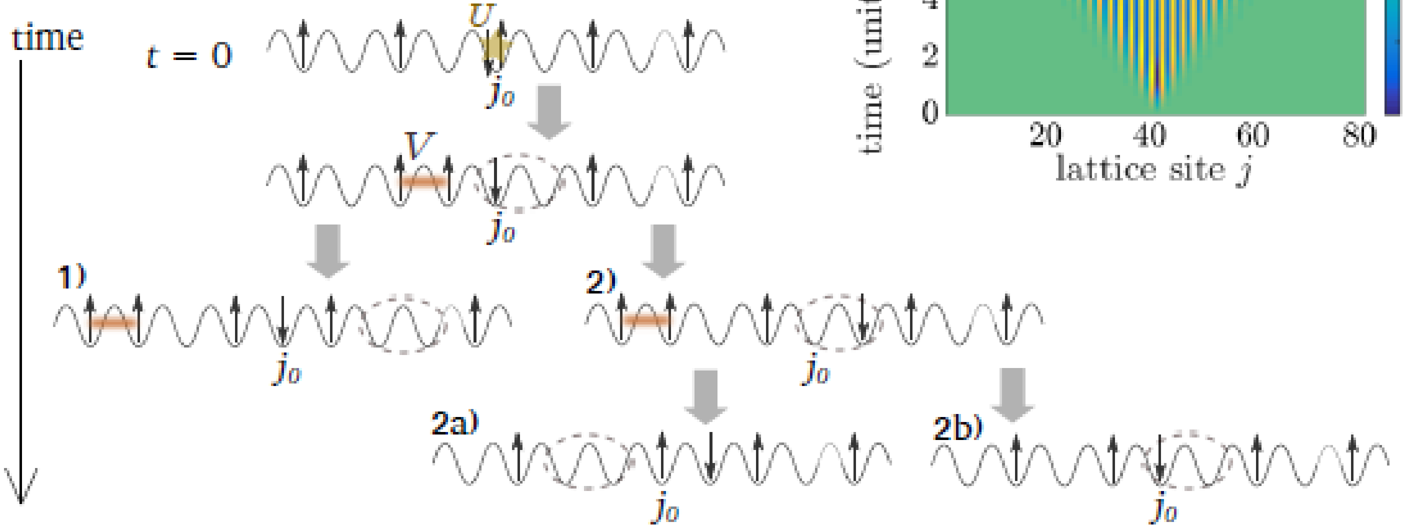
Sebastiano Peotta

Tuomas Vanhala

Aleksi Julku

POSTER: Excitations and impurity dynamics in a fermionic Mott insulator with nearest-neighbour interactions





A.-M. Visuri, T. Giamarchi, P. Törmä, Phys. Rev. B 93, 125110 (2016)

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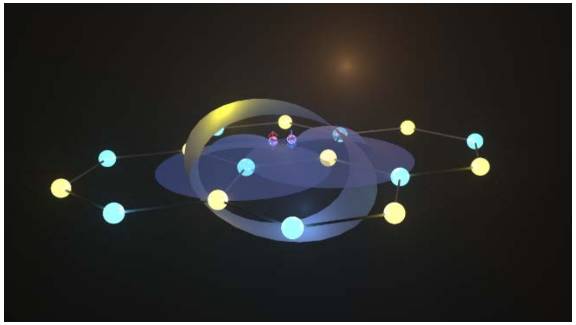

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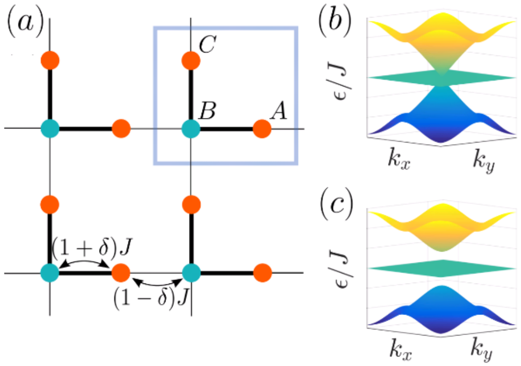

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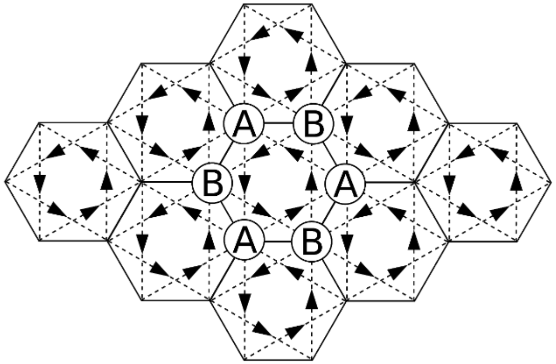

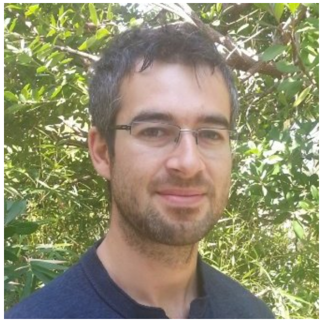

Sebastiano Peotta

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Flat bands:
a route to room-temperature superconductivity?

Parabolic band

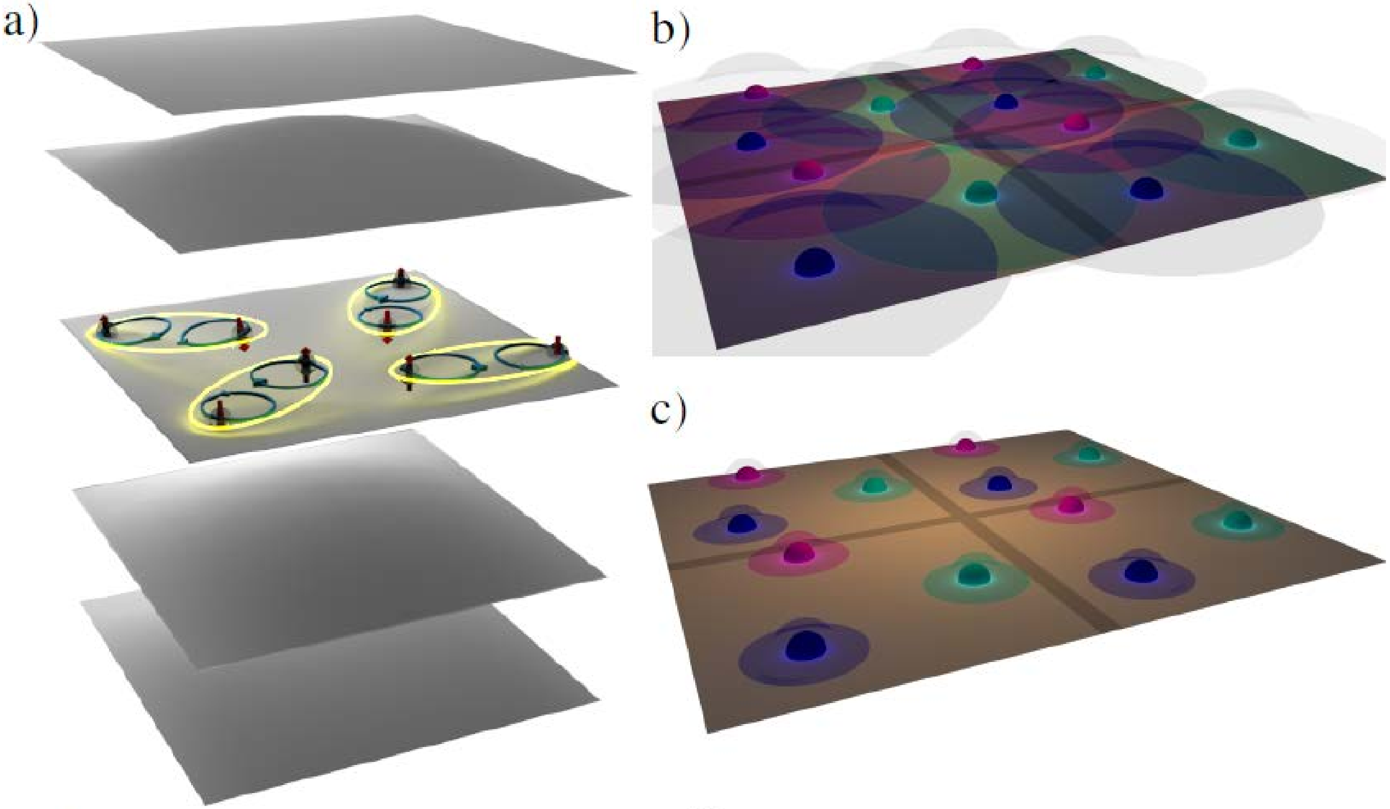
$$U \ll J \Rightarrow T_c \propto e^{-\frac{1}{U n_0(E_F)}}$$

Flat band

$$U \gg J \Rightarrow T_c \propto U n_0(E_F) \propto U/J$$

Our question:
if the band is flat, is there any supercurrent?

Our answer:
yes; superfluid transport is possible in a flat band. It is guaranteed if the band is topologically nontrivial.



Invariant matrix $\mathcal{M}_{ij} = \mathcal{M}^{R}_{ij} + i\epsilon_{ij}C \geq 0$

Real part: superfluid weight

Imaginary part: Chern number

$$D_s \geq |C|$$

M is Brillouin-zone averaged quantum geometric tensor.

We show that superfluidity and quantum metric are connected!

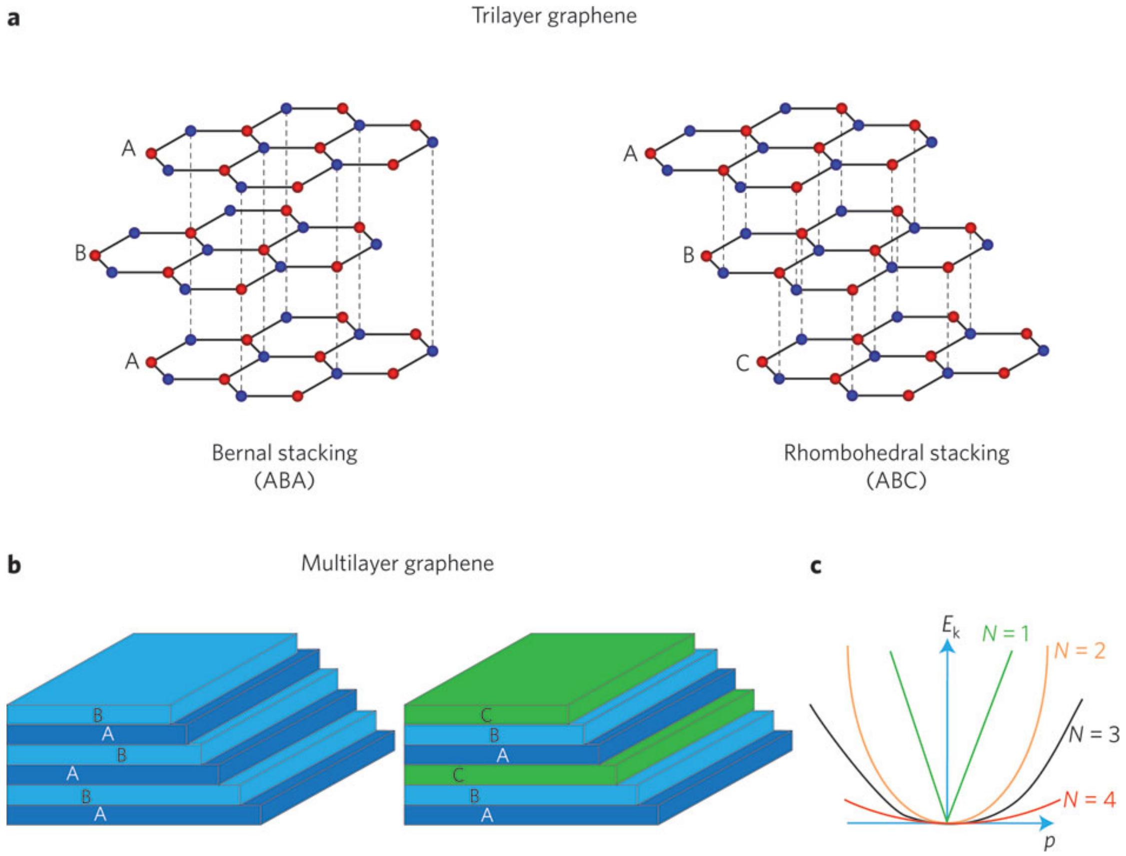
Flat bands and room-temperature superconductivity

Parabolic band

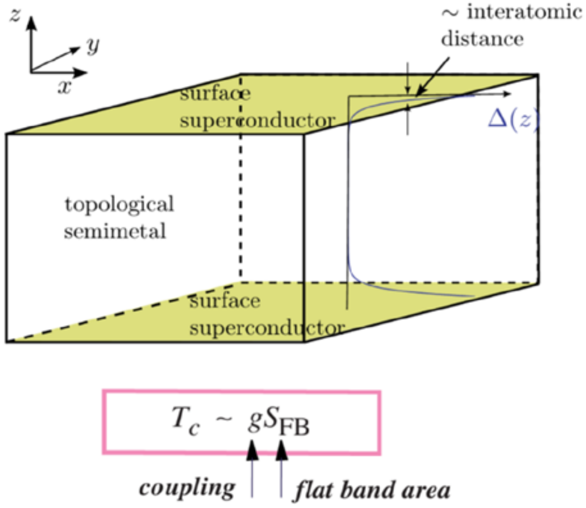
$$U \ll J \Rightarrow T_c \propto e^{-\frac{1}{U n_0(E_F)}}$$

Flat band

$$U \gg J \Rightarrow T_c \propto U n_0(E_F) \propto U/J$$



Partial flat band of surface states in rhombohedral graphite



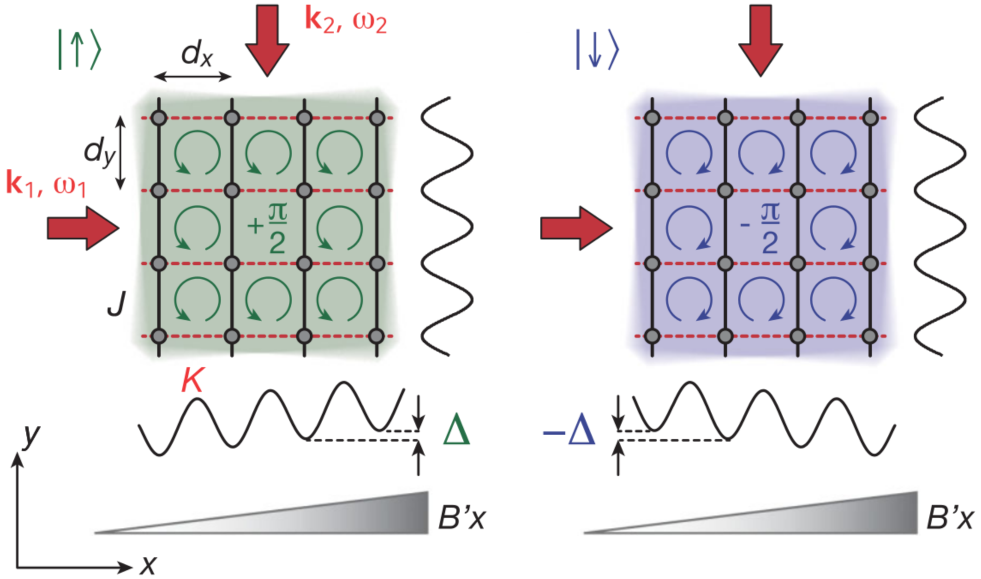
Volovik, J Supercond Nov Magn (2013) **26**:2887
Kopnin, Heikkilä, Volovik, PRB **83**, 220503(R) (2011)
Khodel, Shaginyan, JETP Lett. 51, 553 (1990)

Realization of the Harper Hamiltonian in ultracold gases

$$^{87}\mathrm{Rb} \qquad |\uparrow\rangle = |F=1, m_F=-1\rangle$$

$$|\downarrow\rangle = |F=2, m_F=1\rangle$$

Flux quanta per unit cell $n_\phi = \frac{1}{7}$



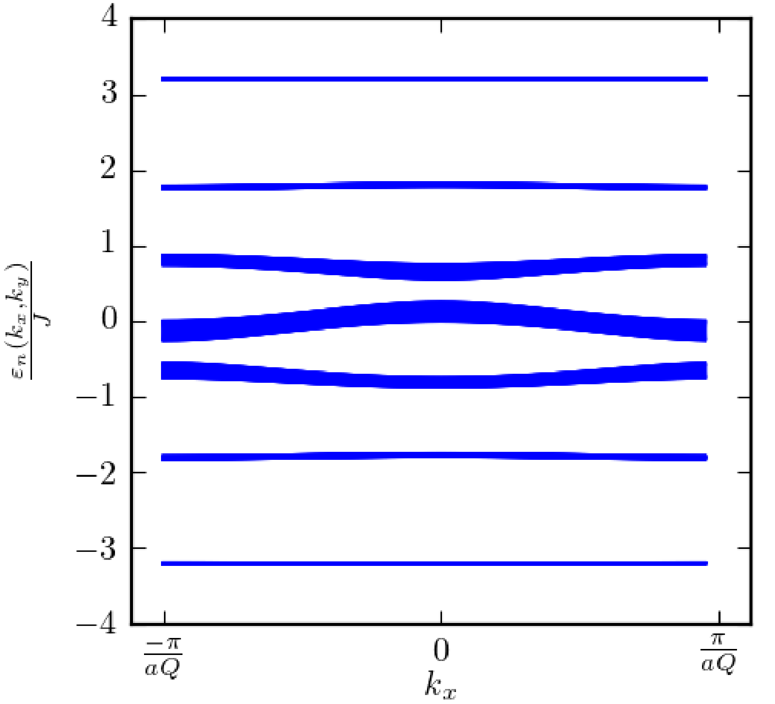


Fig. taken from Ref. 1

- Time-reversal symmetry is preserved
- Lowest bands are quasi-flat (Landau levels)
- Non-zero Chern number

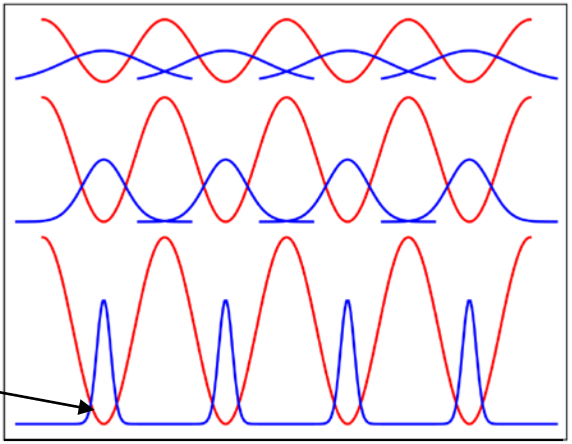
[1] Aidelsburger et al., PRL **111**, 185301 (2013)
[2] Miyake et al., PRL **111**, 185302 (2013)

Superfluid density and superfluid weight

Nonzero pairing potential does not guarantee superconductivity

Example of a flat band composed of localized Wannier functions

$$\Delta \neq 0$$
$$D_s = 0$$



Definition of superfluid density and superfluid weight

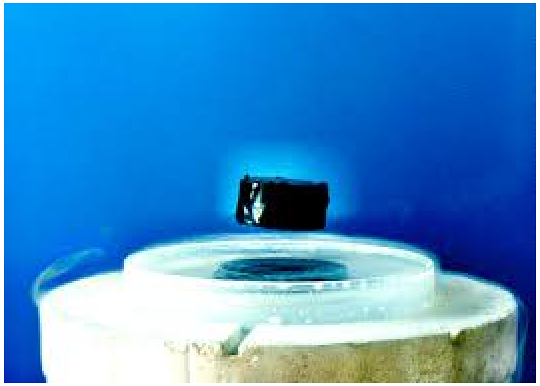
$$\frac{\Delta F}{V} = \frac{1}{2}\rho_s v_s^2 = \frac{1}{2} D_s p_s^2$$

$$v_s = \frac{\hbar \mathbf{q}}{m}$$ Cooper pair velocity

$$2p_s = 2\hbar \mathbf{q}$$ Cooper pair momentum

$$\mathbf{J} = D_s \hbar \mathbf{q}$$ Supercurrent density

$$\left.\begin{array}{l}\Delta \neq 0 \\ D_s \neq 0\end{array}\right\} \Rightarrow$$



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See e.g. M. Holzmann and G. Baym, Phys. Rev. B 76, 092502 (2007)

Superfluid weight from the grand potential

$$[D_s]_{i,j} = \frac{4\pi^2}{Vh^2}\frac{\partial^2\Omega}{\partial q_i \partial q_j}\bigg|_{\mu,\Delta\ ,\boldsymbol{q}=0}$$

Supercurrent $\Delta = |\Delta| e^{2i\boldsymbol{q}\cdot\boldsymbol{r}}$

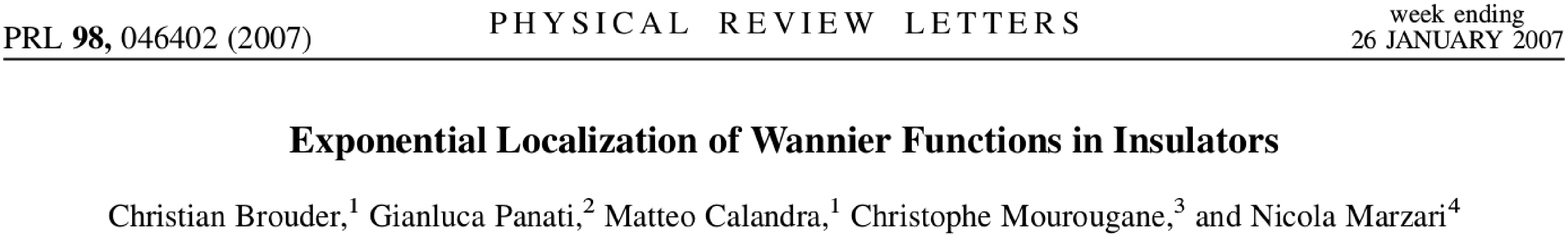
Gauge transformation to make the order parameter real
and to move q to the kinetic Hamiltonian (Peierls substitution):

$$K_{i,j} \rightarrow K_{i,j} e^{i\boldsymbol{q}\cdot(\boldsymbol{r}_i - \boldsymbol{r}_j)}$$

$$K_{i,j} \propto J$$

NOTE: Peierls substitution is justified only if there exist exponentially localized Wannier functions!

Chern number and Wannier functions



PRL **98,** 046402 (2007) PHYSICAL REVIEW LETTERS week ending 26 JANUARY 2007

Exponential Localization of Wannier Functions in Insulators

Christian Brouder,[1] Gianluca Panati,[2] Matteo Calandra,[1] Christophe Mourougane,[3] and Nicola Marzari[4]

Theorem:
Exponentially localized Wannier functions can be constructed in 2D or 3D iff the Chern number(s) is zero

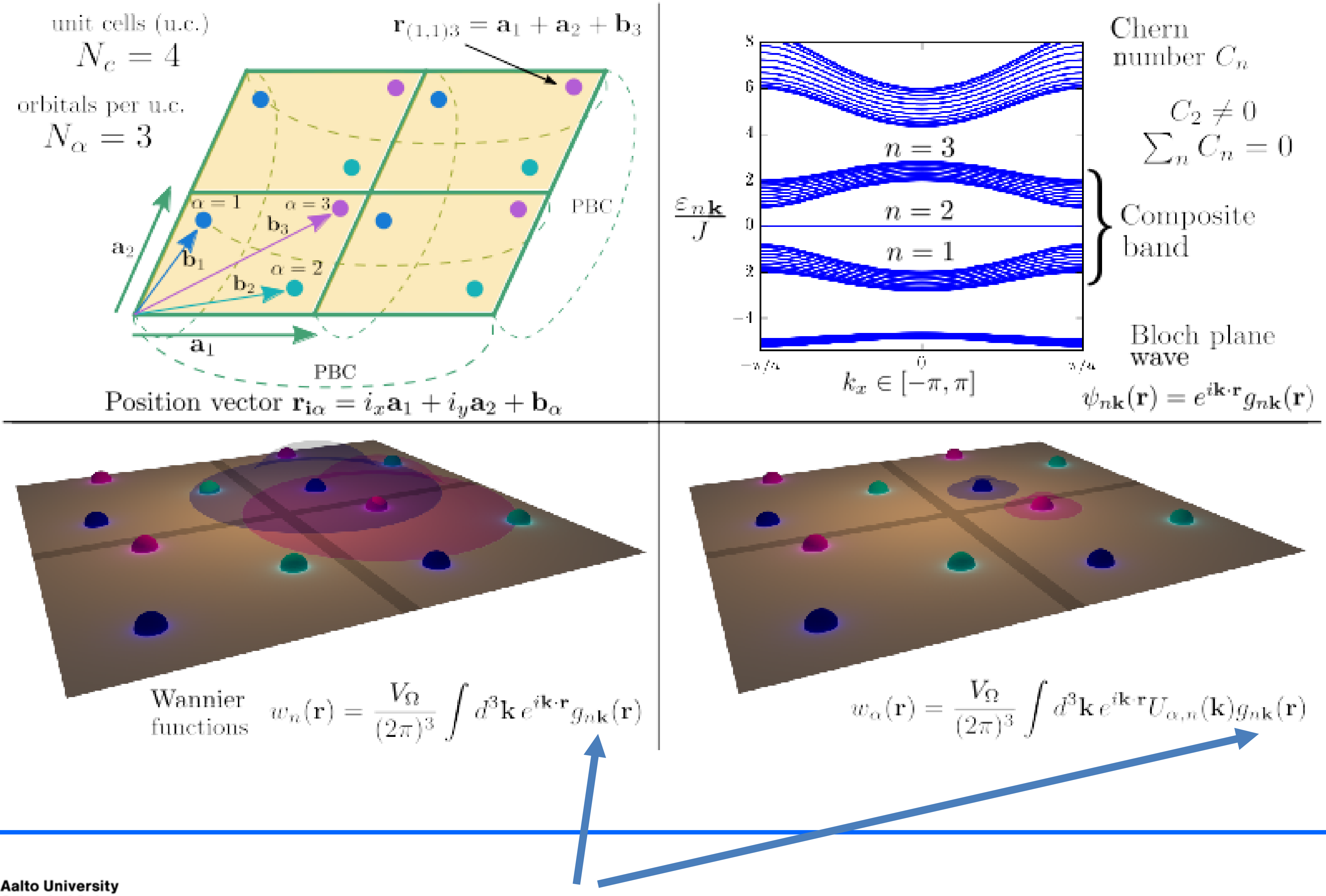
$\mathbf{A}(\mathbf{k}) = i \langle g_{n\mathbf{k}} | \partial_{\mathbf{k}} g_{n\mathbf{k}} \rangle$ Berry connection

$\Omega(\mathbf{k}) = \hat{z} \cdot \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{k})$ Berry curvature

$C = \frac{1}{2\pi} \int_{\mathrm{B.Z.}} d^2\mathbf{k}\, \Omega(\mathbf{k})$ Chern number

Therefore Peierls substitution is not justified for topologically non-trivial (nonzero Chern number) bands.

The way out: multiband approach



Bloch function for band n

Superfluid weight in a BCS multiband superconductor

$$\frac{\Delta F}{V} = \frac{1}{2}\rho_s v_s^2 = \frac{1}{2}D_s(\hbar\mathbf{q})^2$$

$$[D_s]_{i,j} = \frac{4\pi^2}{Vh^2}\frac{\partial^2\Omega}{\partial q_i \partial q_j}\bigg|_{\mu,\Delta\ ,\boldsymbol{q}=0}$$

$$[\mathcal{G}_{\mathbf{k}}]_{\alpha,n} = g_{n\mathbf{k}\uparrow}(\alpha)$$

Bloch function

$$D_s = D_{s,1} + D_{s,2} + D_{s,3}$$

$$[D_{s,1}]_{i,j} = \frac{2}{V\hbar^2}\sum_{\mathbf{k}}\mathrm{Tr}\left[\mathcal{V}_{\mathbf{k}}\mathcal{V}_{\mathbf{k}}^\dagger \partial_{k_i}\partial_{k_j}\varepsilon_{\mathbf{k}}\right]$$

Conventional term present in the single band case

$$\mathcal{D}_{\mathbf{k}}(\mathbf{q}) = -\mathcal{G}_{\mathbf{k}-\mathbf{q}}^\dagger \Delta \mathcal{G}_{\mathbf{k}+\mathbf{q}}$$

$$[D_{s,2}]_{i,j} = \frac{2}{V\hbar^2}\sum_{\mathbf{k}}\mathrm{Tr}\left[\mathcal{V}_{\mathbf{k}}\mathcal{U}_{\mathbf{k}}^\dagger \partial_{q_i}\partial_{q_j}\mathcal{D}_{\mathbf{k}}\right]$$

$$[D_{s,3}]_{i,j} = \frac{2}{V\hbar^2}\sum_{\mathbf{k}}\sum_{n,n'}\frac{[B_{\mathbf{k},i}]_{n,n'}[B_{\mathbf{k},j}]_{n',n}}{E_{n\mathbf{k}} + E_{n'\mathbf{k}}}$$

$$B_{\mathbf{k},i} = \mathcal{U}_{\mathbf{k}}^\dagger \partial_{q_i}\mathcal{D}_{\mathbf{k}}\mathcal{U}_{\mathbf{k}} + \mathcal{V}_{\mathbf{k}}^\dagger \partial_{q_i}\mathcal{D}_{\mathbf{k}}\mathcal{V}_{\mathbf{k}} + \mathcal{V}_{\mathbf{k}}^\dagger \partial_{k_i}\varepsilon_{\mathbf{k}}\mathcal{U}_{\mathbf{k}} - \mathcal{U}_{\mathbf{k}}^\dagger \partial_{k_i}\varepsilon_{\mathbf{k}}\mathcal{V}_{\mathbf{k}}$$

Geometric term present only in the multiband case

Can be nonzero in a flat band

Superfluid weight in a flat band

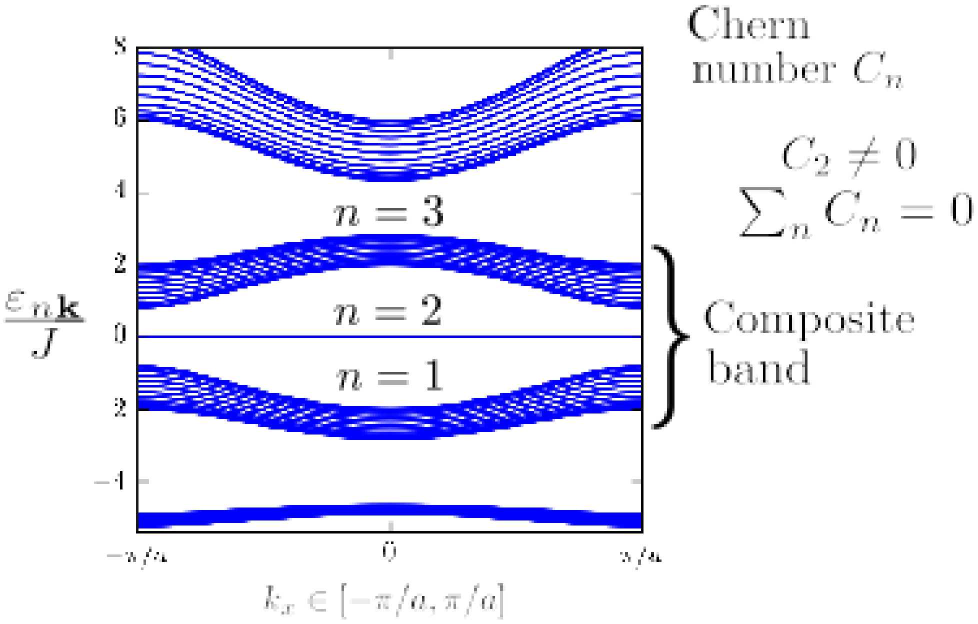
Assumption: $\Delta_\alpha = \Delta = \text{const.}$

$n_\phi^{-1} = N_\alpha$ Number of orbitals with nonzero pairing order parameter

$\bar{n}$ partially filled flat band

ν flat band filling

$$\Delta = U n_\phi \sqrt{\nu(1-\nu)}$$



Isolated flat-band approximation

bandwidth $J \ll U \ll E_g$ band gap

flat band

isolated band

Superfluid weight in a flat band

$$[D_s]_{i,j} = [D_{s,2}]_{i,j} + [D_{s,3}]_{i,j} = \frac{2Un_\phi}{\pi\hbar^2}\nu(1-\nu)\mathcal{M}^R_{ij}$$

Real part of a matrix-valued invariant of the band structure

QGT

$$g_{ij} + i\sigma_{ij}$$

Quantum metric

Berry curvature

Distance between two nearby quantum states

$$d\ell^2 = g_{ij}\, ds_i\, ds_j$$

$$\mathcal{M}_{ij} = \frac{1}{2\pi}\int_{\mathrm{B.Z.}} d^2\mathbf{k}\, \mathcal{B}_{ij}(\mathbf{k})$$

$$\mathcal{B}_{ij}(\mathbf{k}) = 2\mathrm{Tr}[(\partial_{k_i}\bar{\mathcal{G}}_{\mathbf{k}}^\dagger)(\partial_{k_j}\bar{\mathcal{G}}_{\mathbf{k}})] + 2\mathrm{Tr}[\bar{\mathcal{G}}_{\mathbf{k}}^\dagger(\partial_{k_i}\bar{\mathcal{G}}_{\mathbf{k}})\bar{\mathcal{G}}_{\mathbf{k}}^\dagger(\partial_{k_j}\bar{\mathcal{G}}_{\mathbf{k}})]$$

We show that the flat band superfluid weight is proportional to the Brillouin-zone averaged quantum metric, and derive a lower bound to it

$$\boxed{\mathcal{M}_{ij} = \mathcal{M}_{ij}^R + i\epsilon_{ij}C \geq 0 \Rightarrow \det(\mathcal{M}^R) \geq |C|^2}$$

Chern number

$$D_s \geq |C|$$

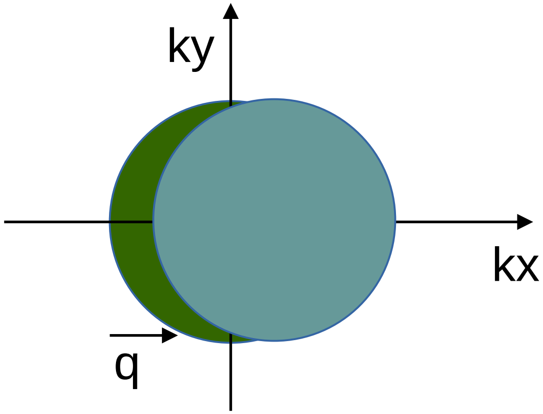
Parabolic band vs. flat band

Parabolic band

$$D_s = \frac{n_\mathrm{p}}{m_\mathrm{eff}}\left(1 - \left(\frac{2\pi\Delta}{k_\mathrm{B}T}\right)^{1/2} e^{-\Delta/(k_\mathrm{B}T)}\right)$$

n_p	Particle density
$\frac{1}{m_\mathrm{eff}} \propto J$	Bandwidth

Physical picture: global shift of the Fermi surface



Flat band

$$[D_s]_{i,j} = \frac{2Un_\phi}{\pi\hbar^2}\nu(1-\nu)\mathcal{M}_{ij}^R$$

Linearly proportional to the coupling constant U!

Physical picture:

Delocalization of Wannier functions

Overlapping Cooper pairs: pairing fluctuations support transport if pairs can be created and destroyed at distinct locations

c.f. A.J. Leggett, Phys. Rev. Lett. 25, 1544 (1970);
Bounds on supersolids related to (dis)connectedness of the density

Commun. Math. Phys. 76, 289-301 (1980)

Riemannian Structure on Manifolds of Quantum States

J. P. Provost and G. Vallee
Physique Theorique, Universite de Nice*·**

$$\langle \quad | \quad \rangle$$

$$\|\psi(s+ds)-\psi(s)\|^2=(\partial_i\psi,\partial_j\psi)ds_i\,ds_j\;\left(\partial_i=\frac{\partial}{\partial s_i}\right)$$

$$(\partial_i\psi,\partial_j\psi)=\gamma_{ij}+i\sigma_{ij}$$

Gauge invariant version:

$$g_{ij}(s)=\gamma_{ij}(s)-\beta_i(s)\,\beta_j(s) \qquad \beta_j(s)=-i(\psi(s),\partial_j\psi(s))$$

QGT

Distance between two nearby quantum states

$$d\ell^2=g_{ij}\,ds_i\,ds_j$$

$$g_{ij}+i\sigma_{ij}$$

Quantum metric

Berry curvature

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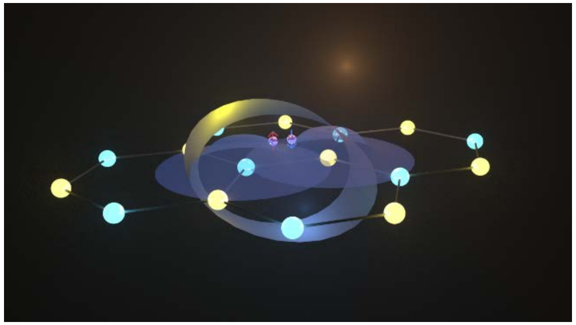

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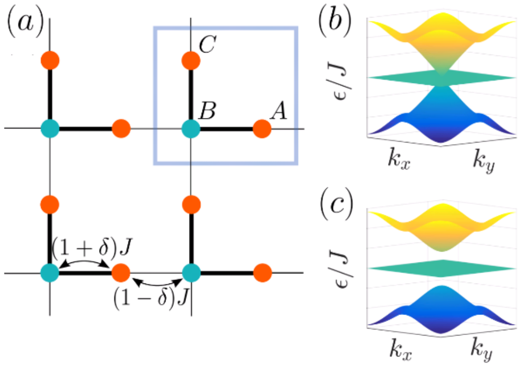

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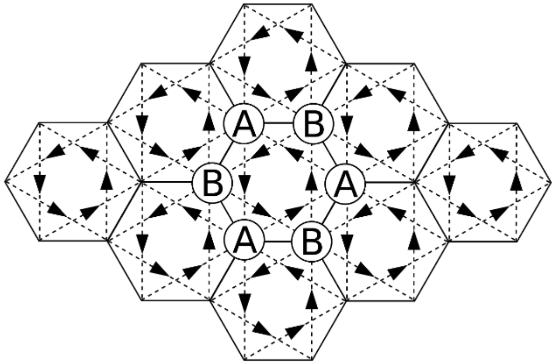

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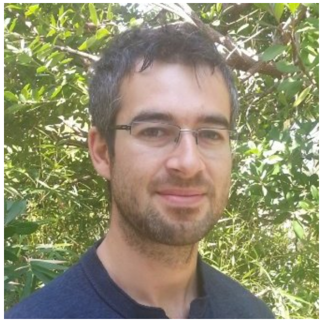

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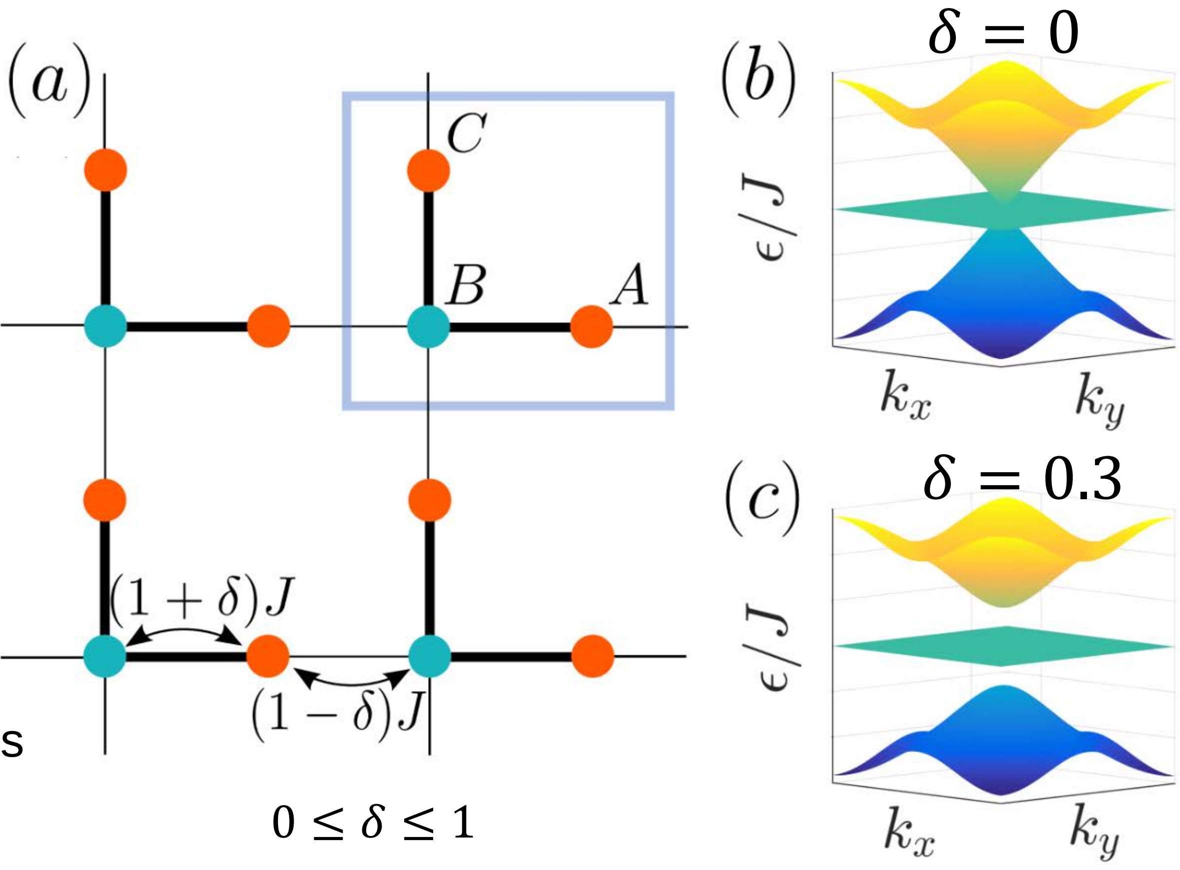
Lieb lattice

What about superfluidity in topologically trivial flat bands?

Three bands: two dispersive and one strictly flat band with $C = 0$

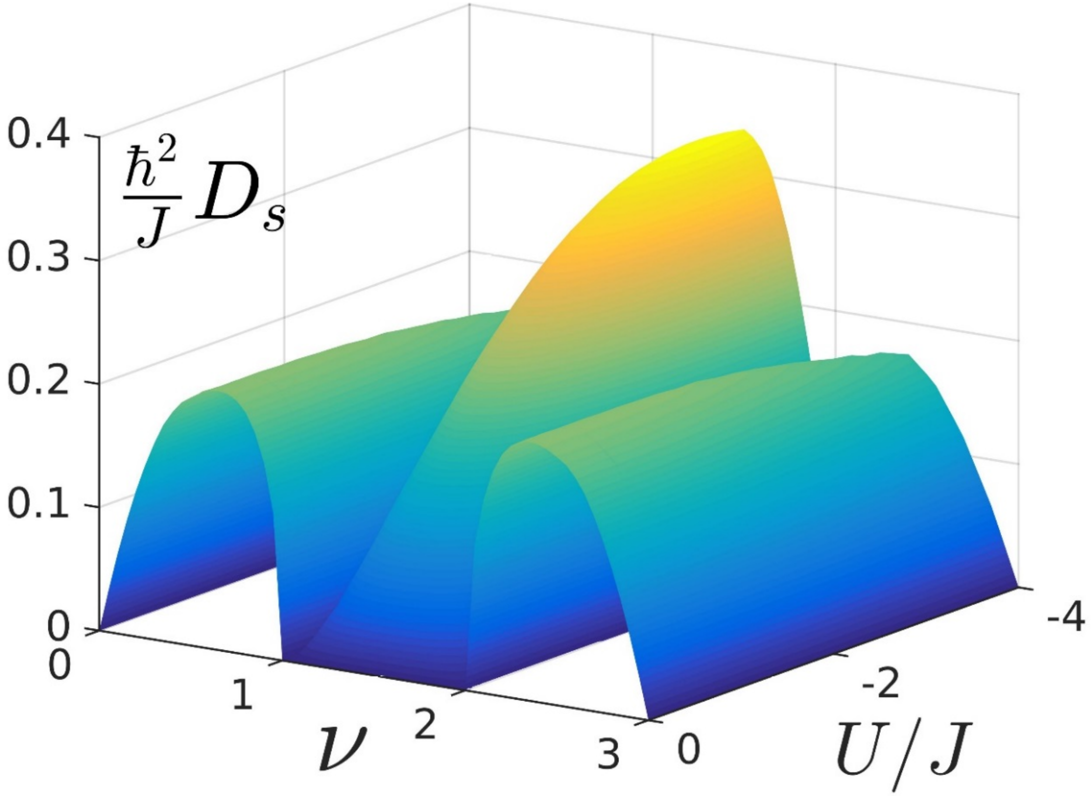


Staggered hopping coefficients enable the isolation of the flat band

$0 \leq \delta \leq 1$

Lieb lattice: superfluid weight

ν Total filling (1< ν <2 for the flat band)



D_s on the flat band depends strongly on *U*

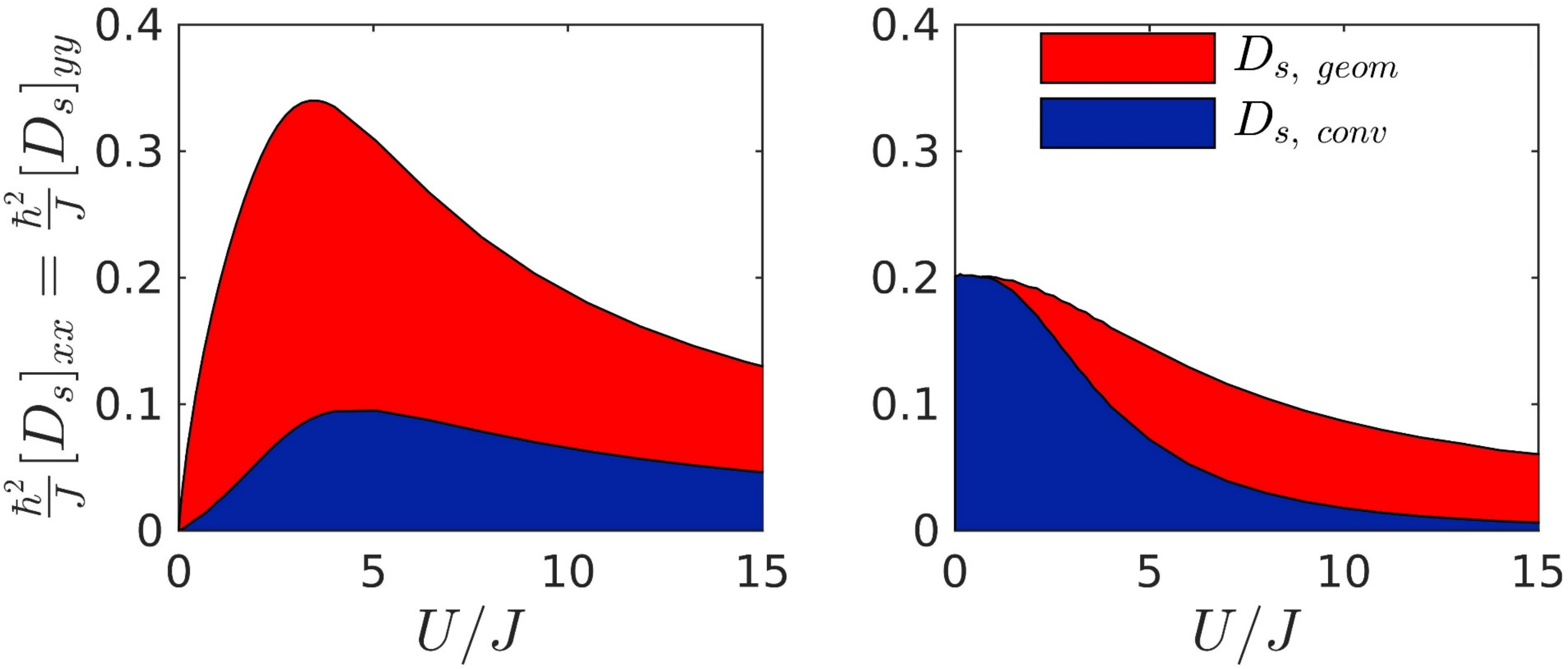
D_s on dispersive bands roughly constant

D_s for the trivial flat bands is non-zero and large!

Lieb lattice: geometric contribution

Large D_s and its strong dependence on U within the flat band is explained by the geometric superfluid weight contribution

$$D_{s,geom} = D_{s,2} + D_{s,3}$$



Half-filled flat band (ν = 1.5)

Half-filled upper band (ν = 2.5)

For isolated flat band, the BCS wavefunction is exact!

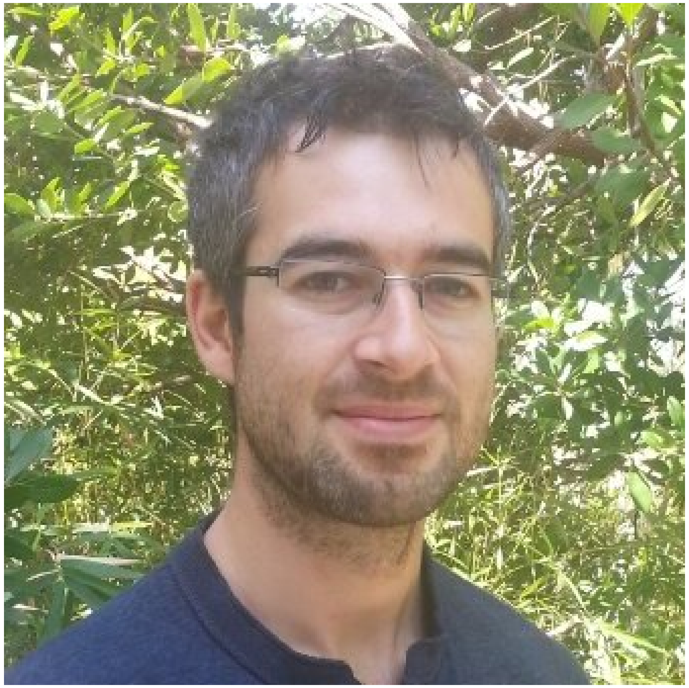
(for bipartite lattices)

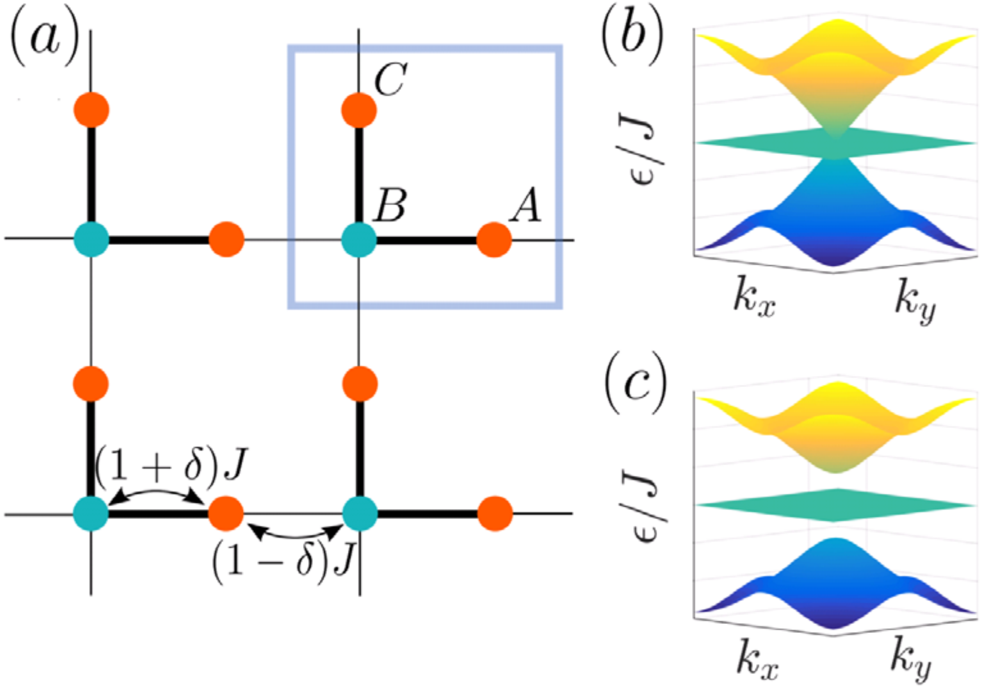
The mean-field prediction of finite flat band superfluidity is exact
for bipartite lattices and isolated flat bands.

For partially flat bands we have confirmed the results by DMFT and ED.

See POSTER 30 on the Lieb lattice superfluidity

Sebastiano Peotta





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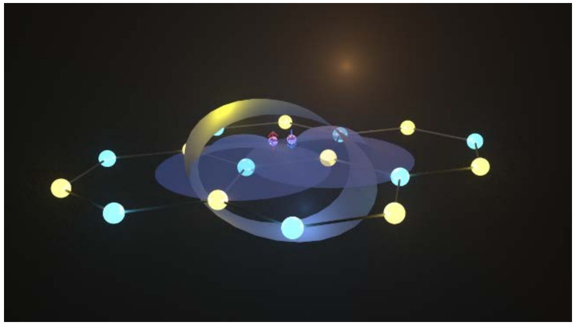

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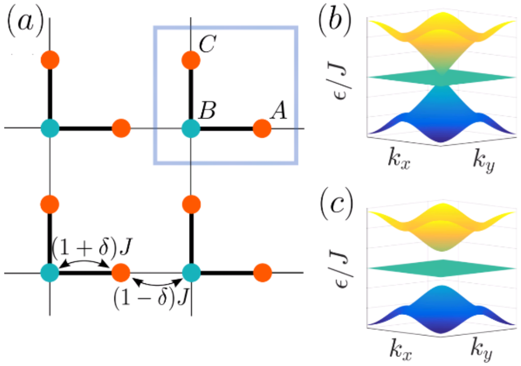

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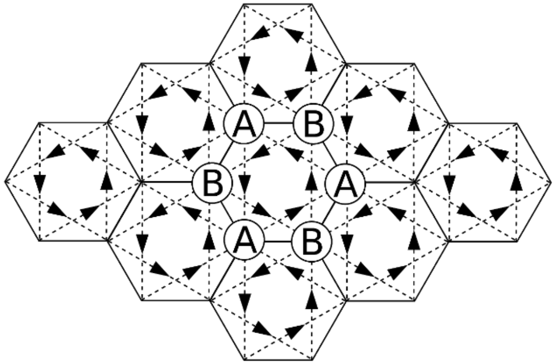

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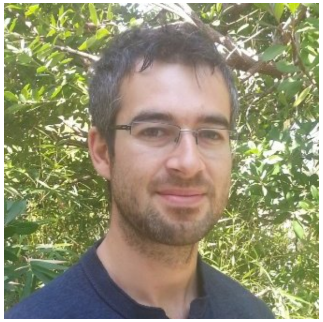

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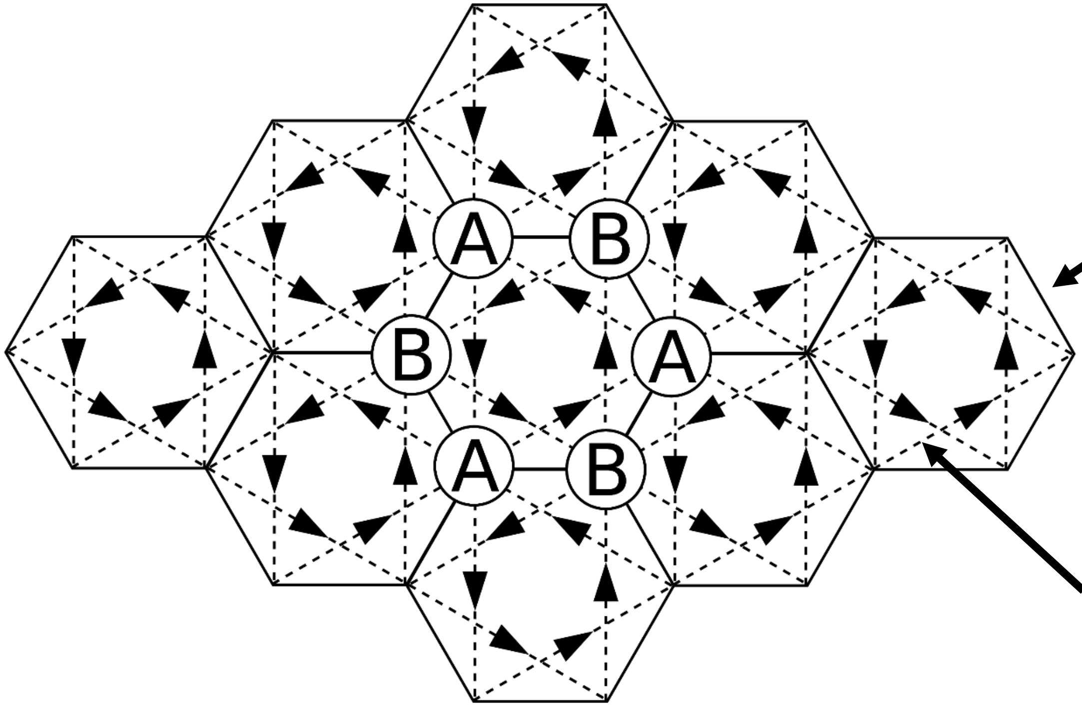
Haldane-Hubbard model

Potential difference Δ_{AB} between the sublattices



Nearest neighbour hopping t=1

Next-nearest neighbour hopping: $t'\exp(i\phi)$

Noninteracting model a well known model of a topological insulator

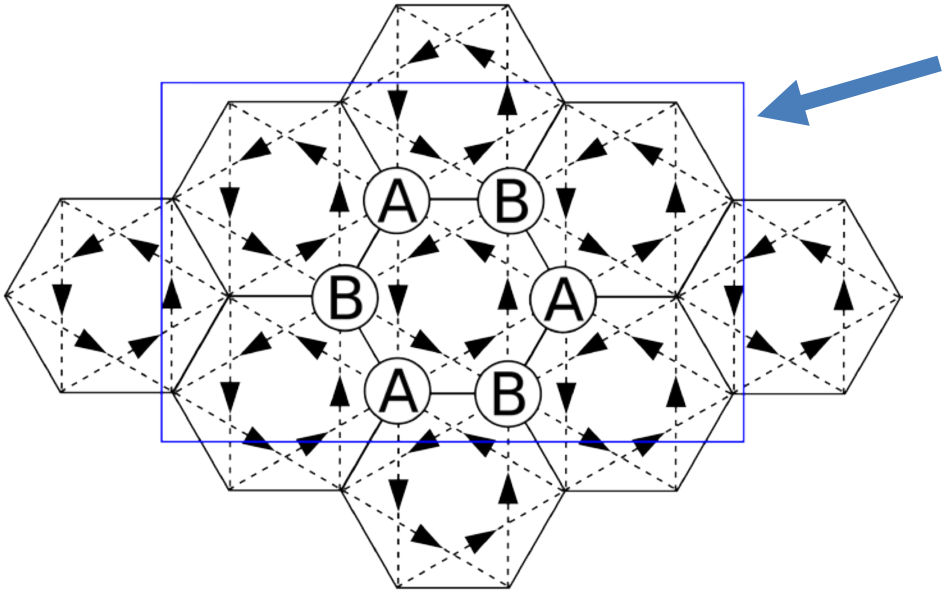
Jotzu et al., Nature **515**, 237 (2014)

Take a two-component version:

What are the effects of an on-site Hubbard repulsion?

Mean field: Physica B 481, 53-58; PRB 84, 035127; PRB 85, 205107; J. Phys B. 26, 175601
Feng, Kou, Huber groups

Exact diagonalization and DMFT: two complementary methods

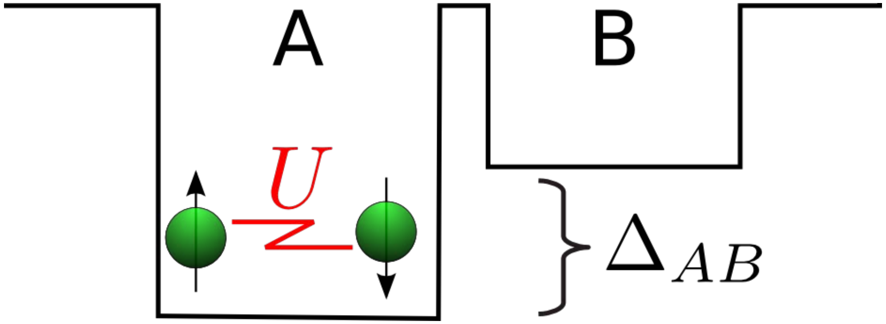


ED cluster (16 sites)

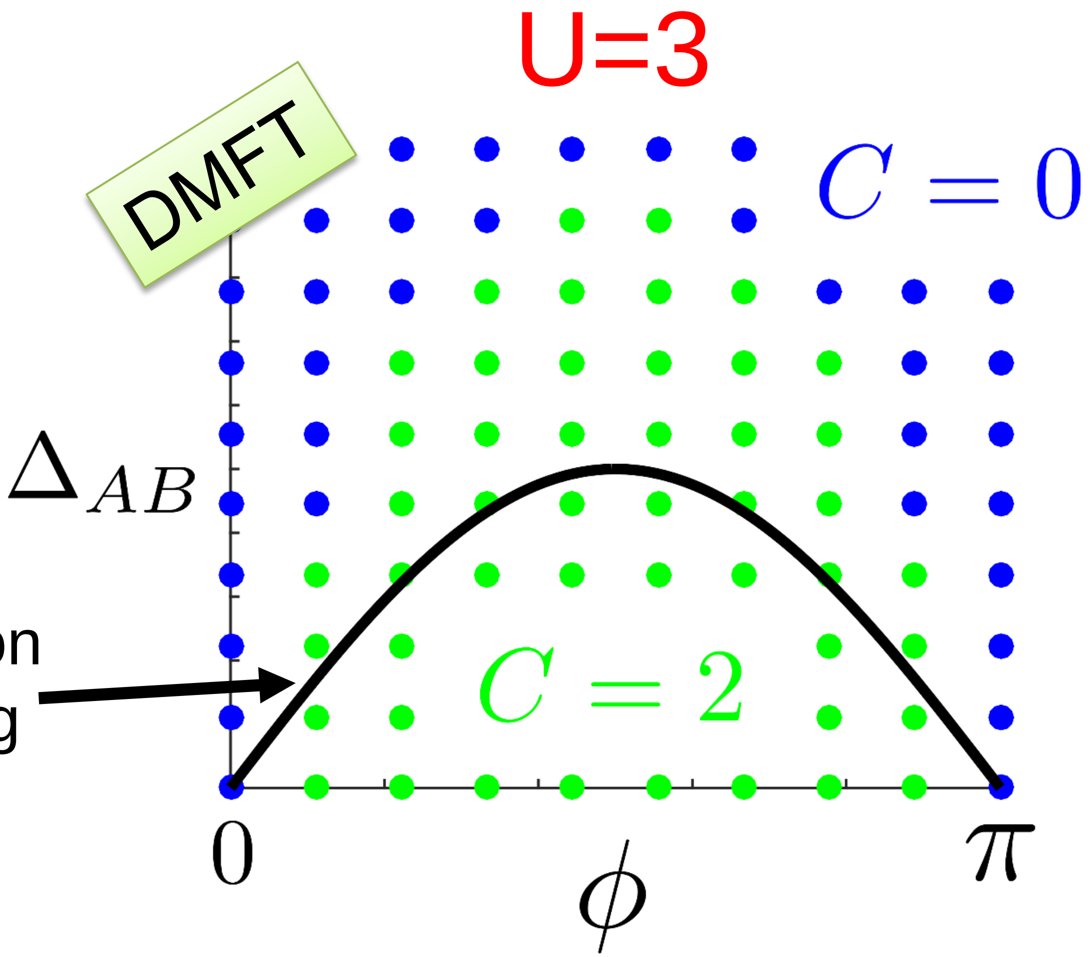
- ED: Finite system, but correlations treated exactly
    - Can map out topological phase boundaries using twisted boundary conditions

- DMFT: Thermodynamic limit, but only local quantum correlations treated exactly
    - Chern numbers using the ”topological Hamiltonian” (Z. Wang and S.-C. Zhang, Phys. Rev. X 2, 031008, 2012)

Interactions oppose the effect of the sublattice potential difference

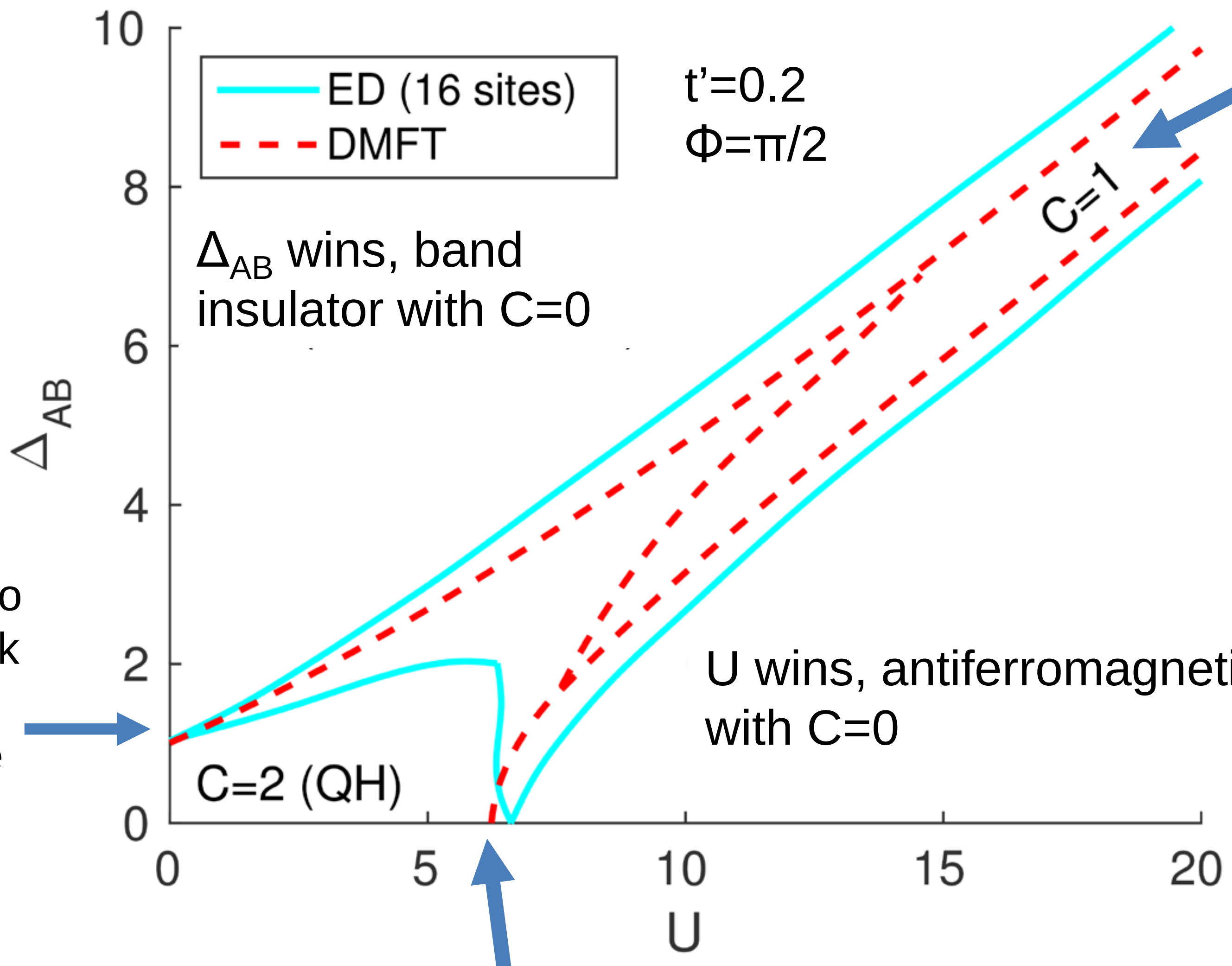


Weak interactions: Enlarged topological phase



Topological transition in the noninteracting case

Strong interactions: U drives antiferromagnetism, Δ_{AB} density wave



Δ_{AB} wins, band insulator with C=0

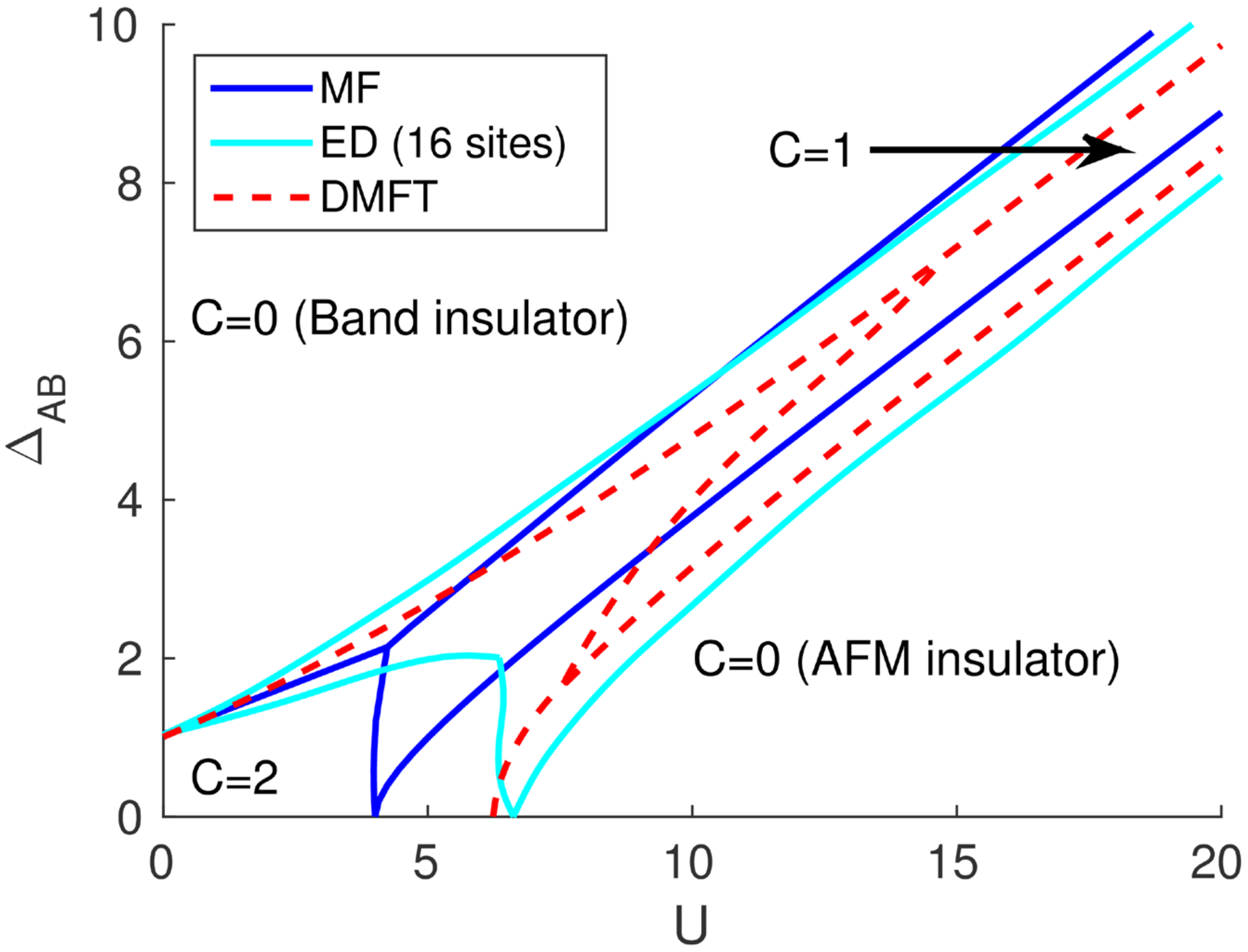
U wins, antiferromagnetic phase with C=0

Intermediate insulating phase with C=1

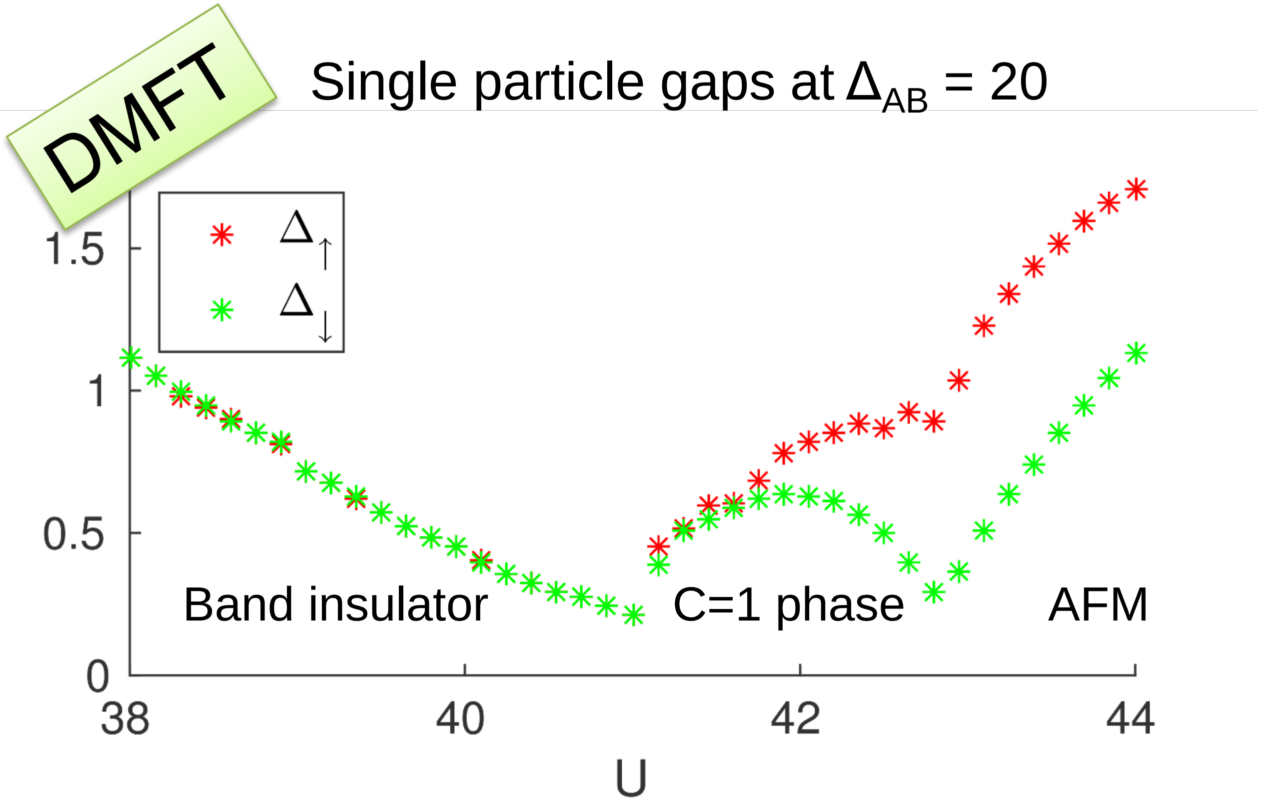
DMFT close to exact for weak coupling, ED has finite size effects

First order transition to antiferromagnetism at Δ_{AB}=0?

Comparison with pure mean field



Intermediate insulating phase with C=1



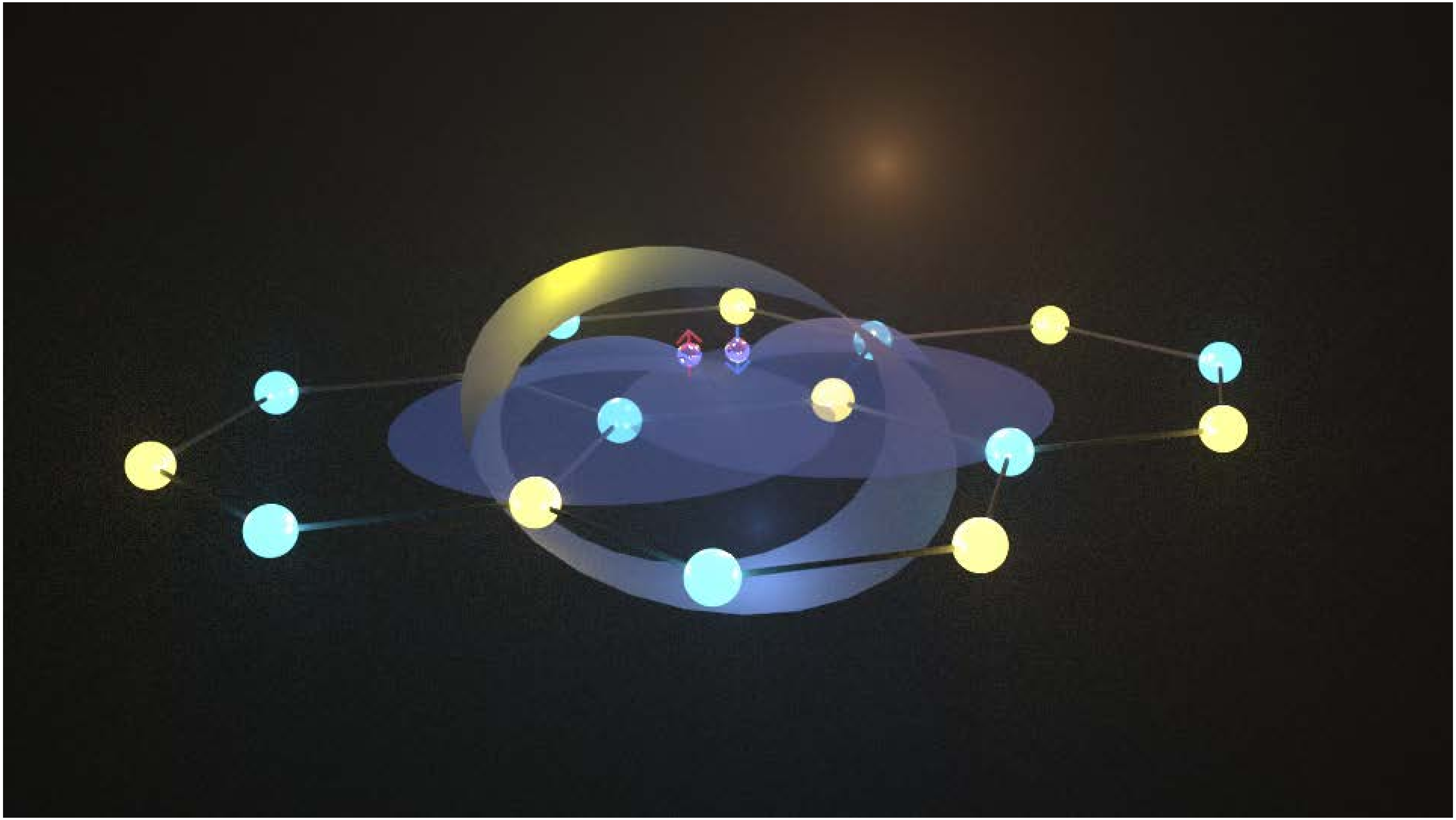
One spin component in the quantum Hall phase, the other one in the band insulating (density wave) phase.

arXiv:1512.08804

Summary

Geometry and topology make superfluidity possible even in flat bands

Chern number C=1 phase in the Haldane-Hubbard model



Blender art: Antti Paraoanu