Universiteit Antwerpen

Ultracold Quantum Gases - Current Trends and Future Perspectives
616th WE-Heraus Seminar
Bad Honnef, May 9th – 13th 2016

Theory of Quantum and Complex systems

Vortices and Solitons in Fermi Superfluids

or rather:

Our search for an easy, yet versatile way to describe them

People involved in this project:

J. Tempere, G. Lombardi, W. Van Alphen, N. Verhelst, S. N. Klimin, J. T. Devreese





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FWO VLAANDEREN
Fonds Wetenschappelijk Onderzoek
Research Foundation – Flanders

Motivation: The (unreasonable?) efficiency of Ginburg-Landau equations* for superconductors

$$-\frac{\hbar^2}{2M}\left[\nabla_{\mathbf{r}}-\frac{iQ}{\hbar}\mathbf{A}(\mathbf{r})\right]^2\Psi(\mathbf{r})+\underline{a(T)}\Psi(\mathbf{r})+\underline{b(T)}\left|\Psi(\mathbf{r})\right|^2\Psi(\mathbf{r})=0$$

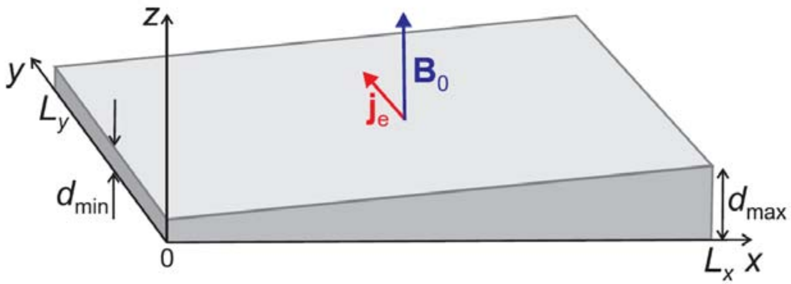
Phenomenological

$$a = -\frac{2\,(\mu e)^2}{m_e}\lambda_L^2 H_c^2,$$
$$b = \frac{4\mu^3 e^4}{m_e^2}\lambda_L^4 H_c^2.$$

Gor'kov

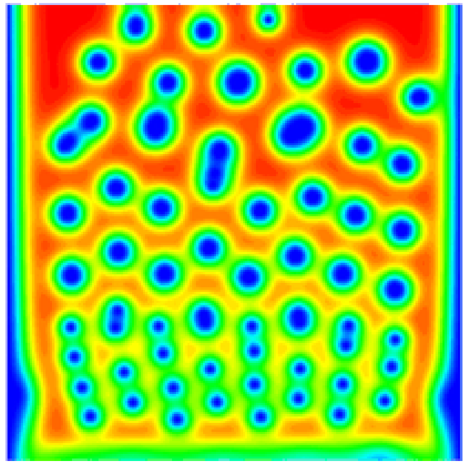
$$a = \frac{(T-T_c)}{\eta T_c}$$
$$b = \frac{1}{\eta N(0)} \qquad \eta = \frac{7\zeta(3)}{6(\pi T_c)^2}\varepsilon_F$$

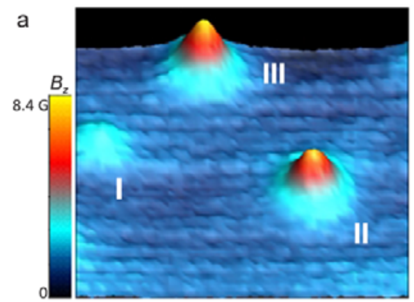
<u>Vortices in the "crossover"</u>



Effective κ = bulk κ × (λ/*d*)

Type II ⟵⟶ Type I





Gladilin, Ge, Gutierrez, Timmermans, Van de Vondel, Tempere, Devreese and Moshchalkov, NJP **17**, 063032 (2015).

* Note that supercurrents feed back into the vector potential:

$$\frac{1}{\mu}\nabla\times(\nabla\times\mathbf{A})=\frac{iQ\hbar}{2M}\left[\Psi(\mathbf{r})\boldsymbol{\nabla}_{\mathbf{r}}\Psi^*(\mathbf{r})-\Psi^*(\mathbf{r})\boldsymbol{\nabla}_{\mathbf{r}}\Psi(\mathbf{r})\right]-\frac{Q^2}{M}\left|\Psi(\mathbf{r})\right|^2\mathbf{A}(\mathbf{r})+\mathbf{j}_{ext}$$

Motivation: The (unreasonable?) efficiency of Ginburg-Landau equations for superconductors

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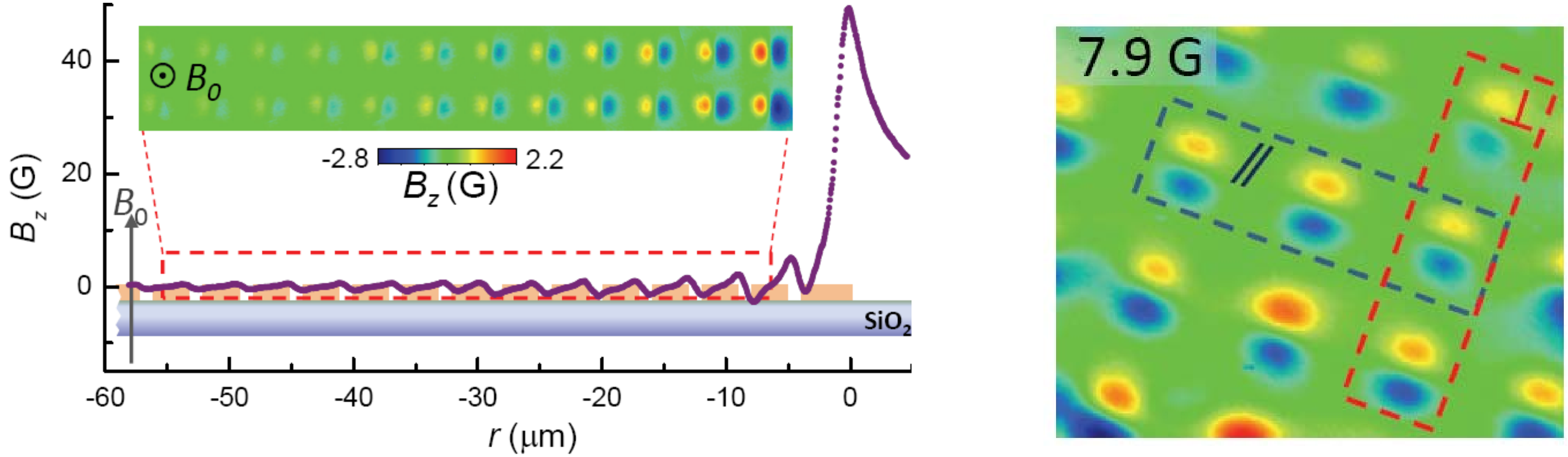
Long-lived vortex dipoles (arXiv: 1604.02341)



and also motivated by the (unreasonable?) success of Gross-Pitaevskii for bosons…

Goal: ***an effective field theory for fermionic superfluids*** – including mixtures and finite-T effects.

Similar efforts by:

- Ginzburg-Landau type equation for the atomic Fermionic superfluid: C.A.R. Sa de Melo, M. Randeria, and J.R. Engelbrecht, Phys. Rev. Lett. **71**, 3202 (1993).
- K. Huang, Z.-Q. Yu and L. Yin, Phys. Rev. A **79**, 053602 (2009).
- “Coarse-grained” BdG : S. Simonucci and G. C. Strinati, Phys. Rev. B **89**, 054511 (2014).

Theoretical part: our effective field theory for the superfluid Fermi gas

Functional integral description of the superfluid Fermi gas

The thermodynamic potential is calculated in the functional integral formalism:

$$\mathcal{Z} = e^{-\beta\Omega(T,V,\mu_\sigma)} = \int \mathcal{D}\bar{\phi}\mathcal{D}\phi \exp\left\{-S[\bar{\phi},\phi]\right\}$$

The action functional for the fermionic fields is given by:

$$S\left[\bar{\phi},\phi\right] = \int_0^\beta d\tau \int d\mathbf{r} \left\{ \sum_{\sigma=\uparrow,\downarrow} \bar{\phi}_{\mathbf{r},\tau,\sigma} \left(\partial_\tau - \nabla_{\mathbf{r}}^2 - \mu_\sigma\right) \phi_{\mathbf{r},\tau,\sigma} + g\bar{\phi}_{\mathbf{r},\tau,\uparrow}\bar{\phi}_{\mathbf{r},\tau,\downarrow}\phi_{\mathbf{r},\tau,\downarrow}\phi_{\mathbf{r},\tau,\uparrow} \right\}$$

(units $\hbar = 2m = k_F = 1$)

Application of path integral description to BEC-BCS crossover, see:
C.A.R. Sa de Melo, M. Randeria, and J.R. Engelbrecht, Phys. Rev. Lett. **71**, 3202 (1993).
Additional details can be found for example in Stoof, Dickerscheid & Gubbels, *Ultracold Quantum Fields* (Springer, 2009).

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The Hubbard-Stratonovic action functional is given by:

$$S_{HS} = S_B - \int_0^\beta d\tau \int d\mathbf{r} \begin{pmatrix} \bar{\phi}_\uparrow & \phi_\downarrow \end{pmatrix} \begin{pmatrix} -\partial_\tau - H_\uparrow & \Psi_{\mathbf{r},\tau} \\ \bar{\Psi}_{\mathbf{r},\tau} & -\partial_\tau + H_\downarrow \end{pmatrix} \begin{pmatrix} \phi_\uparrow \\ \bar{\phi}_\downarrow \end{pmatrix}$$

with $H_\sigma = -\nabla_{\mathbf{r}}^2 - \mu_\sigma$ and $S_B = -\int_0^\beta d\tau \int d\mathbf{r} \frac{\bar{\Psi}_{\mathbf{r},\tau}\Psi_{\mathbf{r},\tau}}{g}$

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The effective action obtained after integrating out fermions is given by:

$$S_{eff} = S_B - \mathrm{Tr}\ln\left(-\mathbb{G}^{-1}\right)$$

$$\downarrow$$

$$-\mathbb{G}^{-1} = -\mathbb{G}_0^{-1} + \mathbb{F}$$

split up in free field and pairing

with $-\mathbb{G}_0^{-1} = \begin{pmatrix} -\partial_\tau - H_\uparrow & 0 \\ 0 & -\partial_\tau + H_\downarrow \end{pmatrix}$ and $\mathbb{F}(\mathbf{r},\tau) = \begin{pmatrix} 0 & -\Psi_{\mathbf{r},\tau} \\ -\bar{\Psi}_{\mathbf{r},\tau} & 0 \end{pmatrix}$

$$\Rightarrow \quad \mathrm{Tr}\ln\left(-\mathbb{G}^{-1}\right) = \mathrm{Tr}\ln\left(-\mathbb{G}_0^{-1}\right) + \sum_{p=1}^{\infty}\frac{1}{p}\mathrm{Tr}\left\{\left[\mathbb{G}_0\mathbb{F}(\mathbf{r},\tau)\right]^p\right\}$$

A schematic overview of the different ways to approximate

The exact series $\mathrm{Tr}\left[\mathbb{G}_0\mathbb{F}(x_1)\right] + \frac{1}{2}\mathrm{Tr}\left[\mathbb{G}_0\mathbb{F}(x_1)\mathbb{G}_0\mathbb{F}(x_2)\right] + \frac{1}{3}\mathrm{Tr}\left[\mathbb{G}_0\mathbb{F}(x_1)\mathbb{G}_0\mathbb{F}(x_2)\mathbb{G}_0\mathbb{F}(x_3)\right] + \ldots$

is approximated in different ways:

$$\mathbb{F}(\mathbf{r},\tau) = \begin{pmatrix} 0 & -\Psi_{\mathbf{r},\tau} \\ -\bar{\Psi}_{\mathbf{r},\tau} & 0 \end{pmatrix}$$

1. The saddle-point approximation[1]:

$$\sum_{p=1}^{\infty}\frac{1}{p}\mathrm{Tr}\left[(\mathbb{G}_0\mathbb{F})^p\right] \approx \mathrm{Tr}\left[\mathbb{G}_0\mathbb{F}_{\mathrm{sp}}\right]+\frac{1}{2}\mathrm{Tr}\left[\mathbb{G}_0\mathbb{F}_{\mathrm{sp}}\mathbb{G}_0\mathbb{F}_{\mathrm{sp}}\right]+\frac{1}{3}\mathrm{Tr}\left[\mathbb{G}_0\mathbb{F}_{\mathrm{sp}}\mathbb{G}_0\mathbb{F}_{\mathrm{sp}}\mathbb{G}_0\mathbb{F}_{\mathrm{sp}}\right]+\ldots$$

2. Gaussian pair fluctuations[2]:

$$\sum_{p=1}^{\infty}\frac{1}{p}\mathrm{Tr}\left[(\mathbb{G}_0\mathbb{F})^p\right] \approx \mathrm{Tr}\left[\mathbb{G}_0\mathbb{F}(x_1)\right]+\frac{1}{2}\mathrm{Tr}\left[\mathbb{G}_0\mathbb{F}(x_1)\mathbb{G}_0\mathbb{F}(x_2)\right]+\frac{1}{3}\mathrm{Tr}\left[\mathbb{G}_0\mathbb{F}_{\mathrm{sp}}\mathbb{G}_0\mathbb{F}_{\mathrm{sp}}\mathbb{G}_0\mathbb{F}_{\mathrm{sp}}\right]+\ldots$$

[1] see eg. A. J. Leggett, in *Modern Trends in the Theory of Condensed Matter* (eds A. Pekalski and R. Przystawa , Springer, 1980).

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3. Gradient expansion[3]:

$$\mathbb{F}(x_2-x_1) \approx \underbrace{\mathbb{F}_0 + (x_2-x_1)\left(\nabla\mathbb{F}\right)_0 + \frac{1}{2}(x_2-x_1)^2\left(\nabla^2\mathbb{F}\right)_0}_{\mathbb{F}_{\mathrm{grad}}} + \ldots$$

Expand around $\mathbb{F}_0 \to 0$ (i.e. near T=T_c) to get the usual Ginzburg-Landau formalism.

Expand around $\mathbb{F}_0 \to \mathbb{F}_{\mathrm{sp}}$ and determine $\mathbb{F}_{\mathrm{sp}}$ self-consistently from gap and number equations to extend the validity domain beyond the usual Ginzburg-Landau validity.

[1] see eg. A. J. Leggett, in *Modern Trends in the Theory of Condensed Matter* (eds A. Pekalski and R. Przystawa , Springer, 1980).

[2] P. Nozières and S. Schmitt-Rink, J. Low Temp. Phys. **59**, 195 (1985); C. A. R. Sá de Melo, M. Randeria, and J. R. Engelbrecht, Phys. Rev. Lett. **71**, 3202 (1993).

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4. Current proposal: replace in all $p>2$ terms up to two $\mathbb{F}_{\mathrm{sp}}$'s by $\mathbb{F}_{\mathrm{grad}}$

$$\sum_{p=1}^{\infty}\frac{1}{p}\mathrm{Tr}\left[(\mathbb{G}_0\mathbb{F})^p\right] \approx \mathrm{Tr}\left[\mathbb{G}_0\mathbb{F}_{\mathrm{sp}}\right]+\frac{1}{2}\mathrm{Tr}\left[\mathbb{G}_0\mathbb{F}_{\mathrm{sp}}\mathbb{G}_0\mathbb{F}_{\mathrm{grad}}\right]+\frac{1}{3}\mathrm{Tr}\left[\mathbb{G}_0\mathbb{F}_{\mathrm{sp}}\mathbb{G}_0\mathbb{F}_{\mathrm{grad}}\mathbb{G}_0\mathbb{F}_{\mathrm{grad}}\right]+\ldots$$

[1] see eg. A. J. Leggett, in *Modern Trends in the Theory of Condensed Matter* (eds A. Pekalski and R. Przystawa , Springer, 1980).
[2] P. Nozières and S. Schmitt-Rink, J. Low Temp. Phys. **59**, 195 (1985); C. A. R. Sá de Melo, M. Randeria, and J. R. Engelbrecht, Phys. Rev. Lett. **71**, 3202 (1993).
[3] Kun Huang, Zeng-Qiang Yu, and Lan Yin, Phys. Rev. A 79, 053602 (2009).

The gradient expansion in the pair field

The thermodynamic potential is calculated in the functional integral formalism:

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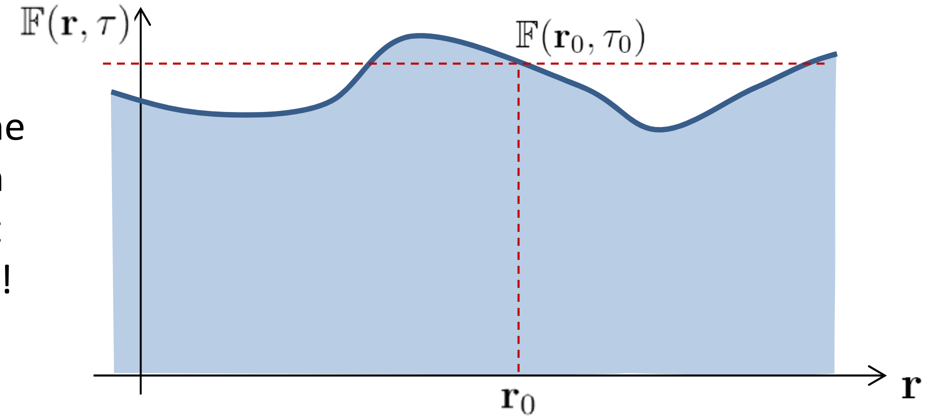
all others are kept as $\mathbb{F}_0$

expand at most 2 by $\mathbb{F}(\mathbf{r},\tau) = \mathbb{F}_0 + (\tau-\tau_0)\left.\frac{\partial\mathbb{F}}{\partial\tau}\right|_{\mathbf{r}_0,\tau_0} + (\mathbf{r}-\mathbf{r}_0)\cdot\nabla\mathbb{F}|_{\mathbf{r}_0,\tau_0} + \ldots$

We include all second order terms, neglecting third and higher order.

Here we assume that the pair fields vary slowly in time and space, but not necessarily around zero!



S.N. Klimin, J.Tempere, J.T. Devreese, Phys. Rev. A **90**, 053613 (2014).
S.N. Klimin, G.Lombardi, J.Tempere, J.T. Devreese, Eur. Phys. Journ. B **88**, 122 (2015) .

Effective field theory obtained after gradient expansion

The action obtained after the gradient expansion is the basis of our effective field theory:

$$S_{EFT}\left[\Psi(\mathbf{r},\tau)\right]=\int_0^\beta d\tau\int d\mathbf{r}\left\{\Omega_s+\frac{D}{2}\left(\frac{\partial\bar{\Psi}}{\partial\tau}\Psi-\bar{\Psi}\frac{\partial\Psi}{\partial\tau}\right)+C\left|\nabla_{\mathbf{r}}\Psi\right|^2-E\left(\nabla_{\mathbf{r}}\left|\Psi\right|^2\right)^2\right\}$$

Analytic results were obtained for the coefficients:

$$\Omega_s\left(|\Psi|\right)=-\frac{|\Psi|^2}{8\pi k_F a_s}-\int\frac{d\mathbf{k}}{(2\pi)^3}\left[\frac{1}{\beta}\ln\left[2\cosh\left(\beta E_{\mathbf{k}}\right)+2\cosh\left(\beta\zeta\right)\right]-\xi_{\mathbf{k}}-\frac{|\Psi|^2}{2k^2}\right]$$

$$C\left(|\Psi|\right)=\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{2k^2}{3}f_2\left(\beta,E_{\mathbf{k}},\zeta\right)$$

$$D\left(|\Psi|\right)=\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{\xi_{\mathbf{k}}^2}{|\Psi|^2}\left[f_1\left(\beta,\xi_{\mathbf{k}},\zeta\right)-f_1\left(\beta,E_{\mathbf{k}},\zeta\right)\right]$$

$$E\left(|\Psi|\right)=\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{4k^2}{3}\xi_{\mathbf{k}}^2 f_4\left(\beta,E_{\mathbf{k}},\zeta\right)$$

where $f_1\left(\beta,\varepsilon,\zeta\right)=\frac{1}{2\varepsilon}\frac{\sinh(\beta\varepsilon)}{\cosh(\beta\varepsilon)+\cosh(\beta\zeta)}$ and $f_{n+1}=-\frac{1}{2n\varepsilon}\frac{\partial f_n(\beta,\varepsilon,\zeta)}{\partial\varepsilon}$

and $E_{\mathbf{k}}=\sqrt{\xi_{\mathbf{k}}^2+|\Psi|^2}=\sqrt{\left(k^2-\mu\right)^2+|\Psi|^2}$

Results are given in units where $\hbar=2m=k_F=1$

For details on the derivation and a discussion of the $(\partial_\tau^2\Psi)$ and $(\partial_\tau\Psi)^2$ terms, see:
S.N. Klimin, J. Tempere, Devreese, European Physical Journal B **88**, 122 (2015).

Effective field theory compared with Ginzburg-Landau

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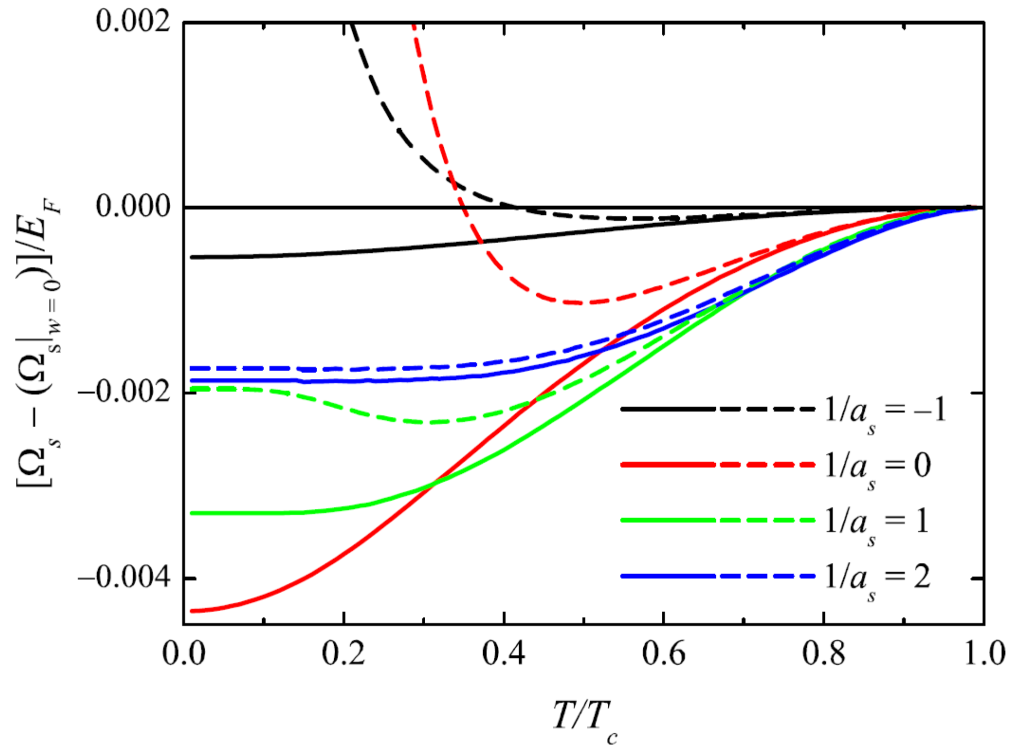
Check the results for against the Ginzburg-Landau energy functional (valid for $T \approx T_c$):

In the seminal BEC-BCS crossover paper [1], the authors propose a fluctuation expansion around $|\Psi|=0$, which corresponds to setting $E_{\mathbf{k}}\to\xi_{\mathbf{k}}$ in our coefficients. In this limit, our coefficient C corresponds to their "*c*" and the coefficients of $|\Psi|^2$ and $|\Psi|^4$ in Ω_s correspond to their $-a$ and b respectively.



[1] C.A.R. Sa de Melo, M. Randeria, and J.R. Engelbrecht, Phys. Rev. Lett. **71**, 3202 (1993).
Note that a more recent approach, K. Huang, Z.-Q. Yu and L. Yin, Phys. Rev. A **79**, 053602 (2009), expands the logarithm up to *p*=2 and performs a gradient expansion, whereas in our approach we take all powers *p* in the logarithm expansion into account.

Application to solitons or vortices

The effective field (real-time[1]) action yields the following Lagrangian

$$\mathcal{L}\left(\Psi,\partial_x\Psi\right)=i\frac{D}{2}\left(\bar{\Psi}\frac{\partial\Psi}{\partial\tau}-\frac{\partial\bar{\Psi}}{\partial\tau}\Psi\right)-\Omega_s-C\left|\nabla_{\mathbf{r}}\Psi\right|^2+E\left(\nabla_{\mathbf{r}}\left|\Psi\right|^2\right)^2$$

Before deriving the field equations, note that for localized excitations such as vortices or solitons, the order parameter may be written as

$$\Psi(\mathbf{r},t)=\left|\Psi_\infty\right|a(\mathbf{r},t)e^{i\theta(\mathbf{r},t)}$$

- $\left|\Psi_\infty\right|$: background amplitude
- $a(\mathbf{r},t)$: amplitude modulation
- $\theta(\mathbf{r},t)$: phase profile

The background amplitude and the chemical potentials are derived from the simultaneous solution of gap and number equations:

$$\frac{\partial\Omega_{sp}}{\partial\left(\left|\Psi_\infty\right|\right)}=0 \qquad n=-\frac{\partial\left(\Omega_{sp}+\Omega_{fl}\right)}{\partial\mu}$$

$$\delta n=-\frac{\partial\left(\Omega_{sp}+\Omega_{fl}\right)}{\partial\zeta}$$

[1] Going from the Euclidean time action to the real time action is performed by the usual formal replacements $\tau\rightarrow it$ and $S(\beta)\rightarrow -iS(t_b,t_a)$.

A first application: solitons and the filling up of the core

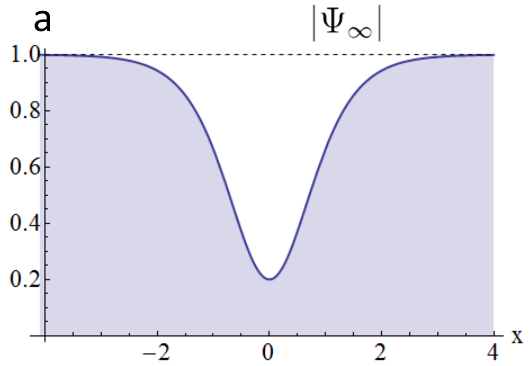
Application to solitons

The effective field action yields the following Lagrangian

$$\mathcal{L}\left(\Psi,\partial_x\Psi\right)=i\frac{D}{2}\left(\bar{\Psi}\frac{\partial\Psi}{\partial\tau}-\frac{\partial\bar{\Psi}}{\partial\tau}\Psi\right)-\Omega_s-C\left|\nabla_{\mathbf{r}}\Psi\right|^2+E\left(\nabla_{\mathbf{r}}\left|\Psi\right|^2\right)^2$$

In particular, for solitons:

$$\Psi(x,t)=\left|\Psi_\infty\right|\ a(x-v_st)e^{i\theta(x-v_st)}$$



Substitution of this form in the Lagrangian yield an effective Lagrangian for *a*(*x*) and θ(*x*) :

$$\mathcal{L}(a,\partial_x a;\theta,\partial_x\theta)=-\int dx\left\{\kappa(a)a^2v_s\partial_x\theta+\Omega_s-\frac{1}{2}\rho_{qp}\left(\partial_x a\right)^2-\frac{1}{2}\rho_{sf}\left(\partial_x\theta\right)^2\right\}$$

with :

$$\kappa(a)=D(a)\left|\Psi_\infty\right|^2$$

$$\rho_{sf}(a)=\frac{\left|\Psi_\infty\right|^2}{m}C(a)a^2$$

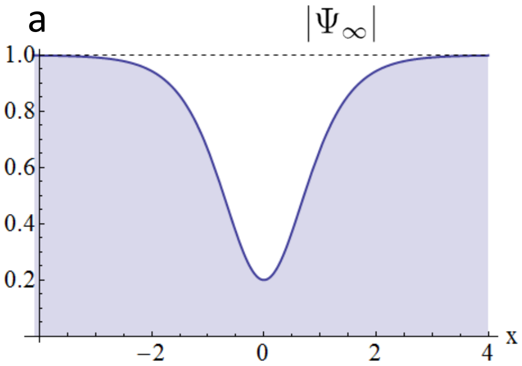
$$\rho_{qp}(a)=\frac{\left|\Psi_\infty\right|^2}{m}\left[C(a)-4\left|\Psi_\infty\right|^2a^2E(a)\right]$$

S.N. Klimin, J. Tempere, J.T. Devreese, Phys. Rev. A **90**, 053613 (2014); also at arXiv:1407.3107

Application to solitons

For solitons:

$$\Psi(x,t) = |\Psi_\infty|^2 a(x - v_s t) e^{i\theta(x - v_s t)}$$



The equations of motion resulting from $\mathcal{L}(a, \partial_x a; \theta, \partial_x \theta)$ can be solved analytically to obtain the relation between *x* and *a* :

$$x = \pm \int_{a_0}^{a} \sqrt{\frac{\rho_{qp}(a)\rho_{sf}(a)}{2\rho_{sf}(a)\Omega_s(a) - v_s^2 \left(\kappa(a)a^2 - \kappa_\infty\right)^2}} da$$

From this we also obtain the phase: $\theta(x) = v_s \int_{-\infty}^{x} \frac{\kappa_\infty / a^2 - \kappa(a)}{\rho_{sf}(a)} dx$

with still: $\kappa(a) = D(a) |\Psi_\infty|^2$

$$\rho_{sf}(a) = \frac{|\Psi_\infty|^2}{m} C(a) a^2$$

$$\rho_{qp}(a) = \frac{|\Psi_\infty|^2}{m} \left[C(a) - 4 |\Psi_\infty|^2 a^2 E(a)\right]$$

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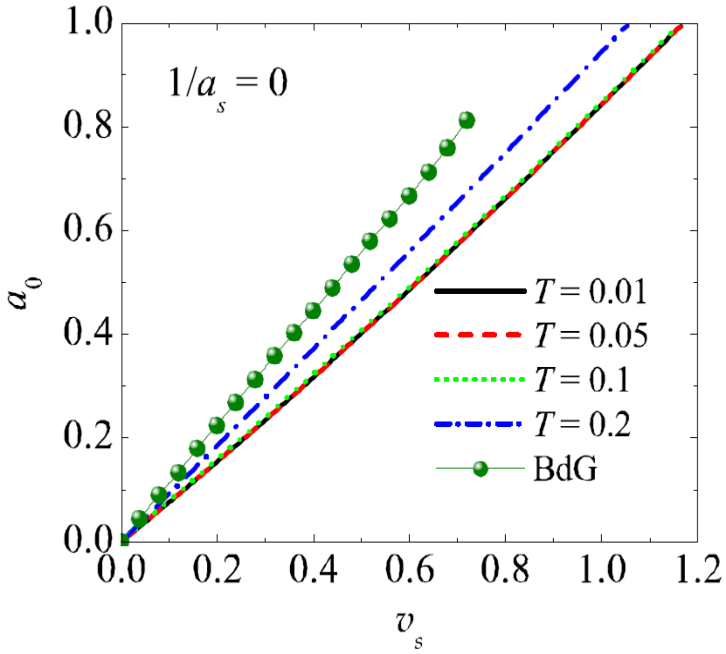
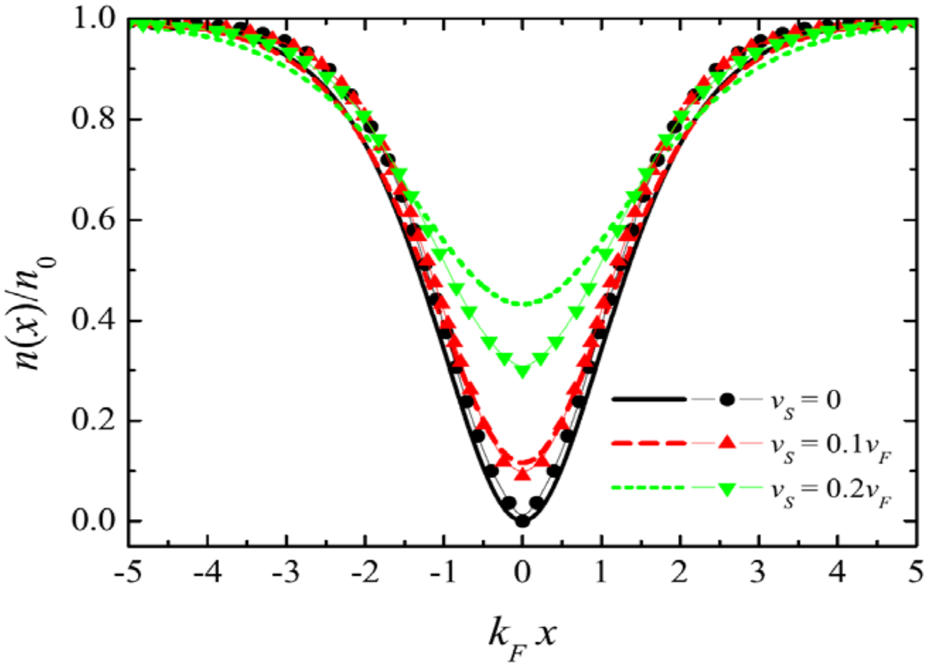
Application to solitons

The effective field (real-time[1]) action yields the following Lagrangian

$$\mathcal{L}\left(\Psi,\partial_x\Psi\right) = i\frac{D}{2}\left(\bar{\Psi}\frac{\partial\Psi}{\partial\tau} - \frac{\partial\bar{\Psi}}{\partial\tau}\Psi\right) - \Omega_s - C\left|\nabla_{\mathbf{r}}\Psi\right|^2 + E\left(\nabla_{\mathbf{r}}\left|\Psi\right|^2\right)^2$$

In particular, for solitons: $\Psi(x,t) = \left|\Psi_\infty\right|^2 a(x - v_s t)e^{i\theta(x - v_s t)}$





[1] S.N. Klimin, J. Tempere, J.T. Devreese, Phys. Rev. A **90**, 053613 (2014); also at arXiv:1407.3107

[2] R. Liao and J. Brand, Phys. Rev. A **83**, 041604 (2011).

Application to vortices in superfluid Fermi gases

Application to vortices

Back to the Lagrangian for the macroscopic wave function:

$$\mathcal{L}\left(\Psi,\partial_x\Psi\right)=i\frac{D}{2}\left(\bar{\Psi}\frac{\partial\Psi}{\partial\tau}-\frac{\partial\bar{\Psi}}{\partial\tau}\Psi\right)-\Omega_s-C\left|\nabla_{\mathbf{r}}\Psi\right|^2+E\left(\nabla_{\mathbf{r}}\left|\Psi\right|^2\right)^2$$

Just as for solitons, for localized excitations such as vortices, the order parameter may be written as

$$\Psi(\mathbf{r},t)=\left|\Psi_\infty\right|a(\mathbf{r},t)e^{i\theta(\mathbf{r},t)}$$

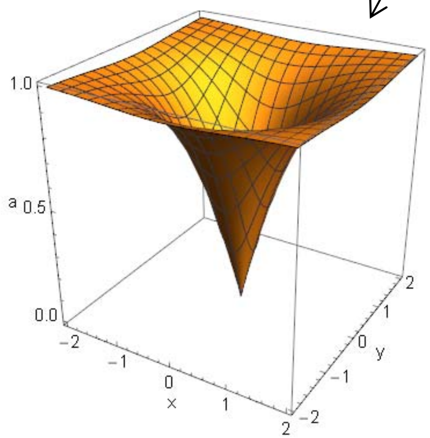
background amplitude

amplitude modulation

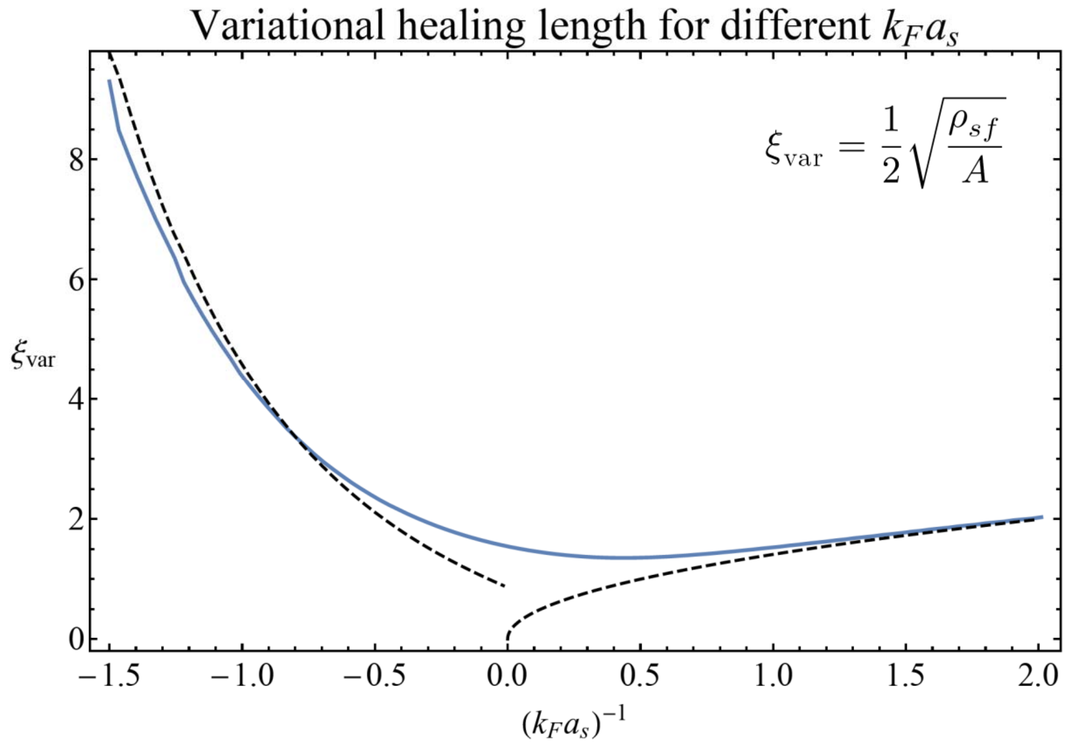
phase = angle around vortex line $e^{i\phi}$

Now there is no analytical solution for *a* – we use a variationally trial shape.

$$a(\mathbf{r})=\tanh\left[r/\left(\sqrt{2}\xi\right)\right]$$



Variational solution for vortex core size



The variational optimal value for ξ depends on the superfluid density $\rho_{sf} = 2C\,|\Psi_\infty|^2$ and the free energy required to make the vortex core:

$$A = \int_0^\infty u \left\{ \Omega_s \left[|\Psi_\infty|^2 \tanh^2\left(u/\sqrt{2}\right) \right] - \Omega_s \left(|\Psi_\infty|^2 \right) \right\} du$$

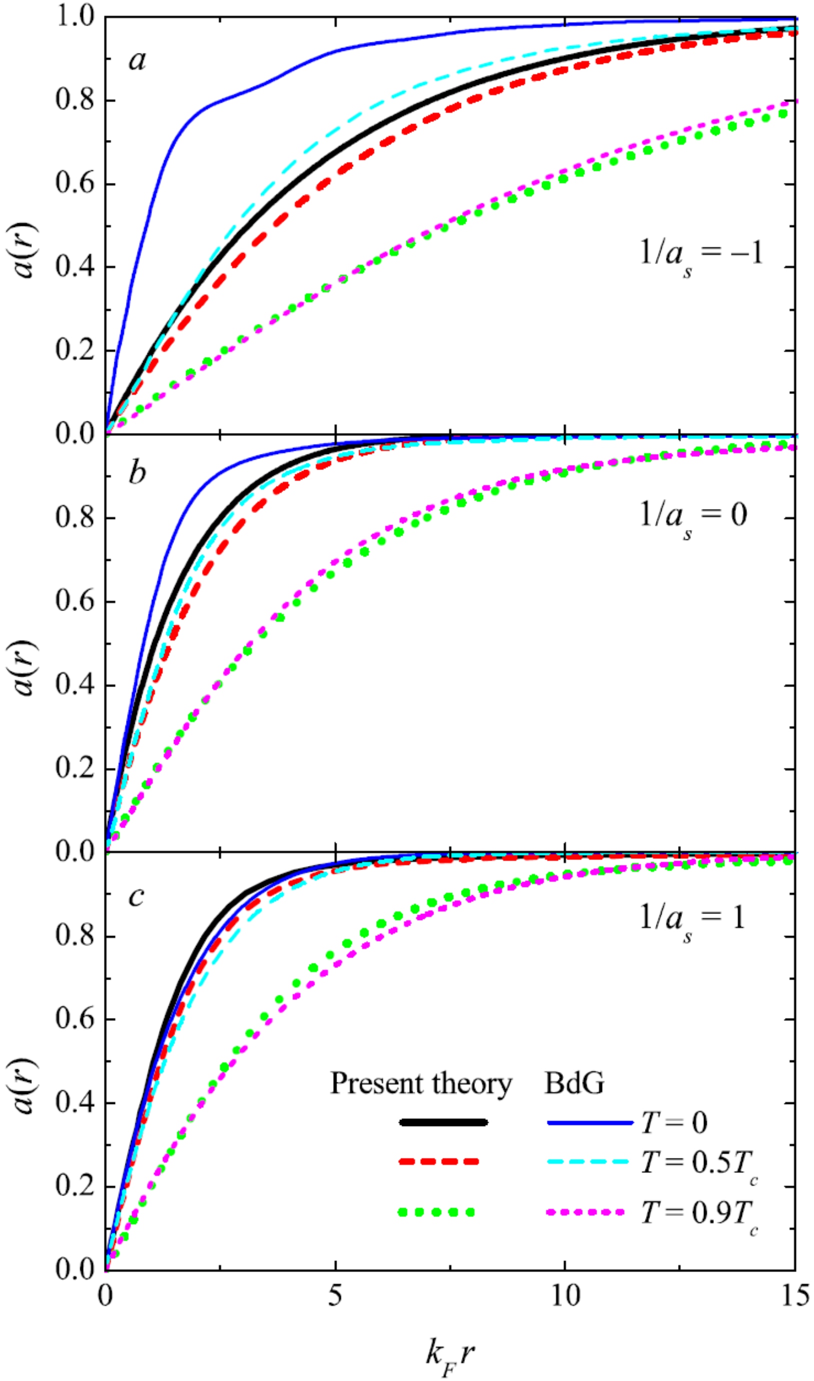
Dashed lines: L. P. Pitaevskii, Sov. Phys. JETP **13**, 451 (1961); M. Marini, F. Pistolesi and G.C. Strinati, Eur. Phys. J. B **1**,151 (1998).
Full line: N. Verhelst, S.N. Klimin, J.T. arXiv: 1603.02523 ; results in argreement with Palestrini and Strinati, Phys. Rev. B **89**, 224508 (2014).

Comparison with BdG at finite T

$$\Psi(r,\varphi,z) = |\Psi_\infty| \; a(r) e^{i\varphi}$$

For a finite-temperature vortex, the effective field theory [1] excellently matches the Bogoliubov – de Gennes solutions [2] in the BCS-BEC crossover everywhere except the BCS case combined with low temperatures.

Modulation of the order parameter amplitude in a vortex



[1] Klimin, Lombardi, JT and Devreese, Eur. Phys. Journ. B **88**, 122 (2015).
[2] S. Simonucci, P. Pieri, and G. C. Strinati, Phys. Rev. B **87**, 214507 (2013).

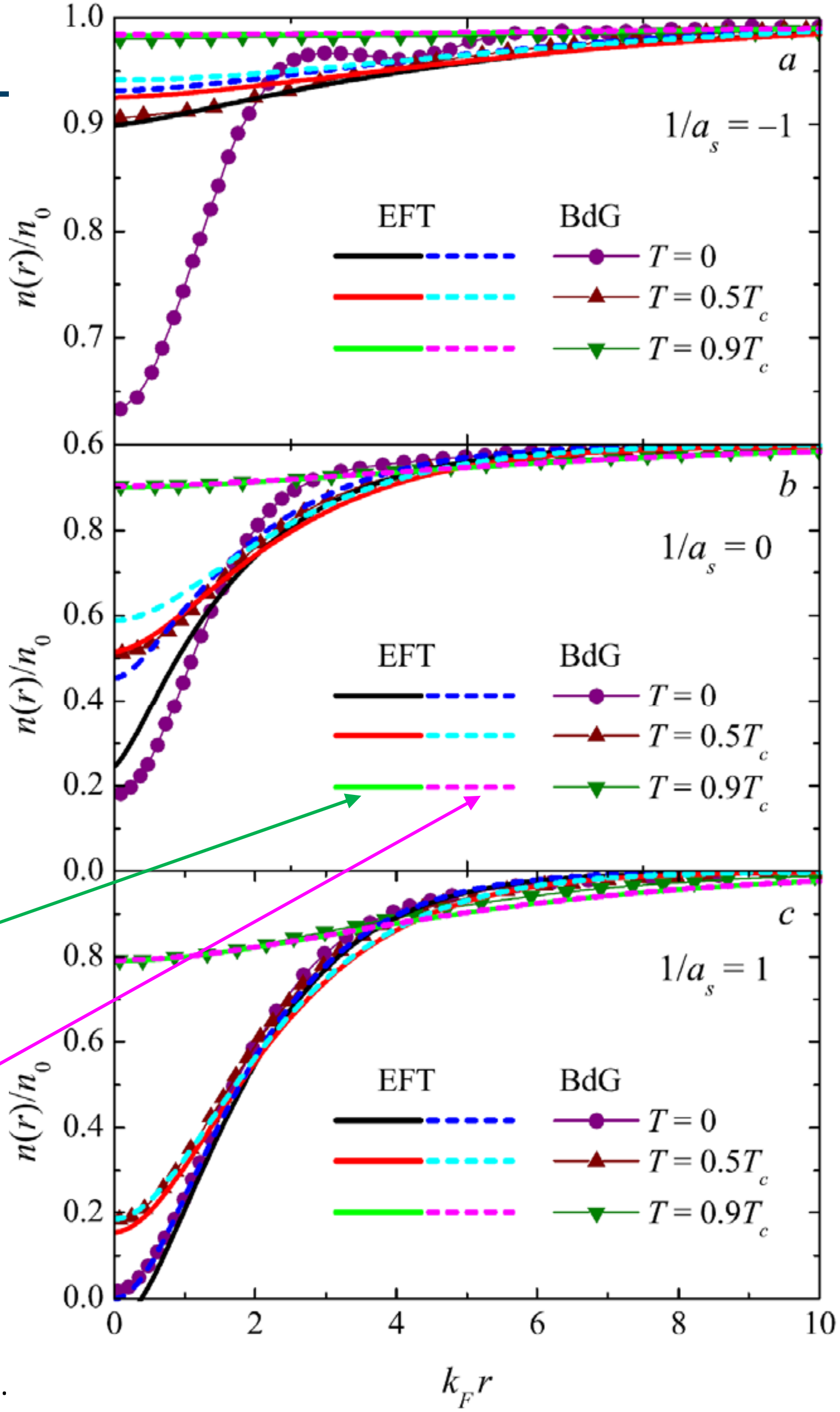
Density profiles

For a finite-temperature vortex, the effective field theory [1] excellently matches the Bogoliubov – de Gennes solutions [2] in the BCS-BEC crossover everywhere except the BCS case combined with low temperatures.

Particle density distribution in a vortex

Density calculated accounting for the gradient term in the effective action

Density calculated within the local density approximation



[1] Klimin, Lombardi, JT and Devreese, Eur. Phys. Journ. B **88**, 122 (2015).
[2] S. Simonucci, P. Pieri, and G. C. Strinati, Phys. Rev. B **87**, 214507 (2013).

Pair correlation length

The pair correlation function

$$g_{\uparrow\downarrow}(\boldsymbol{r}) = -\left(\frac{n}{2}\right)^2 + \left\langle \psi^\dagger_\uparrow\left(\boldsymbol{R}+\frac{\boldsymbol{r}}{2}\right)\psi^\dagger_\downarrow\left(\boldsymbol{R}-\frac{\boldsymbol{r}}{2}\right)\psi_\downarrow\left(\boldsymbol{R}-\frac{\boldsymbol{r}}{2}\right)\psi_\uparrow\left(\boldsymbol{R}+\frac{\boldsymbol{r}}{2}\right)\right\rangle$$

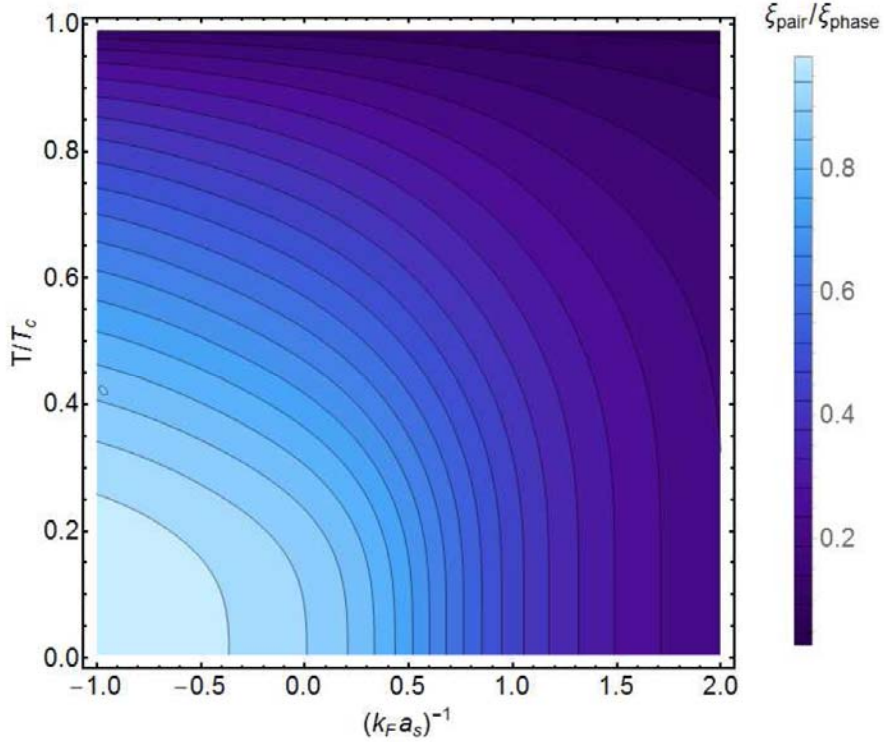
allows to define the pair correlation length[1]

$$\xi_{pair} = \sqrt{\frac{\int \mathrm{d}\boldsymbol{r}\, \boldsymbol{r}^2 g_{\uparrow\downarrow}(\boldsymbol{r})}{\int \mathrm{d}\boldsymbol{r}\, g_{\uparrow\downarrow}(\boldsymbol{r})}}$$

Taking the expectation value with respect to the gradient-expanded action yields[2]:

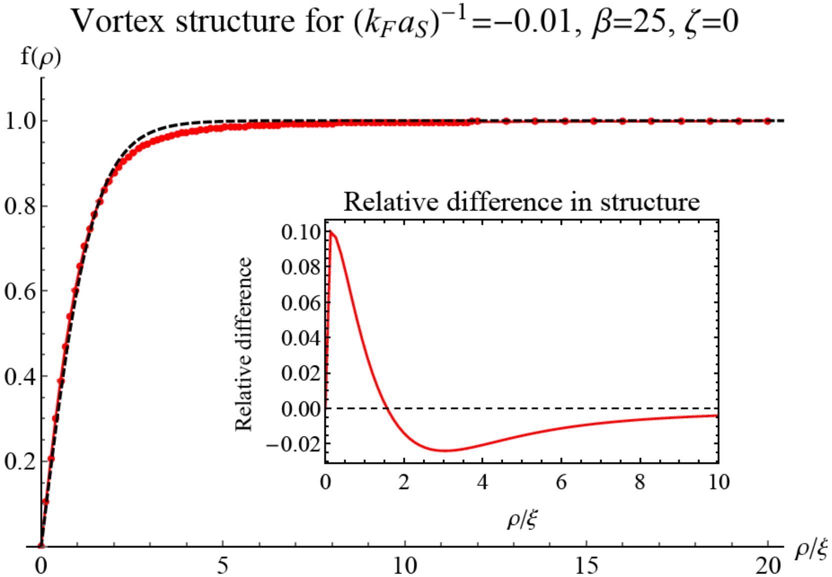
$$\xi_{pair} = \sqrt{\frac{\int \mathrm{d}k\, k^2 \left(4k\, \xi_{\boldsymbol{k}} f_2\left(\beta, E_{\boldsymbol{k}}, \zeta\right)\right)^2}{\int \mathrm{d}k\, k^2 \, \left(f_1\left(\beta, E_{\boldsymbol{k}}, \zeta\right)\right)^2}}$$

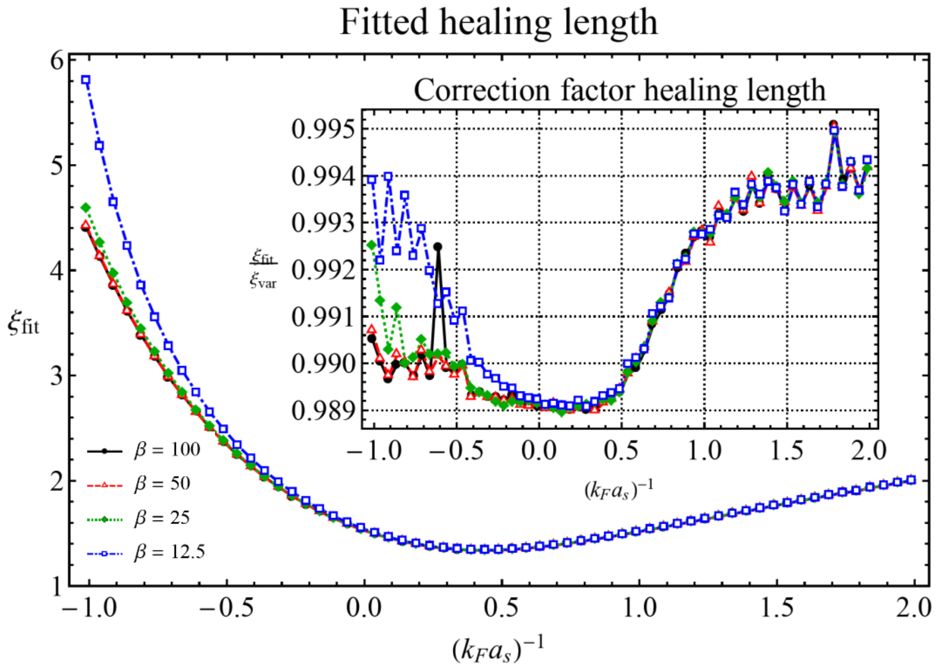
The gradient expansion is expected to hold if the size of the spatial variations of the macroscopic wave function, ξ_{phase}, is larger than the pair correlation length ξ_{pair}.



[1] F. Palestini and G. C. Strinati, Phys. Rev. B **89**, 224508 (2014).
[2] G. Lombardi, W. Van Alphen, S.N. Klimin, J.Tempere, Phys. Rev. A **93**, 013614 (2016)

How good is the hyperbolic tangent?

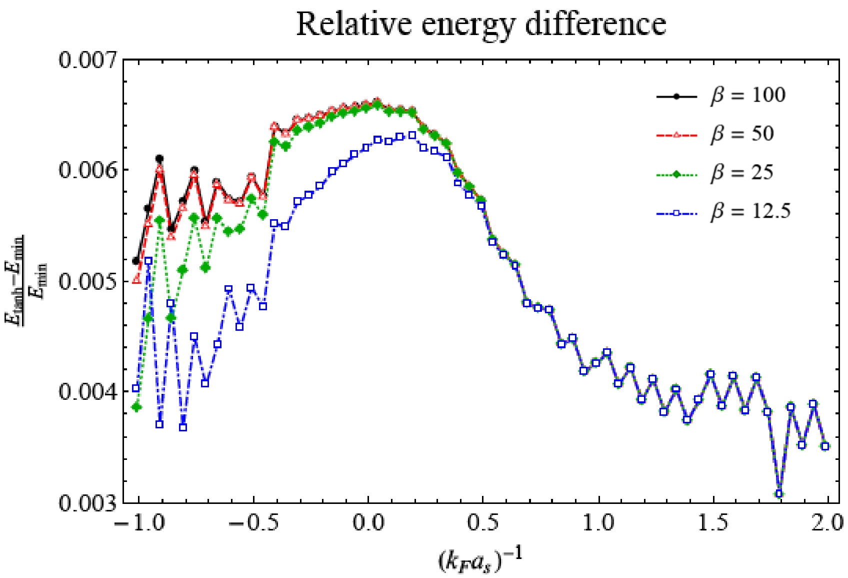


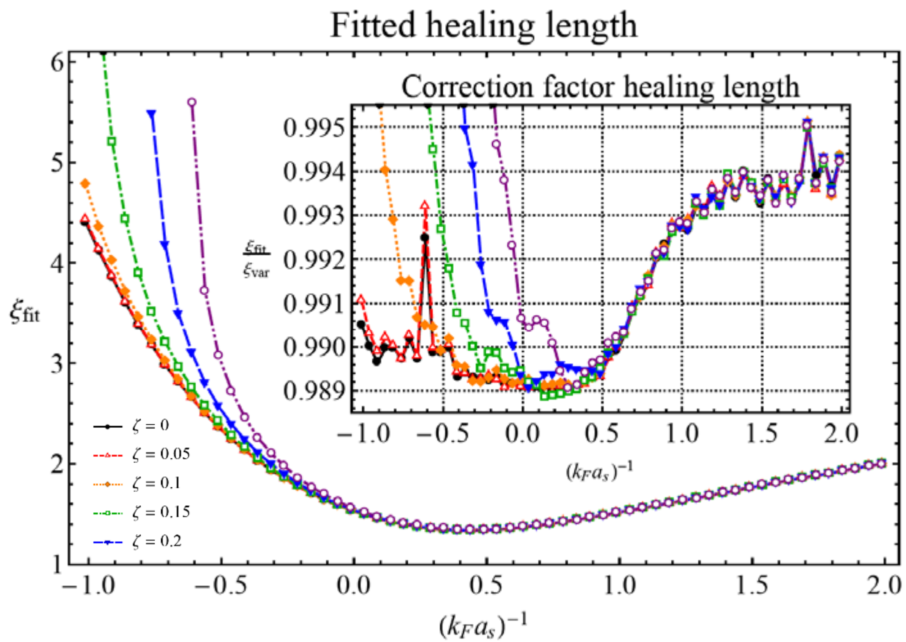


- BdG[1] finds oscillations around a tanh profile in low-T, deep BCS regime – here this is never observed.

Our results: N. Verhelst, S.N. Klimin, J.T. arXiv: 1603.02523

How good is the hyperbolic tangent?





- BdG[1] finds oscillations around a tanh profile in low-T, deep BCS regime – here this is never observed.
- Imbalance increases the deviation from a hyperbolic-tangent form, especially on the BCS side, it also increases the size of the vortex core.

[1] S. Simonucci, P. Pieri, G.C. Strinati, Phys. Rev. B 87, 214507 (2013).
Our results: N. Verhelst, S.N. Klimin, J.T. arXiv: 1603.02523

Critical rotation frequencies for vortices and for superfluidity

Rotating Fermi gases

At the non-interacting, single-particle level, rotating the quantum gas leads to

$$H = \frac{\left[\boldsymbol{\nabla}_{\mathbf{r}} - i\mathbf{A}(\mathbf{r})\right]^2}{2m} + \frac{m\left(\omega_{trap}^2 - \omega^2\right)}{2} r_\perp^2 + \frac{m\omega_{trap,z}^2}{2} z^2$$

with a rotational "vector potential" $\mathbf{A}(\mathbf{r}) = m\left(\boldsymbol{\omega} \times \mathbf{r}\right)$.

One could think that at the level of the effective field theory, rotations can be implemented through the "canonical" subsitution

$$\left|\boldsymbol{\nabla}_{\mathbf{r}}\Psi\right|^2 \rightarrow \left|\left[\boldsymbol{\nabla}_{\mathbf{r}} - 2i\mathbf{A}(\mathbf{r})\right]\Psi\right|^2$$

However, this is wrong. The rotational "charge" need not be twice the atom's.

Rotating Fermi gases

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with a rotational "vector potential" $\mathbf{A}(\mathbf{r}) = m\left(\boldsymbol{\omega} \times \mathbf{r}\right)$.

In the formalism, the rotation does come in through $\mathbb{F}$, but it appears via $\mathbb{G}_0$:

$$\mathbb{G}_0^{-1} = \begin{pmatrix} i\omega_n - \epsilon_{\mathbf{k}} & 0 \\ 0 & i\omega_n + \epsilon_{\mathbf{k}} \end{pmatrix}$$

$\epsilon_{\mathbf{k}} = k^2$, $\zeta_{\mathbf{k}} = \zeta - 2\mathbf{k} \cdot \mathbf{A}(\mathbf{r})$

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with a rotational “vector potential” $\mathbf{A}(\mathbf{r}) = m\left(\boldsymbol{\omega} \times \mathbf{r}\right)$.

Performing the gradient expansion with the changed $\mathbb{G}_0$ yields to leading order

$$\int d\mathbf{r}\left[C\left(\boldsymbol{\nabla}_{\mathbf{r}}\bar{\Psi}\right)\left(\boldsymbol{\nabla}_{\mathbf{r}}\Psi\right)\right] \rightarrow \int d\mathbf{r}\left[C\left(\boldsymbol{\nabla}_{\mathbf{r}}\bar{\Psi}\right)\left(\boldsymbol{\nabla}_{\mathbf{r}}\Psi\right) + iD\mathbf{A}\cdot\left(\bar{\Psi}\boldsymbol{\nabla}_{\mathbf{r}}\Psi - \Psi\boldsymbol{\nabla}_{\mathbf{r}}\bar{\Psi}\right)\right]$$

$$= \int d\mathbf{r}\left[C\left|\left[\boldsymbol{\nabla}_{\mathbf{r}} - i\tilde{e}\mathbf{A}(\mathbf{r})\right]\Psi\right|^2 - Ci\tilde{e}^2 A^2\left|\Psi\right|^2\right]$$

$$\boxed{\tilde{e} = \frac{D}{C} = \frac{\int d\mathbf{k}\ \epsilon_{\mathbf{k}}\left[f_1(\beta,\epsilon_{\mathbf{k}},\zeta_{\mathbf{k}}) - f_1(\beta,E_{\mathbf{k}},\zeta_{\mathbf{k}})\right]/\left|\Psi_\infty\right|^2}{(2/3)\int d\mathbf{k}\ k^2 f_2(\beta,E_{\mathbf{k}},\zeta_{\mathbf{k}})}}$$

where $f_1\left(\beta,\varepsilon,\zeta\right) = \frac{1}{2\varepsilon}\frac{\sinh(\beta\varepsilon)}{\cosh(\beta\varepsilon)+\cosh(\beta\zeta)}$ and $f_{n+1} = -\frac{1}{2n\varepsilon}\frac{\partial f_n(\beta,\varepsilon,\zeta)}{\partial\varepsilon}$

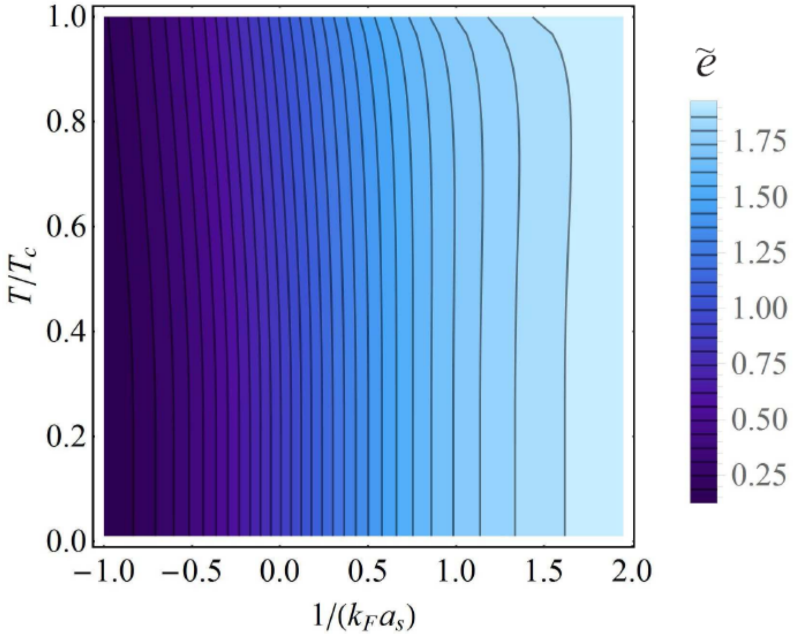
and $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \left|\Psi_\infty\right|^2} = \sqrt{\left(k^2-\mu\right)^2 + \left|\Psi_\infty\right|^2}$

Rotating Fermi gases

At the non-interacting, single-particle level, rotating the quantum gas leads to

$$H = \frac{\left[\nabla_{\mathbf{r}} - i\mathbf{A}(\mathbf{r})\right]^2}{2m} + \frac{m\left(\omega_{trap}^2 - \omega^2\right)}{2} r_\perp^2 + \frac{m\omega_{trap,z}^2}{2} z^2$$

with a rotational "vector potential" $\mathbf{A}(\mathbf{r}) = m\left(\boldsymbol{\omega} \times \mathbf{r}\right)$.



$$\tilde{e} = \frac{D}{C} = \frac{\int d\mathbf{k}\ \epsilon_{\mathbf{k}} \left[f_1(\beta, \epsilon_{\mathbf{k}}, \zeta_{\mathbf{k}}) - f_1(\beta, E_{\mathbf{k}}, \zeta_{\mathbf{k}})\right] / \left|\Psi_\infty\right|^2}{(2/3) \int d\mathbf{k}\ k^2 f_2(\beta, E_{\mathbf{k}}, \zeta_{\mathbf{k}})}$$

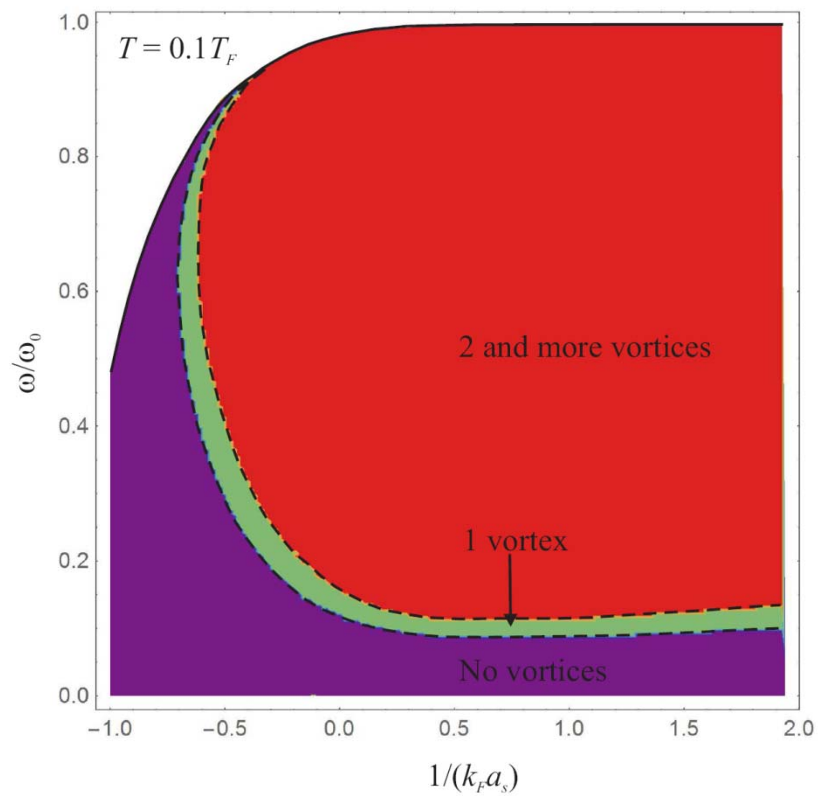
The rotational "charge" (actually rotational moment of inertia) is not necessarily equal to twice that for atoms.

Rotating Fermi gases

Substituting $\Psi(\mathbf{r},t) = |\Psi_\infty|\, a(\mathbf{r},t) e^{i\theta(\mathbf{r},t)}$ one finds the free energy

$$F = \int d\mathbf{r} \left\{ \Omega_s + \frac{1}{2}\rho_{qp} \left[\boldsymbol{\nabla}_{\mathbf{r}} a(\mathbf{r}) \right]^2 + \frac{1}{2}\rho_{sf} a^2(\mathbf{r}) \left[\boldsymbol{\nabla}_{\mathbf{r}} \theta(\mathbf{r}) - \tilde{e}\mathbf{A}(\mathbf{r}) \right]^2 - \frac{1}{2}\rho_{sf} a^2(\mathbf{r}) \tilde{e}^2 \mathbf{A}^2(\mathbf{r}) \right\}$$

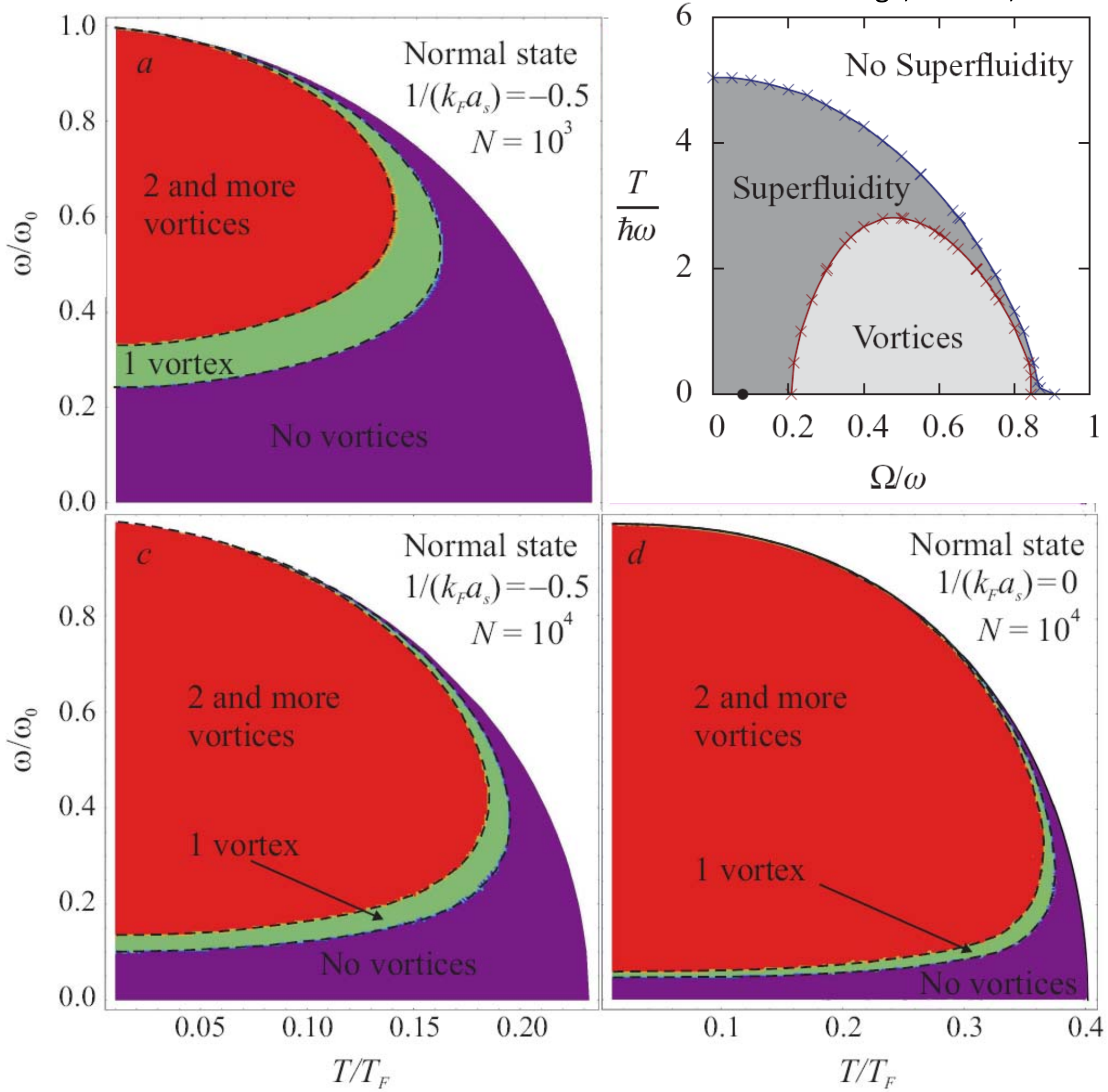
with $\rho_{qp} = 2C\,|\Psi_\infty|^2$ and $\rho_{sf} = 2\,(C - 4E)\,|\Psi_\infty|^2$



Parabolic trap with frequency ω_0

Rotating Fermi gases

H.J. Warringa and A. Sedrakian, PRA **84**, 023609 (2011).
H.J. Warringa, PRA **86**, 043615 (2012).



Rotating Fermi gases

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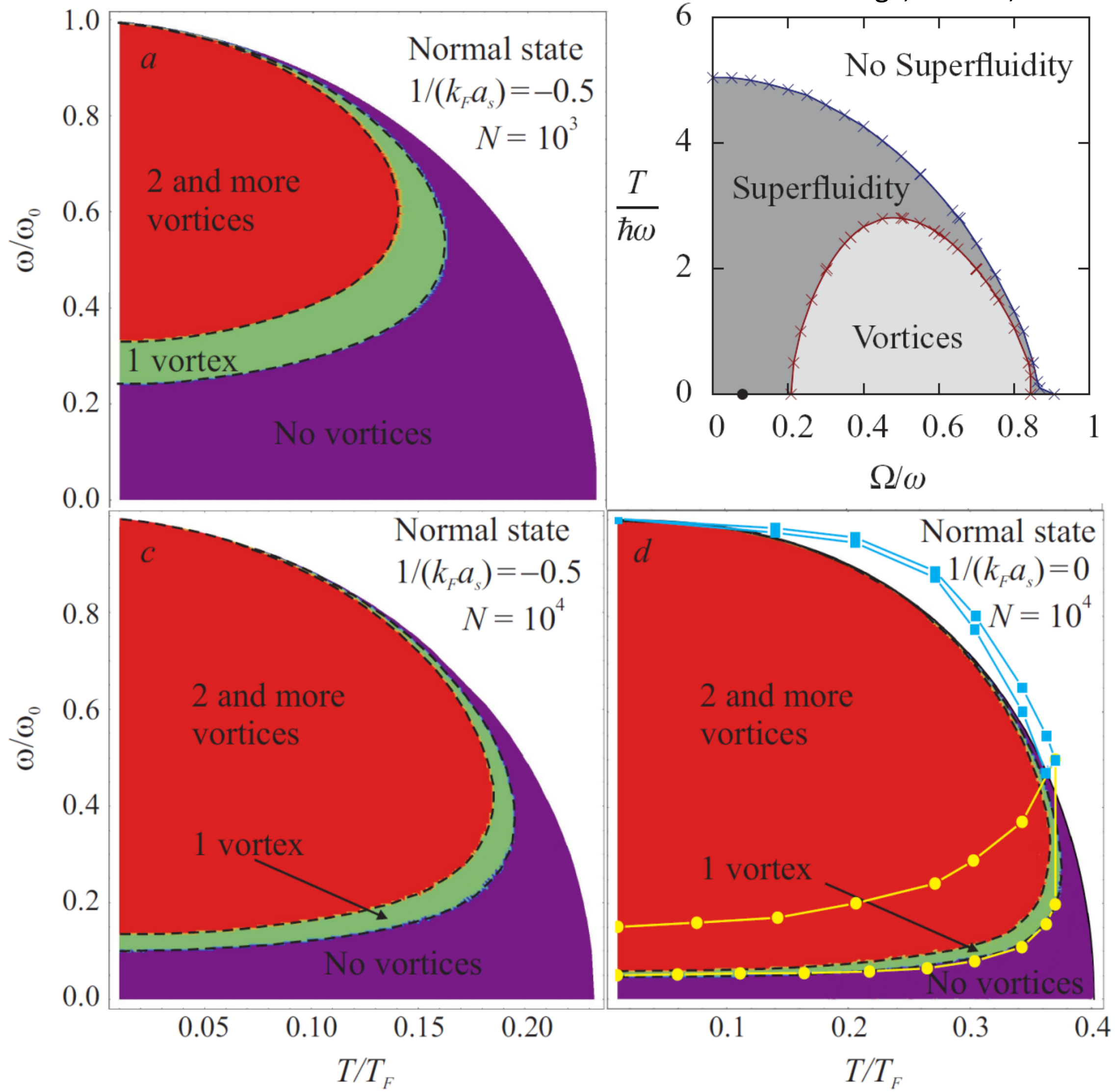

"Coarse-grained" BdG : S. Simonucci and G. C. Strinati, Phys. Rev. B **89**, 054511 (2014),
Applied to rotating gases in Simonucci, Pieri, Strinati, Nat. Phys. **11**, 941 (2015), arXiv: 1509.01130

Conclusions

Development of an description in terms of a macroscopic order parameter for superfluid Fermi gases, valid in the BEC-BCS crossover and for a wide temperature range.

The coefficients in this “Ginzburg-Landau” type of description are related to the microscopic parameters of the Fermi gas.

This description allows to model vortices and solitons well. Next: vortex matter and multivortex dynamics.

Description of the extended gradient expansion and the resulting effective field theory:
extending Ginzburg-Landau for fermi gases: Physica C **503**, 136 (2014) - arXiv: 1508.04693
derivation of the effective field theory: Eur. Phys. Journ. B **88**, 122 (2015) - arXiv: 1309.1421

Solitons:
in the BEC-BCS crossover: Phys.Rev. A **90**, 053613 (2014) - arXiv: 1407.3107
core filling by imbalance: Phys. Rev. A **93**, 013614 (2016) - arXiv: 1506.02527

Vortices:
core profile: arXiv:1512.00214
in rotating Fermi gases: arXiv: 1603.02523

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