Vortices and Solitons in Fermi Superfluids
or rather:
Our search for an easy, yet versatile way to describe them

People involved in this project:


Financial support by the Fund for Scientific Research-Flanders
Motivation: The (unreasonable?) efficiency of Ginburg-Landau equations* for superconductors

\[- \frac{\hbar^2}{2M} \left[ \nabla \cdot \left( \frac{iQ}{\hbar} \mathbf{A}(\mathbf{r}) \right) \right]^2 \Psi(\mathbf{r}) + a(T) \Psi(\mathbf{r}) + b(T) |\Psi(\mathbf{r})|^2 \Psi(\mathbf{r}) = 0\]

Phenomenological

\[a = -\frac{2(\mu e)^2}{m_e^3} \lambda_L H_c^2,\]
\[b = \frac{4\mu^2 e^4}{m_e^2} \lambda_L^4 H_c^2.\]

Gor’kov

\[a = \frac{(T - T_c)}{\eta T_c}, \quad b = \frac{1}{\eta N(0)}, \quad \eta = \frac{7\zeta(3)}{6(\pi T_c)^2 \xi_F} \]

Vortices in the “crossover”

\[\text{Type II} \quad \rightarrow \quad \text{Type I}\]

Effective $\kappa = \text{bulk } \kappa \times (\lambda / d)$

Gladilin, Ge, Gutierrez, Timmermans, Van de Vondel, Tempere, Devreese and Moshchalkov, NJP 17, 063032 (2015).

* Note that supercurrents feed back into the vector potential:

\[\frac{1}{\mu} \nabla \times (\nabla \times \mathbf{A}) = \frac{iQ\hbar}{2M} [\Psi(\mathbf{r}) \nabla \cdot \Psi^*(\mathbf{r}) - \Psi^*(\mathbf{r}) \nabla \cdot \Psi(\mathbf{r})] - \frac{Q^2}{M} |\Psi(\mathbf{r})|^2 \mathbf{A}(\mathbf{r}) + \mathbf{j}_{\text{ext}}\]
Motivation: The (unreasonable?) efficiency of Ginburg-Landau equations for superconductors

\[
- \frac{\hbar^2}{2M} \left[ \nabla_r - \frac{iQ}{\hbar} \mathbf{A}(r) \right]^2 \Psi(r) + a(T)\Psi(r) + b(T)|\Psi(r)|^2 \Psi(r) = 0
\]

and also motivated by the (unreasonable?) success of Gross-Pitaevskii for bosons...

Goal: *an effective field theory for fermionic superfluids* – including mixtures and finite-T effects.

Similar efforts by:

Theoretical part: our effective field theory for the superfluid Fermi gas
The thermodynamic potential is calculated in the functional integral formalism:

\[
Z = e^{-\beta\Omega(T,V,\mu_\sigma)} = \int \mathcal{D}\bar{\phi}\mathcal{D}\phi \exp\left\{ -S[\bar{\phi}, \phi] \right\}
\]

The action functional for the fermionic fields is given by:

\[
S[\bar{\phi}, \phi] = \int_0^\beta d\tau \int d\mathbf{r} \left\{ \sum_{\sigma=\uparrow, \downarrow} \bar{\phi}_{\mathbf{r},\tau,\sigma} \left( \partial_\tau - \nabla^2_\mathbf{r} - \mu_\sigma \right) \phi_{\mathbf{r},\tau,\sigma} + g\bar{\phi}_{\mathbf{r},\tau,\uparrow} \phi_{\mathbf{r},\tau,\downarrow} + g\phi_{\mathbf{r},\tau,\downarrow} \phi_{\mathbf{r},\tau,\uparrow} \right\}
\]

(units \( \hbar = 2m = k_F = 1 \))

Application of path integral description to BEC-BCS crossover, see:
Additional details can be found for example in Stoof, Dickerscheid & Gubbels, *Ultracold Quantum Fields* (Springer, 2009).
Functional integral description of the superfluid Fermi gas

The thermodynamic potential is calculated in the functional integral formalism:

\[ Z = e^{-\beta \Omega(T,V,\mu_\sigma)} = \int \mathcal{D}\phi \mathcal{D}\bar{\phi} \exp \{-S[\phi, \bar{\phi}]\} = \int \mathcal{D}\phi \mathcal{D}\bar{\phi} \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \exp \{-S_{HS}[\phi, \bar{\phi}, \Psi, \bar{\Psi}]\} \]

The Hubbard-Stratonovic action functional is given by:

\[
S_{HS} = S_B - \int_0^\beta d\tau \int d\mathbf{r} \begin{pmatrix} \bar{\phi}_\uparrow & \phi_\uparrow \end{pmatrix} \begin{pmatrix} -\partial_\tau - H_\uparrow & \Psi_{r,\tau} \\ \bar{\Psi}_{r,\tau} & -\partial_\tau + H_\downarrow \end{pmatrix} \begin{pmatrix} \phi_\uparrow \\ \bar{\phi}_\downarrow \end{pmatrix}
\]

with \( H_\sigma = -\nabla_\mathbf{r}^2 - \mu_\sigma \) and \( S_B = -\int_0^\beta d\tau \int d\mathbf{r} \frac{\bar{\Psi}_{r,\tau} \Psi_{r,\tau}}{g} \)
The thermodynamic potential is calculated in the functional integral formalism:

\[ Z = e^{-\beta \Omega(T, V, \mu_\sigma)} = \int \mathcal{D}\phi \mathcal{D}\bar{\phi} \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \exp \{-S_{HS}[\bar{\phi}, \phi, \bar{\Psi}, \Psi]\} = \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \exp \{-S_{eff}[\bar{\Psi}, \Psi]\} \]

The effective action obtained after integrating out fermions is given by:

\[ S_{eff} = S_B - \text{Tr} \ln (-G^{-1}) \]

\[ -G^{-1} = -G_0^{-1} + \mathbb{F} \]

with \( -G_0^{-1} = \begin{pmatrix} -\partial_\tau - H_\uparrow & 0 \\ 0 & -\partial_\tau + H_\downarrow \end{pmatrix} \) and \( \mathbb{F}(r, \tau) = \begin{pmatrix} 0 & -\Psi_{r,\tau} \\ -\bar{\Psi}_{r,\tau} & 0 \end{pmatrix} \)

\[ \Rightarrow \text{Tr} \ln (-G^{-1}) = \text{Tr} \ln (-G_0^{-1}) + \sum_{p=1}^{\infty} \frac{1}{p} \text{Tr} \left\{ [G_0 \mathbb{F}(r, \tau)]^p \right\} \]
The exact series \( \text{Tr} [G_0 F(x_1)] + \frac{1}{2} \text{Tr} [G_0 F(x_1)G_0 F(x_2)] + \frac{1}{3} \text{Tr} [G_0 F(x_1)G_0 F(x_2)G_0 F(x_3)] + \ldots \) is approximated in different ways:

1. The saddle-point approximation\(^1\):

\[
\sum_{p=1}^{\infty} \frac{1}{p} \text{Tr} \left[ (G_0 F)^p \right] \approx \text{Tr} [G_0 F_{sp}] + \frac{1}{2} \text{Tr} [G_0 F_{sp} G_0 F_{sp}] + \frac{1}{3} \text{Tr} [G_0 F_{sp} G_0 F_{sp} G_0 F_{sp}] + \ldots
\]

2. Gaussian pair fluctuations\(^2\):

\[
\sum_{p=1}^{\infty} \frac{1}{p} \text{Tr} \left[ (G_0 F)^p \right] \approx \text{Tr} [G_0 F(x_1)] + \frac{1}{2} \text{Tr} [G_0 F(x_1)G_0 F(x_2)] + \frac{1}{3} \text{Tr} [G_0 F_{sp} G_0 F_{sp} G_0 F_{sp}] + \ldots
\]

The exact series \[ \text{Tr} \left[ G_0 F(x_1) \right] + \frac{1}{2} \text{Tr} \left[ G_0 F(x_1) G_0 F(x_2) \right] + \frac{1}{3} \text{Tr} \left[ G_0 F(x_1) G_0 F(x_2) G_0 F(x_3) \right] + \ldots \]
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\]

3. Gradient expansion\([3]\):

\[
F(x_2 - x_1) \approx F_0 + (x_2 - x_1) (\nabla F)_0 + \frac{1}{2} (x_2 - x_1)^2 (\nabla^2 F)_0 + \ldots
\]

Expand around \( F_0 \rightarrow 0 \) (i.e. near \( T = T_c \)) to get the usual Ginzburg-Landau formalism.

Expand around \( F_0 \rightarrow F_{sp} \) and determine \( F_{sp} \) self-consistently from gap and number equations to extend the validity domain beyond the usual Ginzburg-Landau validity.

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A schematic overview of the different ways to approximate

The exact series

\[ \text{Tr} [G_0 F(x_1)] + \frac{1}{2} \text{Tr} [G_0 F(x_1) G_0 F(x_2)] + \frac{1}{3} \text{Tr} [G_0 F(x_1) G_0 F(x_2) G_0 F(x_3)] + \ldots \]

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1. The saddle-point approximation\(^1\):

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\]

2. Gaussian pair fluctuations\(^2\):

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\sum_{p=1}^{\infty} \frac{1}{p} \text{Tr} [(G_0 F)^p] \approx \text{Tr} [G_0 F_{sp}] + \frac{1}{2} \text{Tr} [G_0 F_{sp} G_0 F(x_2 - x_1)] + \frac{1}{3} \text{Tr} [G_0 F_{sp} G_0 F_{sp} G_0 F_{sp}] + \ldots
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3. Gradient expansion\(^3\):

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\sum_{p=1}^{\infty} \frac{1}{p} \text{Tr} [(G_0 F)^p] \approx \text{Tr} [G_0 F_{sp}] + \frac{1}{2} \text{Tr} [G_0 F_{sp} G_0 F_{grad}] + \frac{1}{3} \text{Tr} [G_0 F_{sp} G_0 F_{sp} G_0 F_{sp}] + \ldots
\]

4. Current proposal: replace in all \( p > 2 \) terms up to two \( F_{sp} \) ’s by \( F_{grad} \)

\[
\sum_{p=1}^{\infty} \frac{1}{p} \text{Tr} [(G_0 F)^p] \approx \text{Tr} [G_0 F_{sp}] + \frac{1}{2} \text{Tr} [G_0 F_{sp} G_0 F_{grad}] + \frac{1}{3} \text{Tr} [G_0 F_{sp} G_0 F_{grad} G_0 F_{grad}] + \ldots
\]


The gradient expansion in the pair field

The thermodynamic potential is calculated in the functional integral formalism:

\[ Z = e^{-\beta \Omega(T,V,\mu_\sigma)} = \int \mathcal{D}\bar{\phi} \mathcal{D}\phi \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \exp \left\{ -S_{HS}[\bar{\phi}, \phi, \bar{\Psi}, \Psi] \right\} = \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \exp \left\{ -S_{eff}[\bar{\Psi}, \Psi] \right\} \]

The effective action obtained after integrating out fermions is given by:

\[ S_{eff} = S_B - \text{Tr} \ln \left( -G_0^{-1} \right) - \sum_{p=1}^{\infty} \frac{1}{p} \text{Tr} \left[ G_0 F(r, \tau) G_0 F(r, \tau) \ldots G_0 F(r, \tau) \right] \]

all others are kept as \( F_0 \)

expand at most 2 by

\[ F(r, \tau) = F_0 + (\tau - \tau_0) \frac{\partial F}{\partial \tau}_{r_0, \tau_0} + (r - r_0) \cdot \nabla F|_{r_0, \tau_0} + \ldots \]

We include all second order terms, neglecting third and higher order.

Here we assume that the pair fields vary slowly in time and space, but not necessarily around zero!

Effective field theory obtained after gradient expansion

The action obtained after the gradient expansion is the basis of our effective field theory:

\[ S_{EFT} [\Psi (\mathbf{r}, \tau)] = \int_0^\beta d\tau \int d\mathbf{r} \left\{ \Omega_s + \frac{D}{2} \left( \frac{\partial \bar{\Psi}}{\partial \tau} \Psi - \bar{\Psi} \frac{\partial \Psi}{\partial \tau} \right) + C \left| \nabla_{\mathbf{r}} \Psi \right|^2 - E \left( \nabla_{\mathbf{r}} \Psi \right)^2 \right\} \]

Analytic results were obtained for the coefficients:

\[ \Omega_s (|\Psi|) = - \frac{|\Psi|^2}{8\pi k_F a_s} - \int \frac{dk}{(2\pi)^3} \left[ \frac{1}{\beta} \ln \left( 2 \cosh (\beta E_{\mathbf{k}}) + 2 \cosh (\beta \zeta) \right) - \xi_k - \frac{|\Psi|^2}{2k^2} \right] \]

\[ C' (|\Psi|) = \int \frac{dk}{(2\pi)^3} \frac{2k^2}{3} f_2 (\beta, E_{\mathbf{k}}, \zeta) \]

\[ D (|\Psi|) = \int \frac{dk}{(2\pi)^3} \frac{\xi_k^2}{|\Psi|^2} \left[ f_1 (\beta, \xi_k, \zeta) - f_1 (\beta, E_{\mathbf{k}}, \zeta) \right] \]

\[ E (|\Psi|) = \int \frac{dk}{(2\pi)^3} \frac{4k^2}{3} \xi_k^2 f_4 (\beta, E_{\mathbf{k}}, \zeta) \]

where

\[ f_1 (\beta, \varepsilon, \zeta) = \frac{1}{2\varepsilon \cosh (\beta \varepsilon) + \cosh (\beta \zeta)} \]

and

\[ f_{n+1} = \frac{1}{2n\varepsilon} \frac{\partial f_n (\beta, \varepsilon, \zeta)}{\partial \varepsilon} \]

and

\[ E_{\mathbf{k}} = \sqrt{\xi_k^2 + |\Psi|^2} = \sqrt{(k^2 - \mu)^2 + |\Psi|^2} \]

Results are given in units where \( \hbar = 2m = k_F = 1 \)

For details on the derivation and a discussion of the \((\partial^2 \Psi)^2\) and \((\partial_{\mathbf{r}} \Psi)^2\) terms, see: S.N. Klimin, J. Tempere, Devreese, European Physical Journal B 88, 122 (2015).
Effective field theory compared with Ginzburg-Landau

The action obtained after the gradient expansion is the basis of our effective field theory:

\[ S_{EFT} [\Psi (r, \tau)] = \int_0^\beta d\tau \int d\mathbf{r} \left\{ \Omega_s + \frac{D}{2} \left( \frac{\partial \Psi}{\partial T} \Psi - \frac{\partial \bar{\Psi}}{\partial T} \bar{\Psi} \right) + C |\nabla_r \Psi|^2 - E \left( \nabla_r |\Psi|^2 \right)^2 \right\} \]

Check the results for against the Ginzburg-Landau energy functional (valid for \( T \approx T_c \)):

In the seminal BEC-BCS crossover paper [1], the authors propose a fluctuation expansion around \( |\Psi|=0 \), which corresponds to setting \( E_k \rightarrow \xi_k \) in our coefficients. In this limit, our coefficient \( C \) corresponds to their “\( c \)” and the coefficients of \( |\Psi|^2 \) and \( |\Psi|^4 \) in \( \Omega_s \) correspond to their \( -a \) and \( b \) respectively.


Note that a more recent approach, K. Huang, Z.-Q. Yu and L. Yin, Phys. Rev. A 79, 053602 (2009), expands the logarithm up to \( p=2 \) and performs a gradient expansion, whereas in our approach we take all powers \( p \) in the logarithm expansion into account.
Application to solitons or vortices

The effective field (real-time\(^1\)) action yields the following Lagrangian

\[
\mathcal{L} (\Psi, \partial_x \Psi) = i \frac{D}{2} \left( \bar{\Psi} \frac{\partial \Psi}{\partial T} - \frac{\partial \bar{\Psi}}{\partial T} \Psi \right) - \Omega_s - C' |\nabla_r \Psi|^2 + E (\nabla_r |\Psi|^2)^2
\]

Before deriving the field equations, note that for localized excitations such as vortices or solitons, the order parameter may be written as

\[
\Psi(r, t) = |\Psi_\infty| a(r, t) e^{i\theta(r, t)}
\]

where the background amplitude, amplitude modulation, and phase profile can be identified.

The background amplitude and the chemical potentials are derived from the simultaneous solution of gap and number equations:

\[
\frac{\partial \Omega_{sp}}{\partial (|\Psi_\infty|)} = 0,
\quad
n = - \frac{\partial (\Omega_{sp} + \Omega_{fi})}{\partial \mu},
\quad
\delta n = - \frac{\partial (\Omega_{sp} + \Omega_{fi})}{\partial \zeta}
\]

\(^1\) Going from the Euclidean time action to the real time action is performed by the usual formal replacements \( \tau \to i\tau \) and \( S(\beta) \to -iS(t_0, t_0) \).
A first application: solitons and the filling up of the core
Application to solitons

The effective field action yields the following Lagrangian

\[ \mathcal{L} (\Psi, \partial_x \Psi) = i \frac{D}{2} \left( \bar{\Psi} \frac{\partial \Psi}{\partial T} - \frac{\partial \bar{\Psi}}{\partial T} \Psi \right) - \Omega_s - C \left| \nabla_r \Psi \right|^2 + E \left( \nabla_r \left| \Psi \right|^2 \right)^2 \]

In particular, for solitons:

\[ \Psi(x, t) = \left| \Psi_\infty \right| \ a(x - v_s t) e^{i\theta(x-v_s t)} \]

Substitution of this form in the Lagrangian yield an effective Lagrangian for \( a(x) \) and \( \theta(x) \):

\[ \mathcal{L}(a, \partial_x a; \theta, \partial_x \theta) = - \int dx \left\{ \kappa(a) a^2 v_s \partial_x \theta + \Omega_s - \frac{1}{2} \rho_{qp} \left( \partial_x a \right)^2 - \frac{1}{2} \rho_{sf} \left( \partial_x \theta \right)^2 \right\} \]

with:

\[ \kappa(a) = D(a) \left| \Psi_\infty \right|^2 \]

\[ \rho_{sf}(a) = \frac{\left| \Psi_\infty \right|^2}{m} C(a) a^2 \]

\[ \rho_{qp}(a) = \frac{\left| \Psi_\infty \right|^2}{m} \left[ C(a) - 4 \left| \Psi_\infty \right|^2 a^2 E(a) \right] \]

Application to solitons

For solitons:

$$\Psi(x, t) = |\Psi_\infty|^2 a(x - v_s t) e^{i\theta(x - v_s t)}$$

The equations of motion resulting from $$\mathcal{L}(a, \partial_x a; \theta, \partial_x \theta)$$ can be solved analytically to obtain the relation between $$x$$ and $$a$$:

$$x = \pm \int_{a_0}^{a} \sqrt{\frac{\rho_{qp}(a)\rho_{sf}(a)}{2\rho_{sf}(a)\Omega_s(a) - v_s^2 (\kappa(a)a^2 - \kappa_\infty)^2}} da$$

From this we also obtain the phase:

$$\theta(x) = v_s \int_{-\infty}^{x} \frac{\kappa_\infty/a^2 - \kappa(a)}{\rho_{sf}(a)} dx$$

with still:

$$\kappa(a) = D(a) |\Psi_\infty|^2$$

$$\rho_{sf}(a) = \frac{|\Psi_\infty|^2}{m} C(a)a^2$$

$$\rho_{qp}(a) = \frac{|\Psi_\infty|^2}{m} \left[ C(a) - 4|\Psi_\infty|^2 a^2 E(a) \right]$$

Application to solitons

The effective field (real-time\(^1\)) action yields the following Lagrangian

\[ \mathcal{L}(\Psi, \partial_x \Psi) = i \frac{D}{2} \left( \bar{\Psi} \frac{\partial \Psi}{\partial \tau} - \frac{\partial \bar{\Psi}}{\partial \tau} \Psi \right) - \Omega_s - C' |\nabla_x \Psi|^2 + E (\nabla_r |\Psi|^2)^2 \]

In particular, for solitons:

\[ \Psi(x, t) = |\Psi_\infty|^2 a(x - v_s t) e^{i\theta(x-v_s t)} \]

Application to vortices in superfluid Fermi gases
**Application to vortices**

Back to the Lagrangian for the macroscopic wave function:

\[
\mathcal{L} (\Psi, \partial_x \Psi) = \frac{i}{2} D \left( \bar{\Psi} \frac{\partial \Psi}{\partial T} - \frac{\partial \bar{\Psi}}{\partial T} \Psi \right) - \Omega_s - C' |\nabla_r \Psi|^2 + E (\nabla_r |\Psi|^2)^2
\]

Just as for solitons, for localized excitations such as vortices, the order parameter may be written as

\[
\Psi(r, t) = |\Psi_\infty| a(r, t) e^{i \theta(r, t)}
\]

- **background amplitude**
- **amplitude modulation**
- **phase = angle around vortex line** \( e^{i \phi} \)

Now there is no analytical solution for \( a \) – we use a variationally trial shape.

\[
a(r) = \tanh \left[ \frac{r}{\sqrt{2} \xi} \right]
\]
The variational optimal value for $\xi$ depends on the superfluid density $\rho_{sf} = 2C|\Psi_\infty|^2$ and the free energy required to make the vortex core:

$$A = \int_0^\infty u \left\{ \Omega_s \left[ |\Psi_\infty|^2 \tanh^2 \left( u/\sqrt{2} \right) \right] - \Omega_s \left( |\Psi_\infty|^2 \right) \right\} du$$


Comparison with BdG at finite $T$

$$\Psi(r, \varphi, z) = |\Psi_\infty| \ a(r)e^{i\varphi}$$

For a finite-temperature vortex, the effective field theory [1] excellently matches the Bogoliubov – de Gennes solutions [2] in the BCS-BEC crossover everywhere except the BCS case combined with low temperatures.

**Modulation of the order parameter amplitude in a vortex**

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Density profiles

For a finite-temperature vortex, the effective field theory [1] excellently matches the Bogoliubov – de Gennes solutions [2] in the BCS-BEC crossover everywhere except the BCS case combined with low temperatures.

Pair correlation length

The pair correlation function

$$g_{\uparrow\downarrow}(r) = -\left(\frac{n}{2}\right)^2 + \langle \psi_{\uparrow}^\dagger \left(R + \frac{r}{2}\right) \psi_{\downarrow} \left(R - \frac{r}{2}\right) \psi_{\downarrow} \left(R + \frac{r}{2}\right) \psi_{\uparrow} \left(R - \frac{r}{2}\right) \rangle$$

allows to define the pair correlation length\[1\]

$$\xi_{\text{pair}} = \sqrt{\frac{\int \mathrm{d}r \ r^2 g_{\uparrow\downarrow}(r)}{\int \mathrm{d}r \ g_{\uparrow\downarrow}(r)}}$$

Taking the expectation value with respect to the gradient-expanded action yields\[2\]:

$$\xi_{\text{pair}} = \sqrt{\frac{\int \mathrm{d}k \ k^2 \left(4k \xi_k f_2(\beta, E_k, \zeta)\right)^2}{\int \mathrm{d}k \ k^2 \left(f_1(\beta, E_k, \zeta)\right)^2}}$$

The gradient expansion is expected to hold if the size of the spatial variations of the macroscopic wave function, $\xi_{\text{phase}}$, is larger than the pair correlation length $\xi_{\text{pair}}$.

How good is the hyperbolic tangent?

- BdG\(^1\) finds oscillations around a tanh profile in low-T, deep BCS regime – here this is never observed.

How good is the hyperbolic tangent?

- BdG\(^1\) finds oscillations around a tanh profile in low-T, deep BCS regime – here this is never observed.
- Imbalance increases the deviation from a hyperbolic-tangent form, especially on the BCS side, it also increases the size of the vortex core.

Critical rotation frequencies for vortices and for superfluidity
Rotating Fermi gases

At the non-interacting, single-particle level, rotating the quantum gas leads to

\[ H = \frac{[\nabla_\mathbf{r} - i \mathbf{A}(\mathbf{r})]^2}{2m} + \frac{m}{2} \left( \omega_{\text{trap}}^2 - \omega^2 \right) r_\perp^2 + \frac{m \omega_{\text{trap},z}^2}{2} z^2 \]

with a rotational “vector potential” \( \mathbf{A}(\mathbf{r}) = m (\mathbf{\omega} \times \mathbf{r}) \).

One could think that at the level of the effective field theory, rotations can be implemented through the “canonical” substitution

\[ |\nabla_\mathbf{r} \Psi|^2 \to |[\nabla_\mathbf{r} - 2i \mathbf{A}(\mathbf{r})] \Psi|^2 \]

However, this is wrong. The rotational “charge” need not be twice the atom’s.
Rotating Fermi gases

At the non-interacting, single-particle level, rotating the quantum gas leads to

\[
H = \frac{[\nabla_r - i A(r)]^2}{2m} + \frac{m(\omega_{\text{trap}}^2 - \omega^2)}{2} r_\perp^2 + \frac{m\omega_{\text{trap},z}^2}{2} z^2
\]

with a rotational “vector potential” \( A(r) = m(\omega \times r) \).

In the formalism, the rotation does come in through \( \mathbb{F} \), but it appears via \( G_0^{-1} \):

\[
G_0^{-1} = \begin{pmatrix}
    i\omega_n - \epsilon_k & 0 \\
    0 & i\omega_n + \epsilon_k
\end{pmatrix}
\]

\( \epsilon_k = k^2 \zeta_k = \zeta - 2k \cdot A(r) \)
Rotating Fermi gases

At the non-interacting, single-particle level, rotating the quantum gas leads to

\[
H = \frac{[\nabla r - iA(r)]^2}{2m} + \frac{m(\omega_{\text{trap}}^2 - \omega^2)}{2} r_1^2 + \frac{m\omega_{\text{trap},z}^2}{2} z^2
\]

with a rotational “vector potential” \( A(r) = m(\omega \times r) \).

Performing the gradient expansion with the changed \( C_0 \) yields to leading order

\[
\int dr \left[ C \left( \nabla r \bar{\Psi} \right) (\nabla r \Psi) \right] \rightarrow \int dr \left[ C \left( \nabla r \bar{\Psi} \right) (\nabla r \Psi) + iDA \cdot (\bar{\Psi} \nabla r \Psi - \Psi \nabla r \bar{\Psi}) \right]
\]

\[
= \int dr \left[ C \left| [\nabla r - i\epsilon A(r)] \Psi \right|^2 - C\epsilon^2 A^2 \left| \Psi \right|^2 \right]
\]

\[
\epsilon = \frac{D}{C} = \frac{\int d\mathbf{k} \epsilon_\mathbf{k} \left[ f_1(\beta, \epsilon_\mathbf{k}, \zeta_\mathbf{k}) - f_1(\beta, E_\mathbf{k}, \zeta_\mathbf{k}) \right] / \left| \Psi_\infty \right|^2}{(2/3) \int d\mathbf{k} k^2 f_2(\beta, E_\mathbf{k}, \zeta_\mathbf{k})}
\]

where \( f_1(\beta, \epsilon, \zeta) = \frac{1}{2\epsilon} \frac{\sinh(\beta\epsilon)}{\cosh(\beta\epsilon) + \cosh(\beta\zeta)} \) and \( f_{n+1} = -\frac{1}{2n\epsilon} \frac{\partial f_n(\beta, \epsilon, \zeta)}{\partial \epsilon} \)

and \( E_\mathbf{k} = \sqrt{\epsilon_\mathbf{k}^2 + \left| \Psi_\infty \right|^2} = \sqrt{(k^2 - \mu)^2 + \left| \Psi_\infty \right|^2} \)
Rotating Fermi gases

At the non-interacting, single-particle level, rotating the quantum gas leads to

\[ H = \frac{[\nabla_r - iA(r)]^2}{2m} + \frac{m(\omega_{\text{trap}}^2 - \omega^2)}{2} r_\perp^2 + \frac{m\omega_{\text{trap}}^2}{2} z^2 \]

with a rotational “vector potential” \( A(r) = m(\omega \times r) \).

\[ \tilde{\epsilon} = \frac{D}{C} = \frac{\int d\mathbf{k} \epsilon_k [f_1(\beta, \epsilon_k, \zeta_k) - f_1(\beta, E_k, \zeta_k)] / |\Psi_\infty|^2}{(2/3) \int d\mathbf{k} k^2 f_2(\beta, E_k, \zeta_k)} \]

The rotational “charge” (actually rotational moment of inertia) is not necessarily equal to twice that for atoms.
Rotating Fermi gases

Substituting $\Psi(\mathbf{r}, t) = |\Psi_\infty| a(\mathbf{r}, t) e^{i\theta(\mathbf{r}, t)}$ one finds the free energy

$$F = \int d\mathbf{r} \left\{ \Omega_s + \frac{1}{2} \rho_{qp} [\nabla_{\mathbf{r}} a(\mathbf{r})]^2 + \frac{1}{2} \rho_{sf} a^2(\mathbf{r}) [\nabla_{\mathbf{r}} \theta(\mathbf{r}) - \tilde{e} A(\mathbf{r})]^2 - \frac{1}{2} \rho_{sf} a^2(\mathbf{r}) \tilde{e}^2 A^2(\mathbf{r}) \right\}$$

with $\rho_{qp} = 2C |\Psi_\infty|^2$ and $\rho_{sf} = 2(C - 4E) |\Psi_\infty|^2$
Rotating Fermi gases

Rotating Fermi gases


Conclusions

Development of an description in terms of a macroscopic order parameter for superfluid Fermi gases, valid in the BEC-BCS crossover and for a wide temperature range.

The coefficients in this “Ginzburg-Landau” type of description are related to the microscopic parameters of the Fermi gas.

This description allows to model vortices and solitons well. Next: vortex matter and multivortex dynamics.

Description of the extended gradient expansion and the resulting effective field theory:

Solitons:

Vortices:

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