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Adiabatic invariants and some statistical properties of the time-dependent linear and nonlinear oscillators

Marko Robnik

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— In Honour of Professor Hermann Haken —

0. Abstract

1. Introduction: *General about adiabatic invariants*

2. Review of the general time dependent linear oscillator and its statistical properties

3. Time dependent nonlinear oscillators:

General and exact considerations and The case of the quartic oscillator

4. Parametrically kicked systems:

The case of a single kick: *General aspects and power law potentials*

The case of kick and anti-kick

The case of periodic kicking

0. PREVIEW

• **adiabatic invariants** denoted by I: long history, important applications, classically and quantally, but rarely rigorous: Einstein and Lorentz 1911

• they are conserved under very slow changes over a time interval of length T

• definition: **adiabatic parameter** $\epsilon = \frac{1}{T}$: **the ideal adiabatic limit:** $\epsilon \to 0$

• 1D harmonic oscillator $\ddot{q} + \omega^2(t)q = 0$: **general** $\omega(t)$

• if $\epsilon \to 0$: it is known (Einstein 1911): $I = E(t)/\omega(t)$

• define an initial ensemble of phase points with sharp energy E_0:
the microcanonical ensemble of initial conditions

• distribution of final energy $P(E_1)$ after time $t = T$: **universal distribution**:
$P(E_1) = \frac{1}{\pi\sqrt{2\mu^2 - x^2}}$, where $x = E_1 - \bar{E}_1$

• $P(E_1)$ is fully determined by the first moment $\bar{E}_1 =$ average final energy

• the variance: $\mu^2 = \frac{E_0^2}{2}\left((\frac{\bar{E}_1}{E_0})^2 - (\frac{\omega_1}{\omega_0})^2\right)$ **for any** $\omega(t)$

• **Positivity of the action growth in the mean**: $\mu^2 = \frac{E_0^2}{2}\left((\frac{\bar{E}_1}{E_0})^2 - (\frac{\omega_1}{\omega_0})^2\right) \geq 0$ and $\bar{E}_1 \geq \omega_1 E_0/\omega_0$ or $I_1 = \bar{E}_1/\omega_1 \geq I_0 = E_0/\omega_0$, and equality holds only if $T = \infty$ or $\epsilon = 0$ (**ideal adiabaticity**).

• finite T: calculate $\bar{E}_1$ and μ^2 in general case by exact WKB-theory to all orders

• **Nonlinear oscillators:** the distribution function can be generally different, although qualitatively similar in adiabatic case, but the positivity of growth of the action at the mean final energy is lost at small ϵ, and nevertheless reappears at larger ϵ (small T), in the limit of parametric kicks $T = 0$.

• Interesting are **resonant phenomena** in the case of a parametric kick and anti-kick, as well as the case of periodic kicking, where the growth of the mean energy can be indefinite.

1. Introduction

Hamilton systems: Phase space (q, p)

Phase flow: $(q_0, p_0) \to (q_1, p_1)$

Hamilton function $H = H(q, p, t)$

Hamilton equations: $\dot{q} = \frac{\partial H}{\partial p} \quad \dot{p} = -\frac{\partial H}{\partial q}$

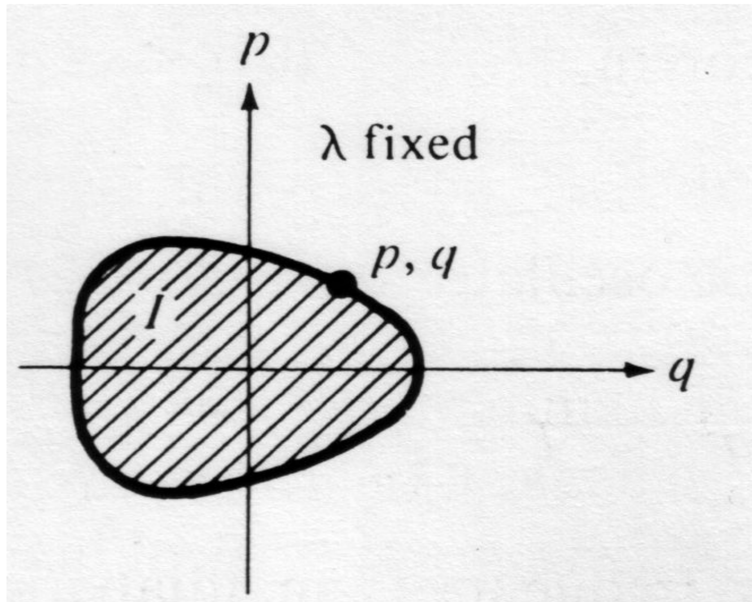
Energy evolution: $\dot{E} = \frac{dE}{dt} = \frac{dH}{dt} = \underbrace{\frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p}}_{=0} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$

Therefore: The energy E is constant only when $\frac{\partial H}{\partial t} = 0$ (autonomous systems)

Liouville theorem: Phase space volume is *always* preserved: phase space flow velocity vector field has zero divergence ("incompressible flow")

In general, in nonautonomous Hamilton systems, the energy $E = E(t) = H(t)$ changes with time.

An **adiabatic invariant** I can be exactly conserved for $\epsilon = 0$: $2\pi I$ is exactly the area in the phase plane (q, p) enclosed by the energy contour of constant E.



In 1-dim system with $\omega(t) \neq 0$ the adiabatic invariant I is

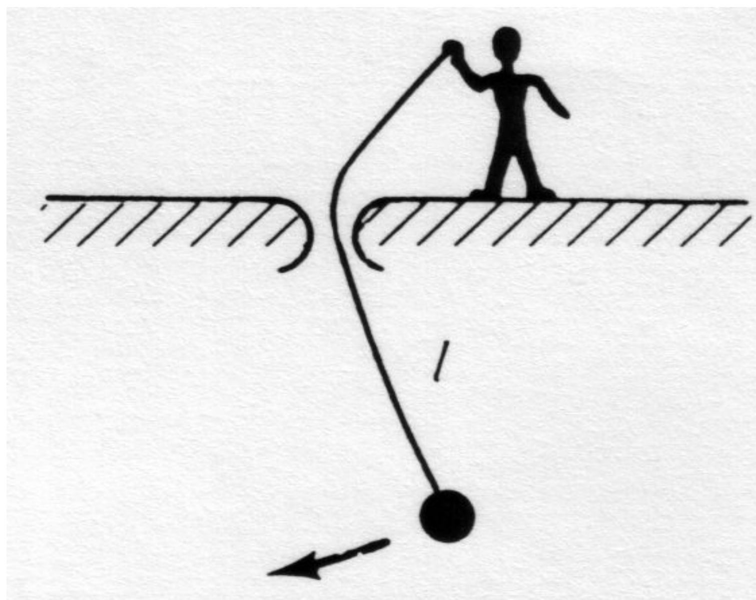
$$I = \frac{1}{2\pi} \oint_{E=H(q,p,t)} p.dq \tag{1}$$

for 1D harmonic oscillator: $I = E(t)/\omega(t) = const.$ if $T = \infty$

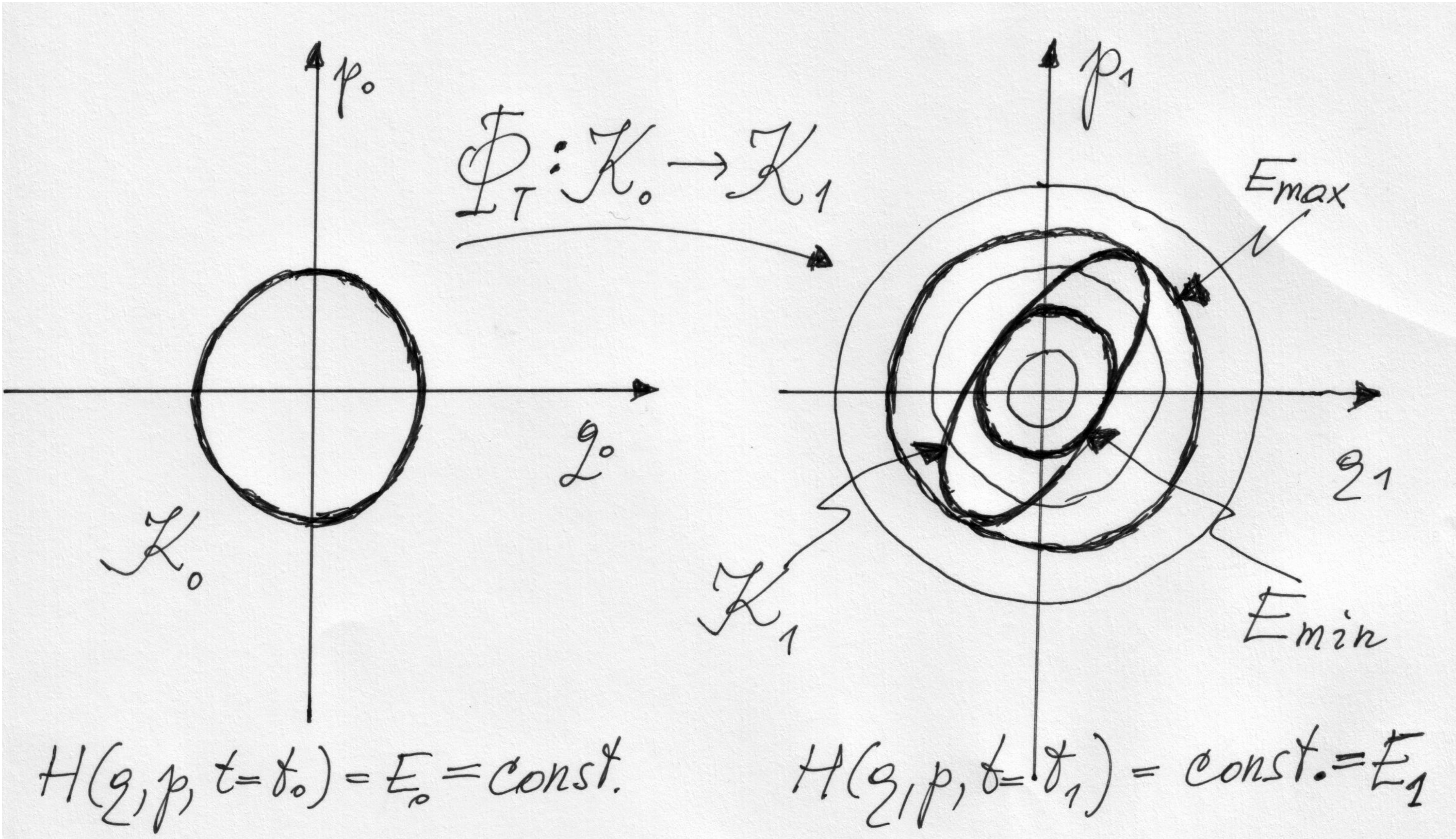
2. Review of the general time dependent linear oscillator and its statistical properties

One-dimensional harmonic oscillator: $\ddot{q} + \omega^2(t)q = 0$

Example: small oscillations of a mathematical pendulum:
$\omega^2(t) = g/l(t)$, where g is the gravitational acceleration and $l(t) =$ the length of the pendulum at time t



Lorentz and Einstein (a paper published in 1911): The adiabatic invariant is
$I = E(t)/\omega(t)$ (= phase space area$/2\pi$)



p₀
q₀
K₀
$\Phi_T: K_0 \to K_1$
p₁
q₁
Emax
Emin
K₁
$H(q,p,t=t_0)=E_0=const.$
$H(q,p,t=t_1)=const.=E_1$

Phase flow map and general exact considerations

The Hamilton function: $H = H(q,p,t) = \frac{p^2}{2M} + \frac{1}{2}M\omega^2(t)q^2$

q, p, M, ω are coordinate, momentum, mass and the frequency.

The numerical value of $H(t)$ is the energy of the system $E(t)$ at time t.

The equation of motion is linear: $\ddot{q} + \omega^2(t)q = 0$

We define the **phase flow map**: $\Phi : \begin{pmatrix} q_0 \\ p_0 \end{pmatrix} \mapsto \begin{pmatrix} q_1 \\ p_1 \end{pmatrix}$.

It is **a linear area preserving map:** $\Phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so that $\mathrm{Det}(\Phi) = \mathrm{ad} - \mathrm{bc} = 1$.

Let $E_0 = H(q_0, p_0, t = t_0)$ be the initial energy and $E_1 = H(q_1, p_1, t = t_1)$ be the final energy, that is,

$$E_1 = \frac{1}{2}\left(\frac{(cq_0 + dp_0)^2}{M} + M\omega_1^2(aq_0 + bp_0)^2\right).$$

We want to study the distribution $P(E_1)$ of final energy E_1.

Define the **microcanonical ensemble of initial conditions:**

$q_0 = \sqrt{\frac{2E_0}{M\omega_0^2}} \cos\phi,\ p_0 = \sqrt{2ME_0}\sin\phi$, where the action is $I_0 = \frac{E_0}{\omega_0}$

Then we obtain: $E_1 = E_0(\alpha\cos^2\phi + \beta\sin^2\phi + \gamma\sin 2\phi)$

with: $\alpha = \frac{c^2}{M^2\omega_0^2} + a^2\frac{\omega_1^2}{\omega_0^2}, \quad \beta = d^2 + \omega_1^2 M^2 b^2, \quad \gamma = \frac{cd}{M\omega_0} + abM\frac{\omega_1^2}{\omega_0}.$

By definition: The distribution of the initial angle variable ϕ is uniform (constant) and equal to $1/(2\pi)$.

The mean value of E_1: $\bar{E}_1 = \frac{1}{2\pi}\oint E_1 d\phi = \frac{E_0}{2}(\alpha+\beta)$.

$x =_{def} E_1 - \bar{E}_1 = E_0(\delta\cos 2\phi + \gamma\sin 2\phi)$, $\delta = \frac{1}{2}(\alpha-\beta)$.

The variance: $\mu^2 = \overline{(E_1-\bar{E}_1)^2} = \frac{E_0^2}{2}\left(\delta^2+\gamma^2\right) = \frac{E_0^2}{2}\left[\left(\frac{\bar{E}_1}{E_0}\right)^2 - \left(\frac{\omega_1}{\omega_0}\right)^2\right]$.

Odd moments: $\overline{(E_1-\bar{E}_1)^{2m-1}} = 0$ Even m.: $\overline{(E_1-\bar{E}_1)^{2m}} = (2m-1)!!\mu^{2m}/m!$

If $m\to\infty$: $\to 2^m/\sqrt{\pi m}$ (to compare with Gaussian: $\to 2^m\Gamma(m+1/2)/\sqrt{\pi}$)

The distribution of final energies E_1: $P(E_1) = \frac{1}{2\pi}\sum_{j=1}^{4}\left|\frac{d\phi}{dE_1}\right|_{\phi=\phi_j(E_1)}$

$P(E_1)$ is distributed on (E_{min}, E_{max}), it is an even function w.r.t. $\bar{E}_1 = (E_{min}+E_{max})/2$, and has $1/\sqrt{x}$ singularity at $E_{min} = \bar{E}_1 - \mu\sqrt{2}$ and $E_{max} = \bar{E}_1 + \mu\sqrt{2}$.

The result: $P(E_1) = \frac{1}{\pi\sqrt{2\mu^2 - x^2}}$, where $x = E_1 - \bar{E}_1$

The variance: $\mu^2 = \overline{(E_1 - \bar{E}_1)^2} = \frac{E_0^2}{2}\left[\left(\frac{\bar{E}_1}{E_0}\right)^2 - \left(\frac{\omega_1}{\omega_0}\right)^2\right] \geq 0$ is positive definite.

Therefore in full generality: $\bar{E}_1 \geq E_0\omega_1/\omega_0$

The final value of the adiabatic invariant (for the average energy!) $\bar{I}_1 = \bar{E}_1/\omega_1$ is always greater or equal to the initial value $I_0 = E_0/\omega_0$.

In other words, the value of the adiabatic invariant at the mean value of the energy never decreases, which is a kind of **irreversibility statement in the mean**.

Moreover, it is conserved only for infinitely slow processes $T = \infty$, which is an ideal adiabatic process, for which $\mu^2 = 0$.

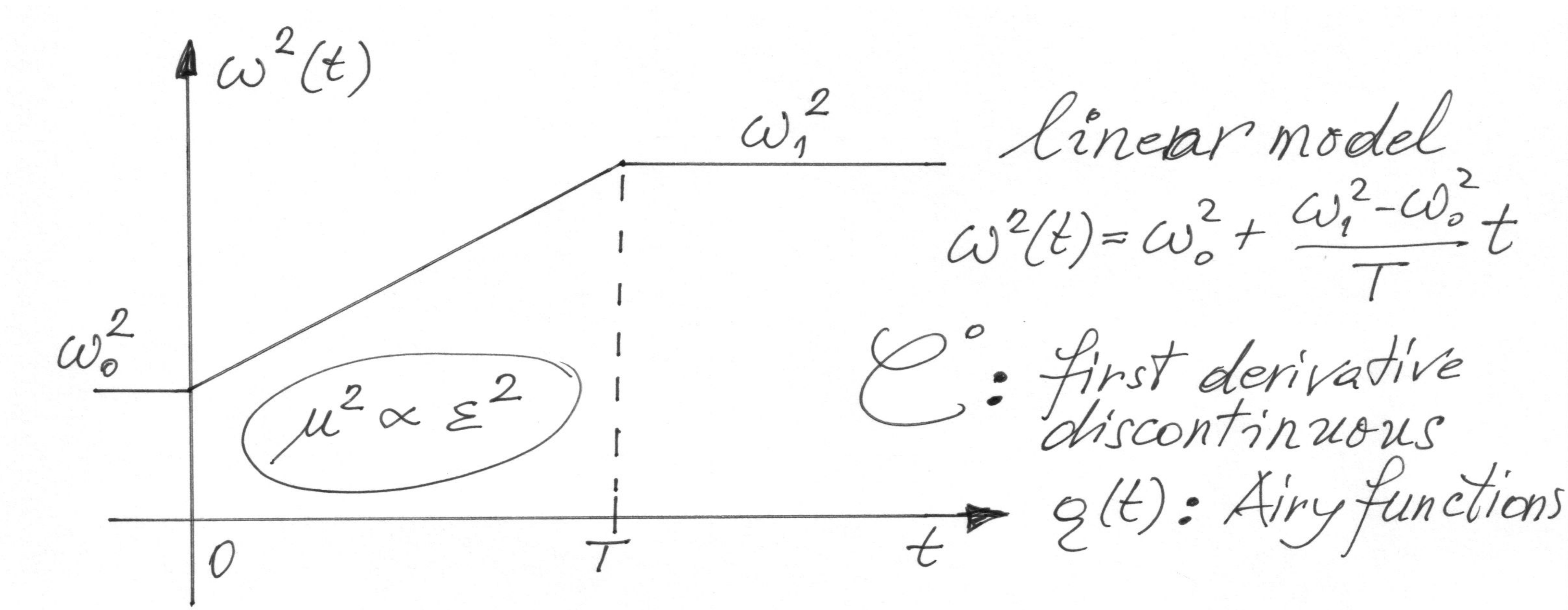
The other extreme opposite to $T = \infty$ is the instantaneous ($T = 0$) jump where ω_0 switches to ω_1 discontinuously (**parametric kick**), whilst q and p remain continuous, and this results in $a = d = 1$ and $b = c = 0$, and then we find

$\bar{E}_1 = \frac{E_0}{2}(\frac{\omega_1^2}{\omega_0^2} + 1), \quad \mu^2 = \frac{E_0^2}{8}\left[\frac{\omega_1^2}{\omega_0^2} - 1\right]^2$. Later: $\omega_1^2 = 2\omega_0^2$, and $\mu^2/E_0^2 = 1/8$.

An exactly solvable case: linear model for $\omega^2(t)$

We assume that function $\omega^2(t)$ is a piecewise linear function of the form

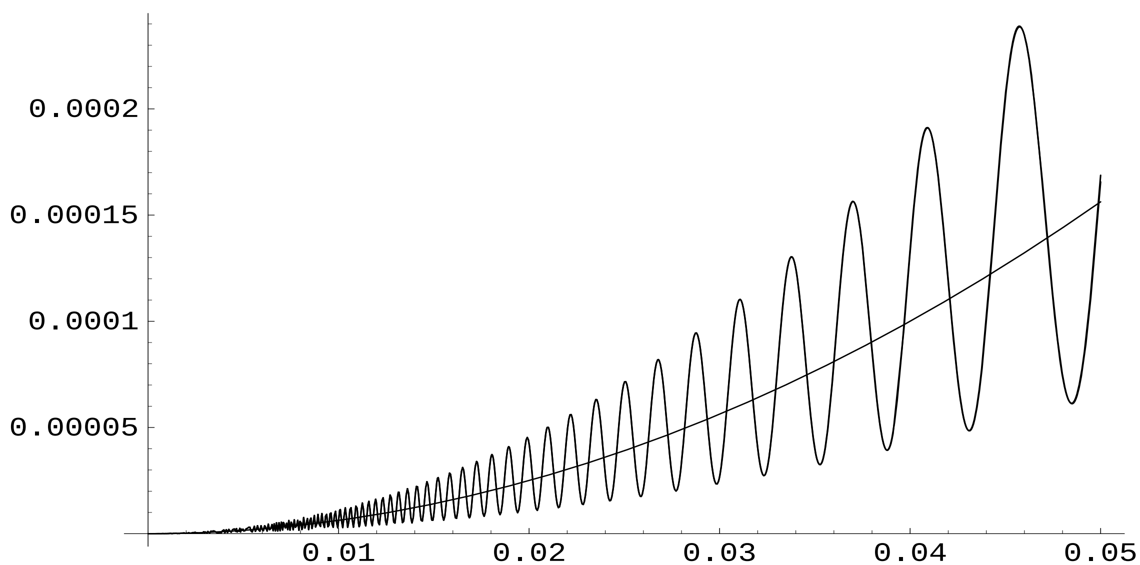
$$\omega^2(t) = \begin{cases} \omega_0^2 & \text{if} \quad t \leq 0 \\ \omega_0^2 + \frac{(\omega_1^2 - \omega_0^2)}{T} t & \text{if} \quad 0 < t < T \\ \omega_1^2 & \text{if} \quad t \geq T \end{cases} . \tag{2}$$



For $\omega_0^2 = 1, \omega_1^2 = 2$, $E_0 = 1$, using the asymptotic expansion of Abramowitz, we obtain the following approximation

$$\mu^2 = \overline{(E_1 - \bar{E}_1)^2} \approx \frac{\epsilon^2}{128} \left(9 - 4\sqrt{2}\,\cos(\frac{4 - 8\sqrt{2}}{3\,\epsilon}) \right), \tag{3}$$

where $\epsilon = \frac{1}{T}$ is the **adiabatic parameter** .



$\mu^2 = \overline{(E_1 - \bar{E}_1)^2}$ for $0 < \epsilon < 0.05$; the lines of the exact expression and the asymptotics practically coincide; the non-oscillating thin line is the parabola $y = \frac{9}{128}\epsilon^2$.

Using the WKB method in linear oscillator: The variance of the energy

$$\frac{\overline{(\Delta E_1)^2}}{E_0^2} = \epsilon^2 \left(\frac{\omega_1^2 \omega'(\lambda_0)^2}{8\omega_0^6} - \frac{\cos\left(\frac{2\int_{\lambda_0}^{\lambda_1} \omega(x)\, dx}{\epsilon}\right) \omega'(\lambda_0)\omega'(\lambda_1)}{4\omega_0^4} + \frac{\omega'(\lambda_1)^2}{8\omega_0^2\omega_1^2} \right) + O(\epsilon^3). \tag{4}$$

$$\frac{\mu^2}{E_0^2} = \frac{\overline{(\Delta E_1)^2}}{E_0^2} = \frac{\epsilon^{2n}}{2^{2n+1}} \left(\frac{\omega_1^2 (\omega_0^{(n)})^2}{\omega_0^{2(n+2)}} + \frac{(\omega_1^{(n)})^2}{(\omega_1)^{2n}\omega_0^2} - 2\frac{\omega_0^{(n)}\omega_1^{(n)}}{\omega_0^{n+3}\omega_1^{n-1}} \cos\left(\frac{2s_1}{\epsilon}\right)\right) + O(\epsilon^{2n+1}). \tag{5}$$

$s_1 = s(\lambda_1) = \int_{\lambda_0}^{\lambda_1} B(x) dx, \quad B(\lambda) = -\mathrm{i} \sum_{k=0}^{\infty} \epsilon^{2k} \sigma'_{2k,+}(\lambda)$

3. Time dependent nonlinear oscillators

The problem: Determine the distribution function of the final energy $P(E_1)$ for an initial microcanonical ensemble, and its main parameters, the mean energy $\bar{E}_1$, the variance μ^2, also the adiabatic invariant at the average energy $I(\bar{E}_1)$.

The property of the non-negative change of the adiabatic invariant at the average energy $I(\bar{E}_1)$ upon the variation of the system parameter is in general lost.

Nevertheless, it is definitely restored for sufficiently large value of the adiabatic parameter ϵ, in particular in a discontinuous change of the system parameter a **parametric kick**, where $\epsilon = \infty$

$H(q,p,t_0) \to H(q,p,t)$

$H_0 = H_0(I_0, \theta_0) = H_0(I_0) \to H_1 = H_1(I_1, \theta_1) = H_1(I_1)$

$(I_0, \theta_0) \to (I_1, \theta_1)$: $I_1 = \sum_{-\infty}^{\infty} a_m e^{im\theta_0}, \quad \theta_1 = \theta_0 + \sum_{-\infty}^{\infty} b_m e^{im\theta_0}$

$\mathcal{A}_1 = \frac{1}{2}\int r_1^2 \, d\theta_1 = \int I_1 \, d\theta_1 = 2\pi a_0 - 2\pi i \sum_{-\infty}^{\infty} a_m b_{-m} m = 2\pi I_0$

Poisson bracket = Jacobian determinant $= \frac{\partial I_1}{\partial \theta_0}\frac{\partial \theta_1}{\partial I_0} - \frac{\partial I_1}{\partial I_0}\frac{\partial \theta_1}{\theta_0} = 1$ (area preserving map)

If higher Fourier terms are zero or neglected: $I_1 = a_0 + a_1 e^{i\theta_0} + a_{-1} e^{-i\theta_0}$

We see immediately that a_0 is the average final action $a_0 = \bar{I}_1$, and writing $a_1 = \Delta I e^{i\alpha}/2$, we find

$$I_1 = \bar{I}_1 + \Delta I \cos(\theta_0 + \alpha).$$

Assuming now that the angle θ_0 is uniformly distributed on the interval $(0, 2\pi)$ with the density $1/(2\pi)$, we derive the quaislinear probability density distribution

$$P(I_1) = \frac{dW}{dI_1} = \frac{1}{\pi} \frac{1}{\sqrt{(\Delta I)^2 - (I_1 - \bar{I}_1)^2}}$$

where the variance of I_1 is equal to $(\Delta I)^2/2$.

If the variance is small, then locally I and E are linearly related by the relationship $\Delta E = (dE/dI)\Delta I$, where $\omega = dE/dI$ is the oscillation frequency at the energy E or action I and we have:

$$P(E_1) = \frac{dW}{dE_1} = \frac{1}{\pi} \frac{1}{\sqrt{2\mu^2 - (E_1 - \bar{E}_1)^2}}$$ = the same as for the linear oscillator

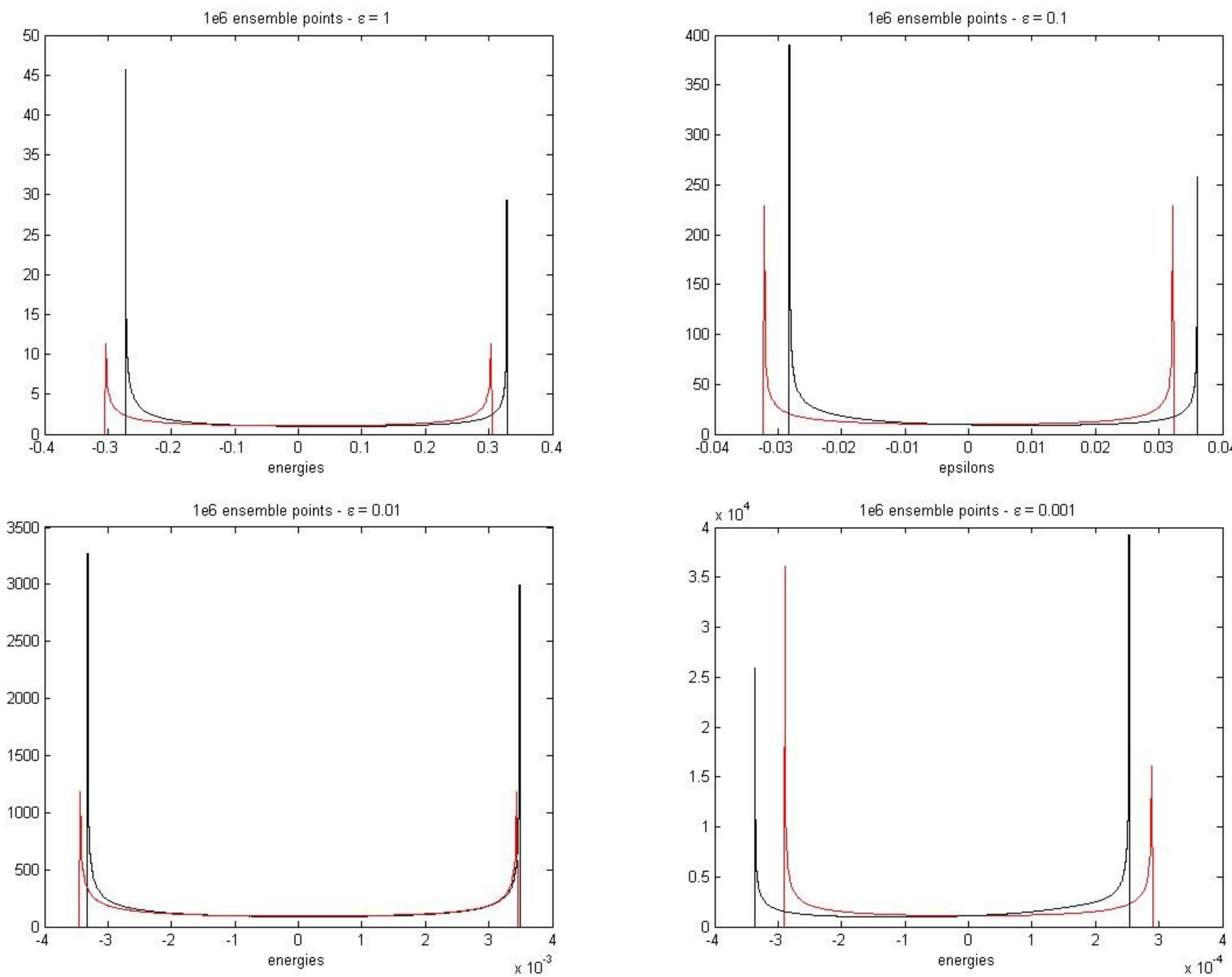
Numerical calculations for the quartic oscillator

$H(q,p,t) = \frac{1}{2}p^2 + \frac{a(t)}{4}q^4,$

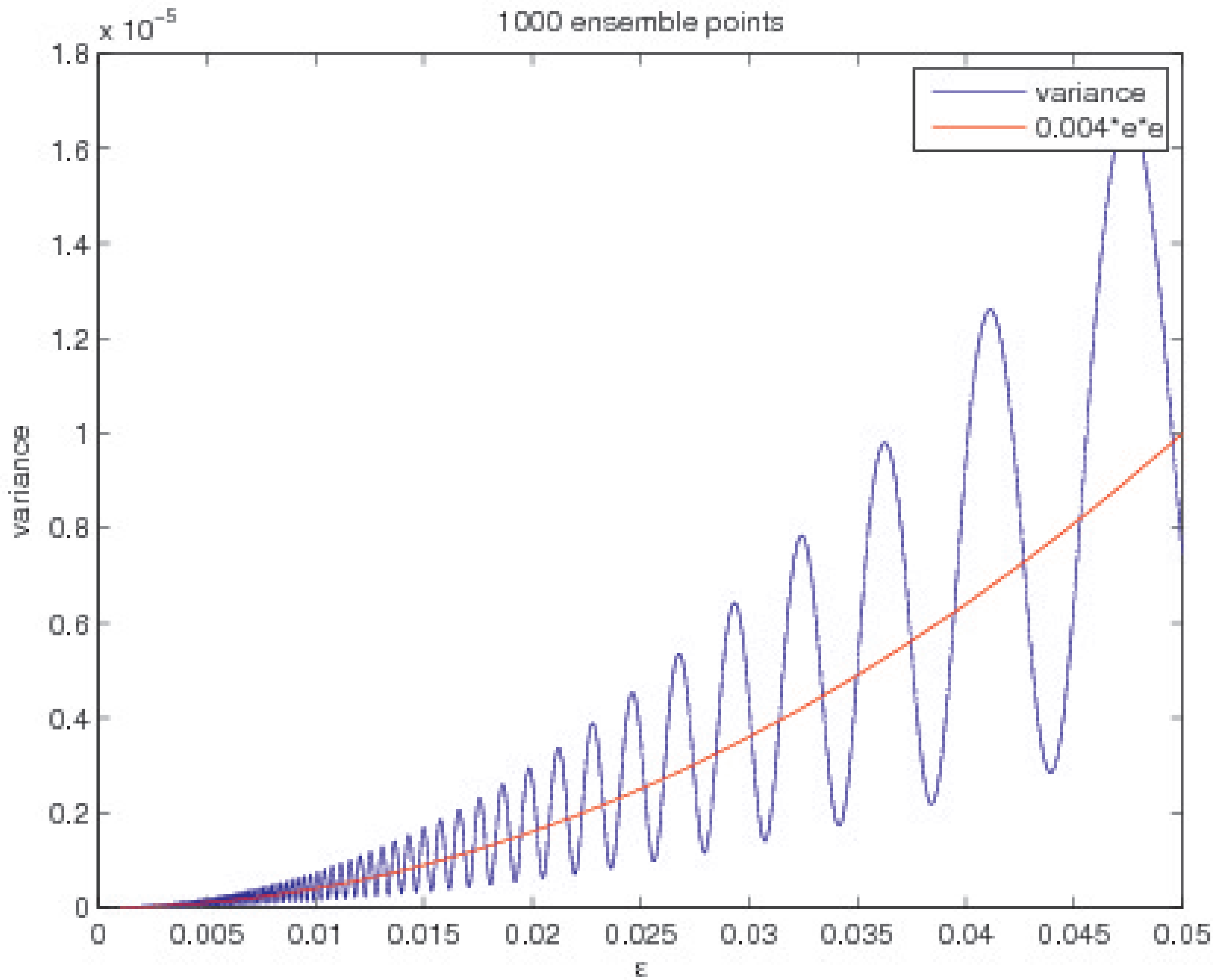
$\epsilon = 1/T$ (the adiabatic parameter) so far an arbitrary positive real number

$$a(t) = \begin{cases} 1 & \text{if} \quad t \le 0 \\ 1+\epsilon t & \text{if} \quad 0 < t < T \\ 2 & \text{if} \quad t \ge T \end{cases} .$$

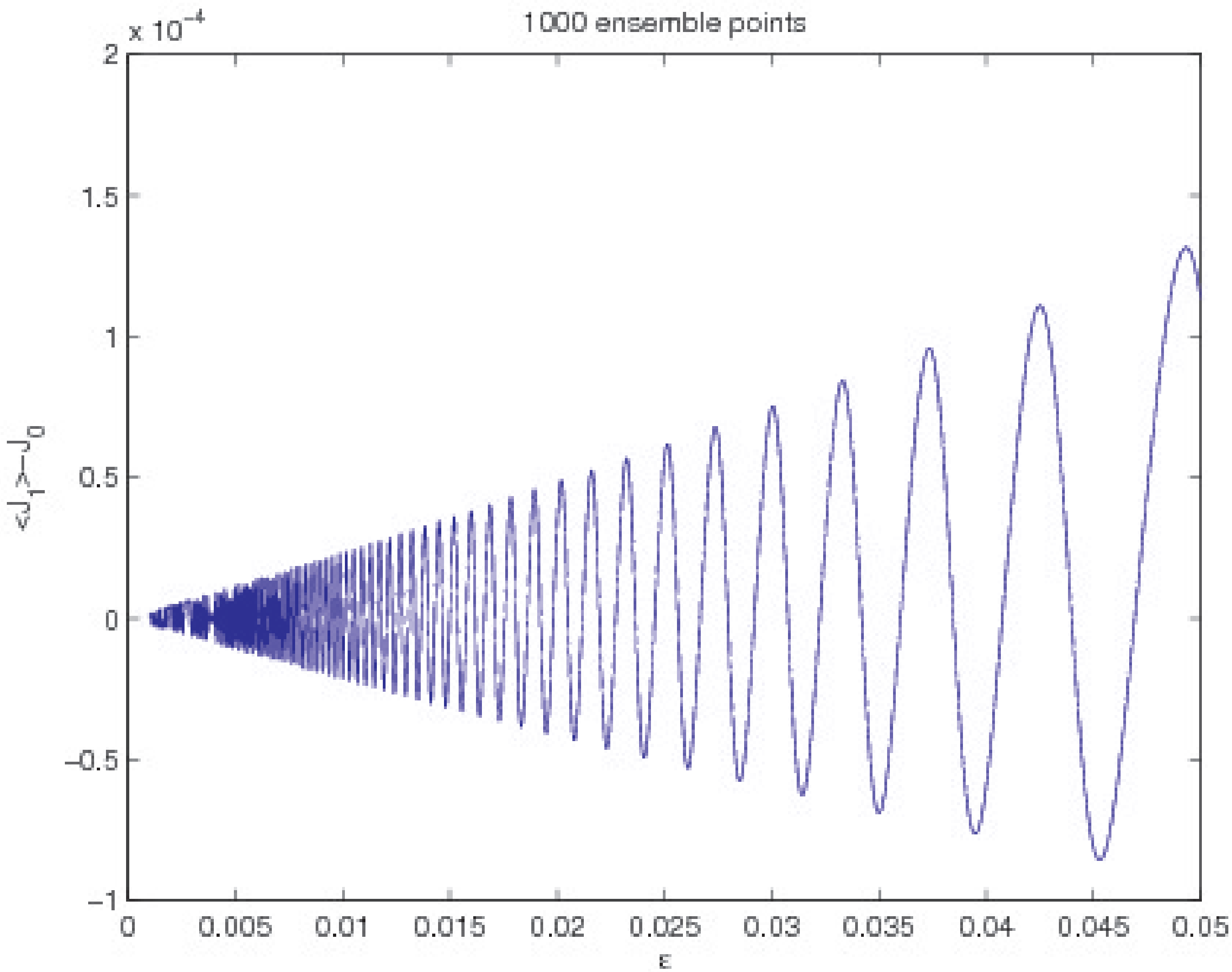
Numerical integration using the 8th order symplectic integrator due to R.I. McLachlan and G.R.W. Quispel, *Acta Numerica* **11** 341 (2002)



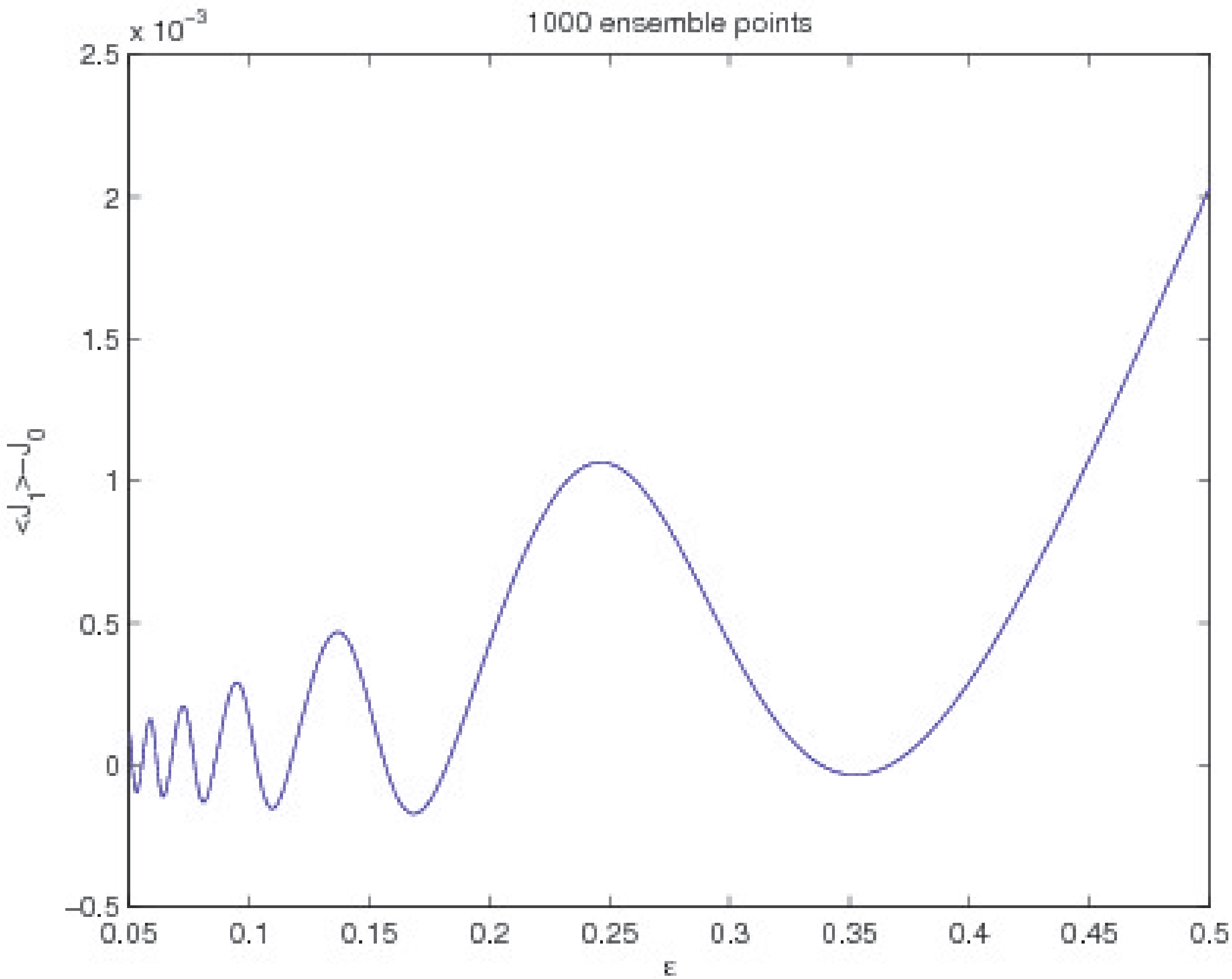
The energy distribution function for the quartic oscillator for four different values of the adiabatic parameter $\epsilon = 1, 0.1, 0.01$ and 0.001. Linear model for a.



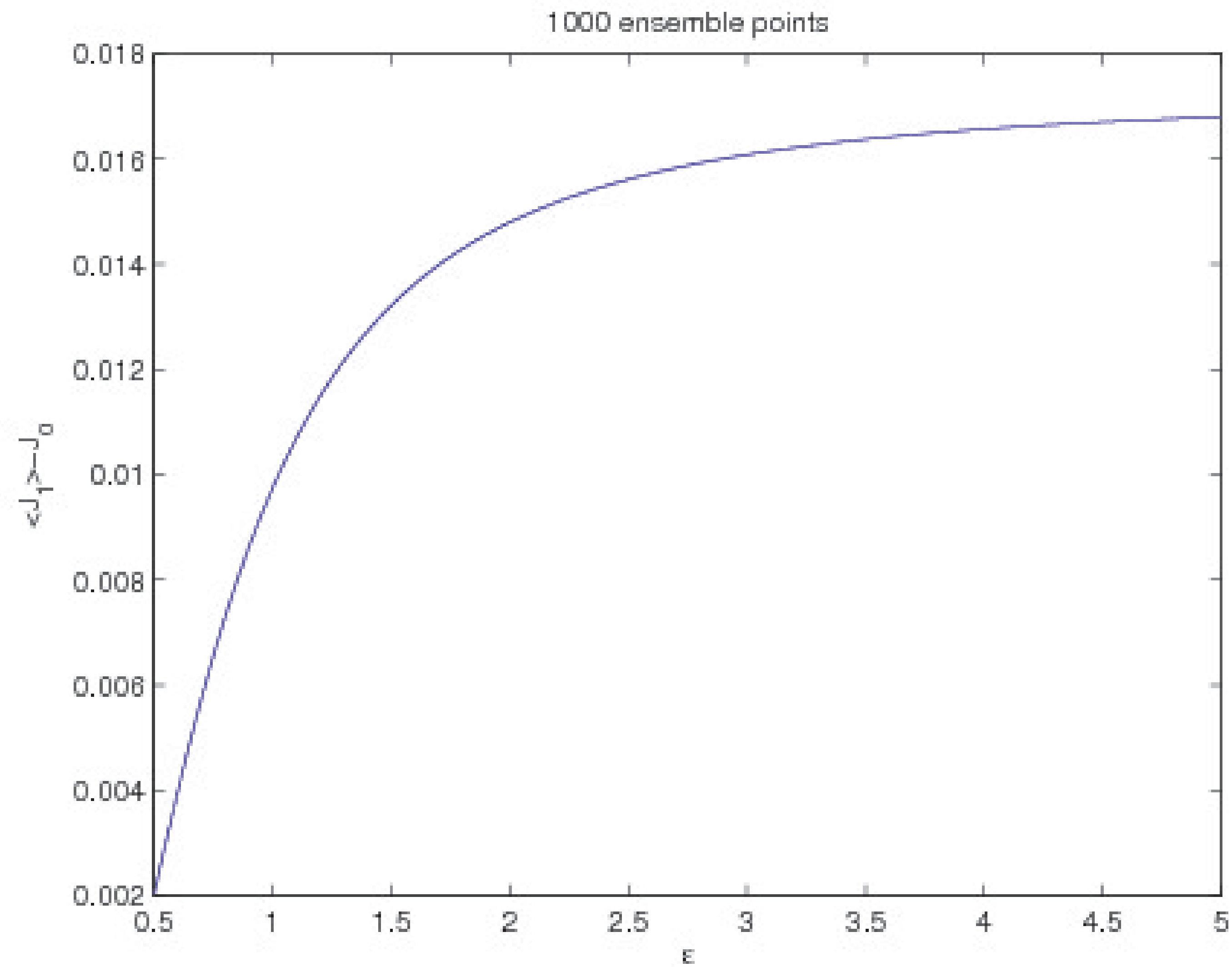
The variance of the energy distribution for the quartic oscillator for the range $\epsilon \in [0, 0.05]$ for 1000 ensemble initial conditions. Linear model for a.



The action difference $I_1(\bar{E}_1) - I_0$ for the quartic oscillator for the range $\epsilon \in [0, 0.05]$ for 1000 ensemble initial conditions at energy $E_0 = 1/4$. Linear model for a.



The action difference $I_1(\bar{E}_1) - I_0$ for the quartic oscillator for the range $\epsilon \in [0.05, 0.5]$ for 1000 ensemble initial conditions at energy $E_0 = 1/4$. Linear model for a.



The action difference $I_1(\bar{E}_1) - I_0$ for the quartic oscillator for the range $\epsilon \in [0.5, 5]$ for 1000 ensemble initial conditions at energy $E_0 = 1/4$. Linear model for a.

4. Parametrically kicked systems (the case of a single kick):

General aspects and power law potentials: $H(q,p,a_0) \to H(q,p,a_1)$

Proposition *(non-negative growth of the action in the mean)*: The action $I_1(\bar{E}_1)$ at the average final energy $\bar{E}_1$ is always greater than the initial action $I_0 = I(E_0)$, except for the equality in case of no change at all $a_1 = a_0$.

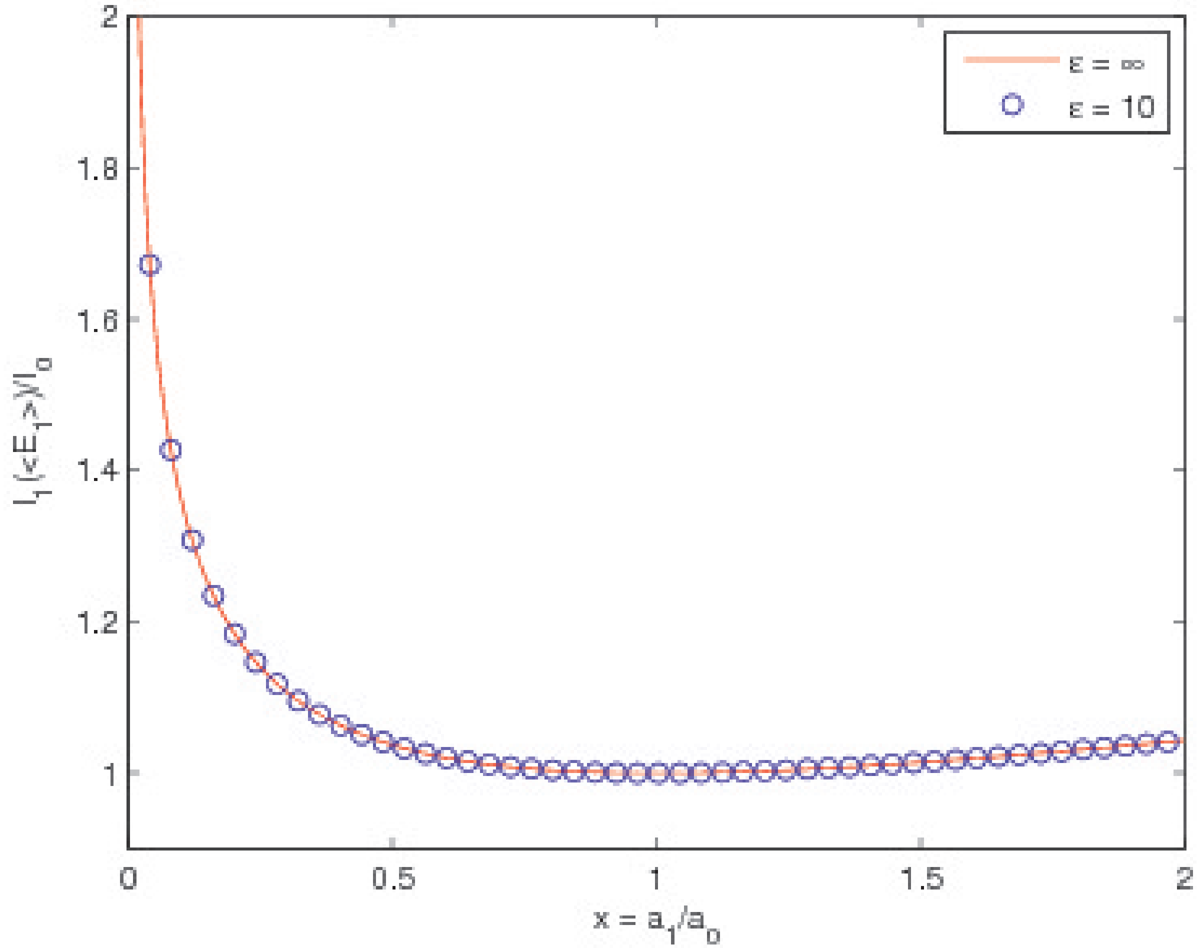
This proposition is difficult to prove in general.

We can prove it in case of a the specific time dependent family of homogeneous nonlinear power law potentials defined by the Hamiltonian

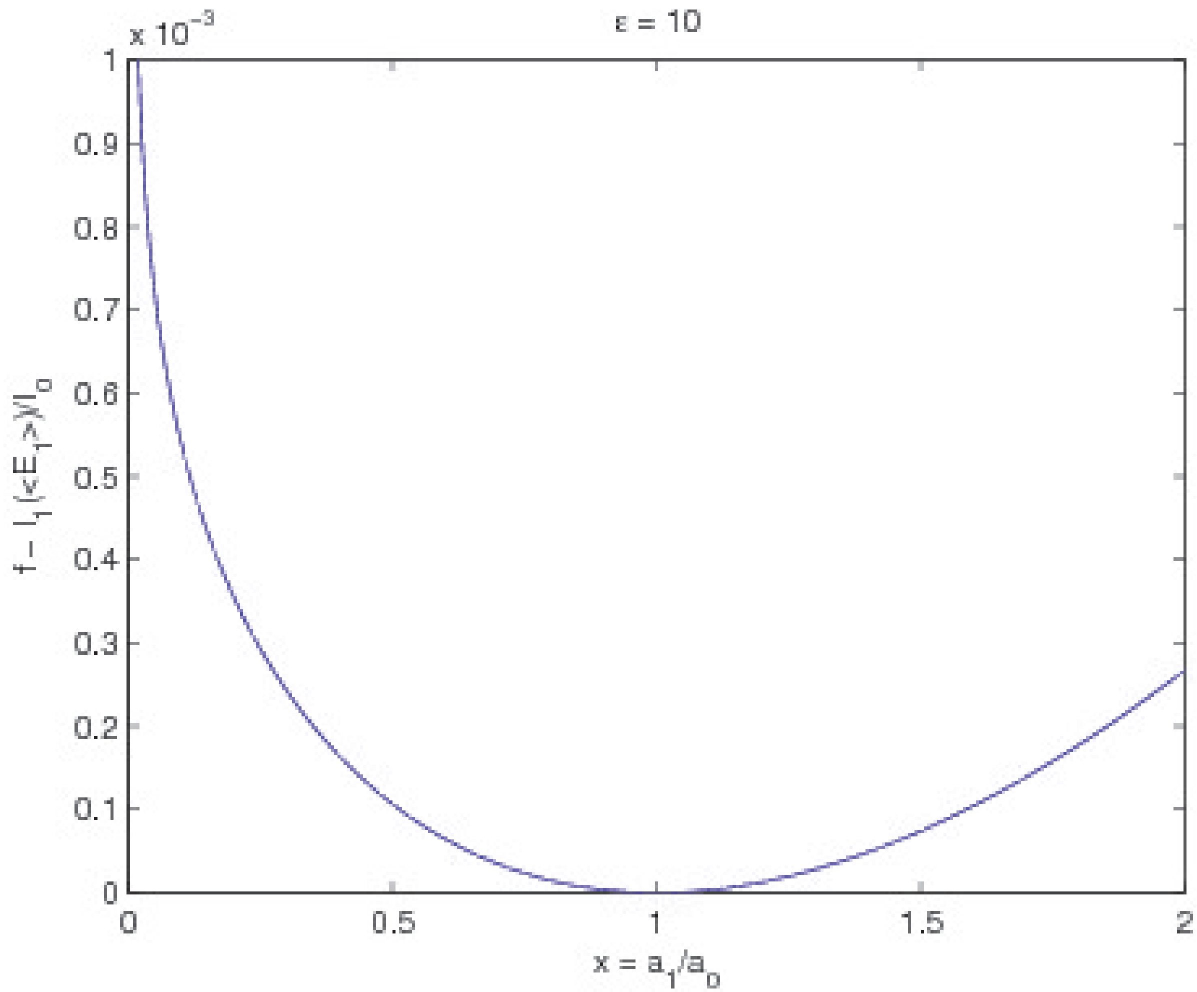
$H(q,p,a) = \frac{p^2}{2} + \frac{a}{2m} q^{2m}$,

where a is the family parameter, and m is a positive integer, and we shall denote: $x = a_1/a_0$:

Universal function indexed by m**:** $I_1(\bar{E}_1) = I_0 f_m(x) = I_0 \frac{\left(1+\frac{1}{m+1}(x-1)\right)^{\frac{m+1}{2m}}}{x^{\frac{1}{2m}}}$



The action ratio $I_1(\bar{E}_1)/I_0$ for the single kicked quartic oscillator for the range $\epsilon = \infty$ (theory) and $\epsilon = 10$ (linear model, numerical integration) versus $x = a_1/a_0$ for 1000 ensemble initial conditions at energy $E_0 = 1/4$.



More details regarding previous figure: The difference of $f_2(x)$ and the action ratio $I_1(\bar{E}_1)/I_0$ for the single kicked quartic oscillator for $\epsilon = 10$ (linear model, numerical integration) versus $x = a_1/a_0$ for 1000 ensemble initial conditions at energy $E_0 = 1/4$.

5. Linear oscillator: *kick and anti-kick*

$$\omega^2(t) = \begin{cases} \omega_0^2 & \text{if} \quad t \leq 0 \\ \omega_1^2 & \text{if} \quad 0 < t < T \\ \omega_0^2 & \text{if} \quad t \geq T \end{cases} .$$

and we find

$$\mu^2 = \frac{E_0^2}{2}\left(\frac{1}{4}\left(\frac{\omega_1^2}{\omega_0^2} - \frac{\omega_0^2}{\omega_1^2}\right)^2 \sin^4 \omega_1 T + \left(\frac{\omega_1}{\omega_0} - \frac{\omega_0}{\omega_1}\right)^2 \cos^2 \omega_1 T \sin^2 \omega_1 T\right) .$$

In such case the variance μ^2 vanishes whenever $\phi_1 = \omega_1 T = n\pi$, where n is any non-negative integer. This is a remarkable result: vanishing μ^2 means that the system is back to the microcanonical ensemble. Thus we conclude that the second kick, in this case anti-kick, can restore the original microcanonical distribution of the energies if the anti-kick happens at the right phase.

This is a kind of parametric resonance: After the anti-kick the entire energy distribution function is recollected back to the microcanonical one.

Periodic kicking of the linear oscillator: parametric resonance

The new element is our statistical analysis, that is the evolution of the energy distribution. We assume that at time $t = 0$ we have a kick from $a_0 = \omega_0^2$ to $a_1 = \omega_1^2$, which at time $t = T$ jumps back to $a_0 = \omega_0^2$, and at time $t = 2T$ back to $a_1 = \omega_1^2$, and this period is repeating itself ad infinitum. Thus the period of the kicking is $2T$.

What happens to the mean final energy $\bar{E}_1$ and the variance μ^2

Using the notation $\phi_0 = \omega_0 T$ and $\phi_1 = \omega_1 T$:

$Det(\Phi_p) = ab - cd = 1$ is unity due to the area preserving property

The trace $B = a + d$ is: $B = Tr(\Phi_p) = 2\cos\phi_0 \cos\phi_1 - \left(\frac{\omega_1}{\omega_0} + \frac{\omega_0}{\omega_1}\right) \sin\phi_0 \sin\phi_1$.

$$\lambda^2 - B\lambda + 1 = 0, \quad \lambda_{1,2} = \tfrac{1}{2}(B \pm \sqrt{B^2 - 4}).$$

6. Quartic oscillator: *kick and anti-kick*

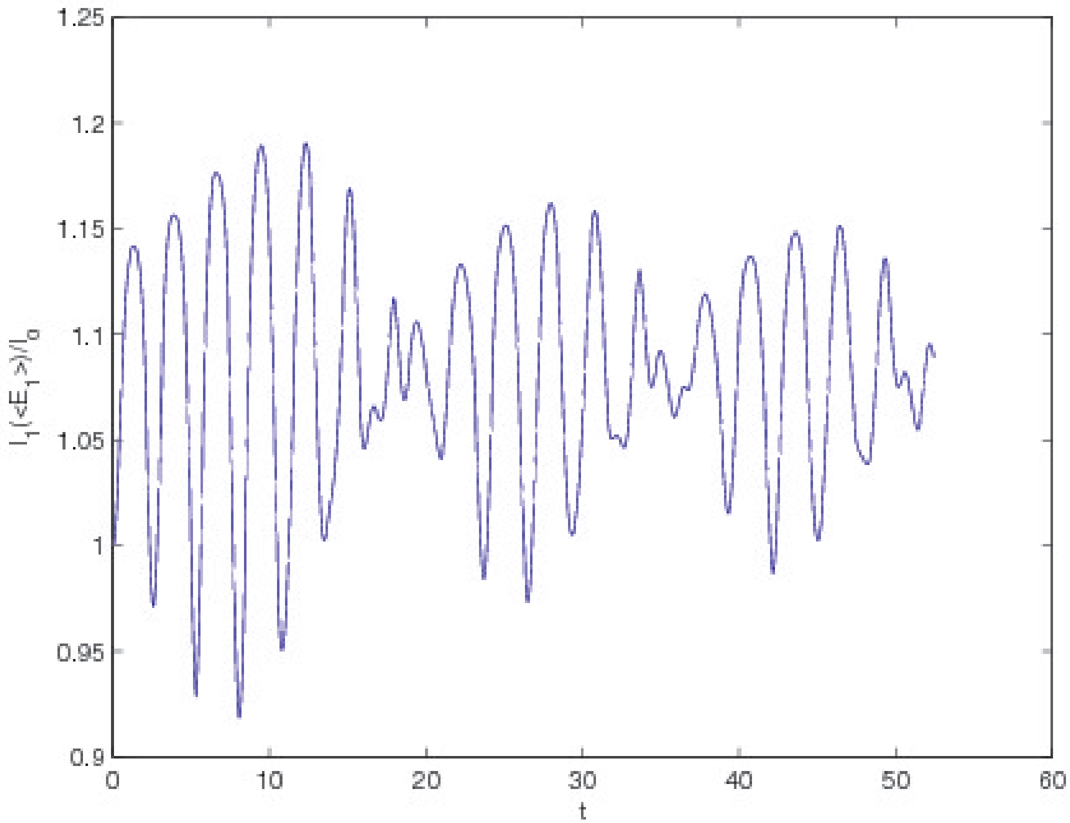
$H(q,p,t) = \frac{1}{2}p^2 + \frac{a(t)}{4}q^4,$

$$a(t) = \begin{cases} a_0 = 1 & \text{if} \quad t \le 0 \\ a_1 = 2 & \text{if} \quad 0 < t < T \\ a_0 = 1 & \text{if} \quad t \ge T \end{cases} .$$

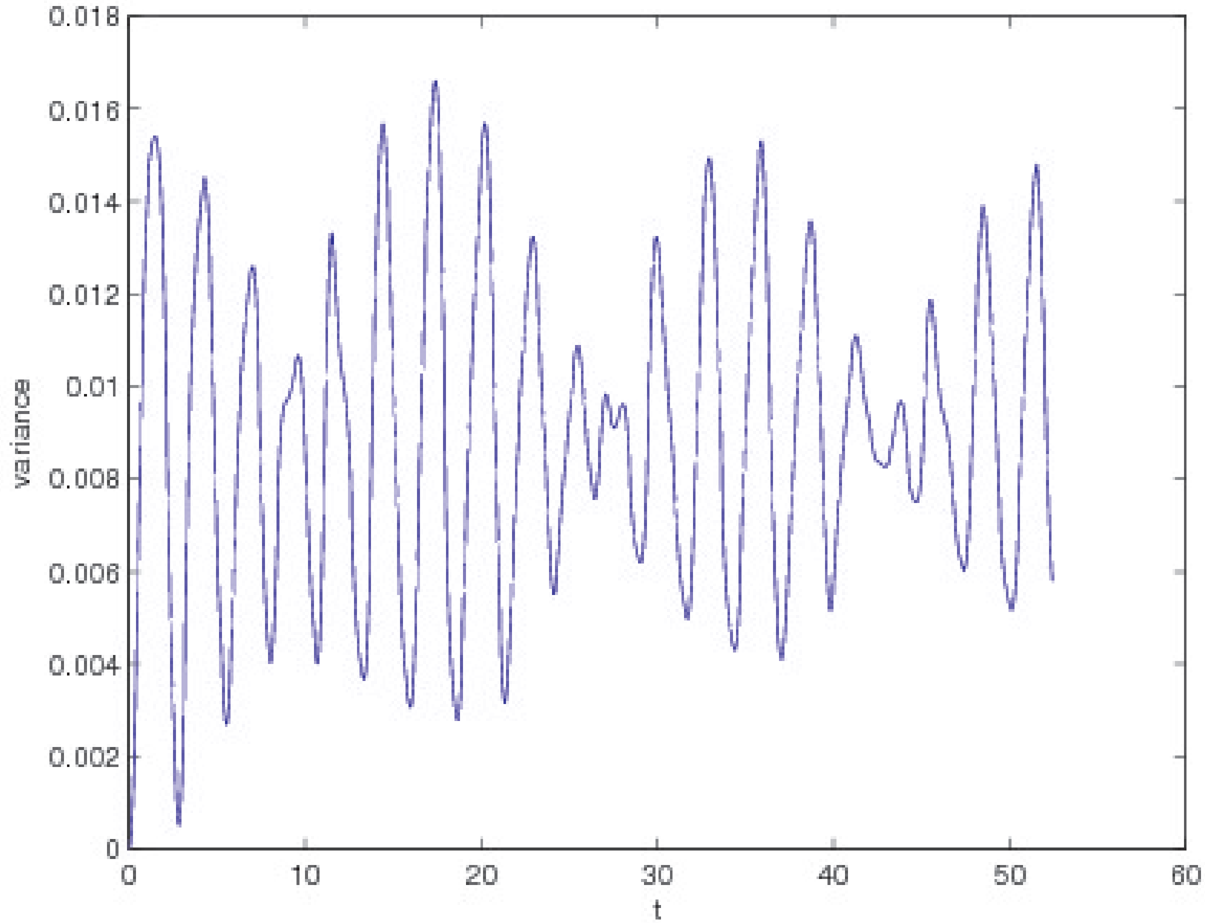
While fixing the amplitude of the size of the parametric kick and anti-kick, we look now at the variance μ^2 of the final energy distribution function as a function of time T.

In the next two figures we show the results: oscillations similar like in the linear oscillator are seen.

However, both $I_1(\bar{E}_1)$ and μ^2 are now **nonperiodic**, which is a consequence of the **nonisochronicity**: the oscillation frequency of the quartic oscillator is a function of the energy, and since after the first kick each ensemble point (orbit) is at a different energy, some kind of a dispersion occurs, destroying the periodicity. Also, the variance μ^2 never goes down to zero, although it comes close to that for certain times.



The action ratio $I_1(\bar{E}_1)/I_0$ for the quartic oscillator for 1000 ensemble initial conditions at energy $E_0 = 1/4$, as a function of time T between the kick and anti-kick $a_0 = 1, \ a_1 = 2$. We observe oscillations below 1 for small T, showing that in nonlinear oscillators the action of the average final energy can decrease.



The variance μ^2 for the quartic oscillator for 1000 ensemble initial conditions at energy $E_0 = 1/4$, as a function of time T between the kick and anti-kick $a_0 = 1,\ a_1 = 2$. The nonperiodicity of the oscillations is a consequence of the nonisochronicty of the quartic oscillator.

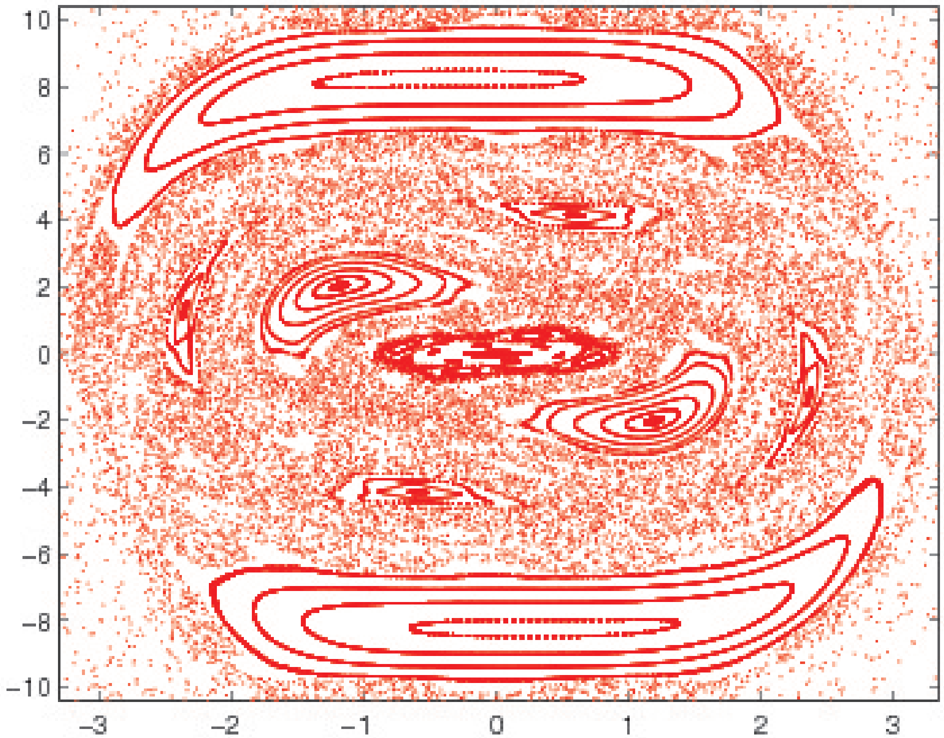
Quartic oscillator: The periodic kicking

Similar to the periodic kicking of the linear oscillator we assume that at time $t = 0$ we have a kick from $a_0 = 1$ to $a_1 = 2$, which at time $t = T$ jumps back to $a_0 = 1$, and at time $t = 2T$ back to $a_1 = 2$, and this period is repeating itself ad infinitum.

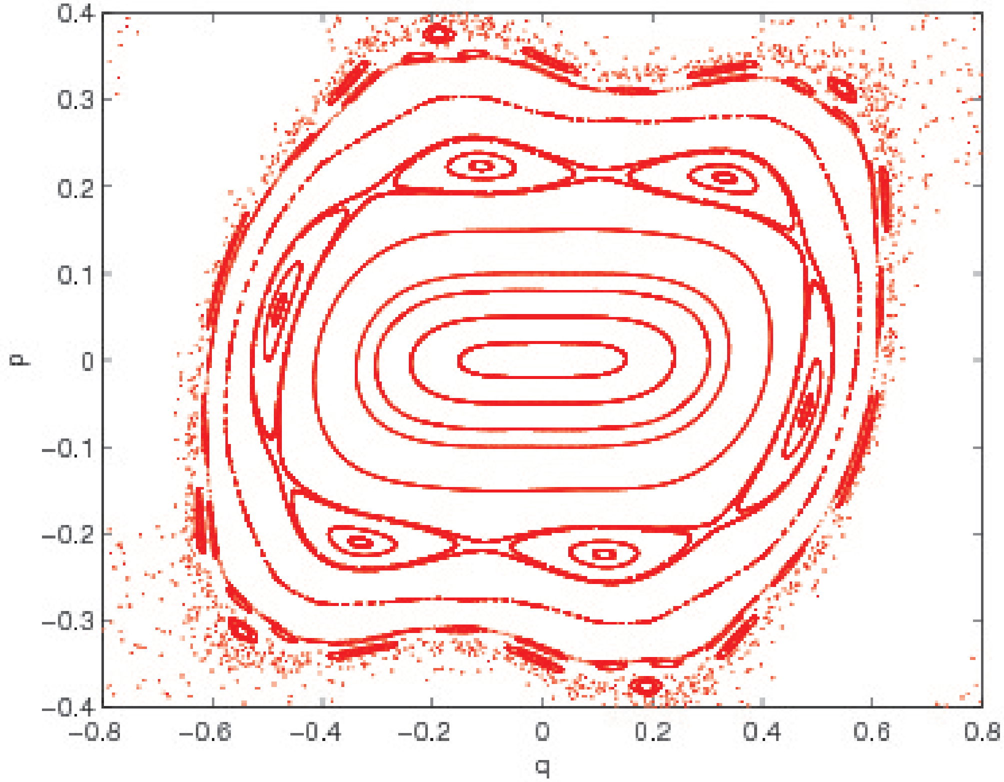
What happens to the mean final energy $\bar{E}_1$, the action ratio $I_1(\bar{E}_1)/I_0$ and the variance μ^2 of the distribution function $P(\bar{E}_1)$ as a function of the discrete number n?

In analogy with the linear oscillator we expect two regimes, the oscillatory one where $\bar{E}_1$ and μ^2 just oscillate, and the unstable regime, where the energy grows indefinitely as well as the $I_1(\bar{E}_1)/I_0$ and variance μ^2.

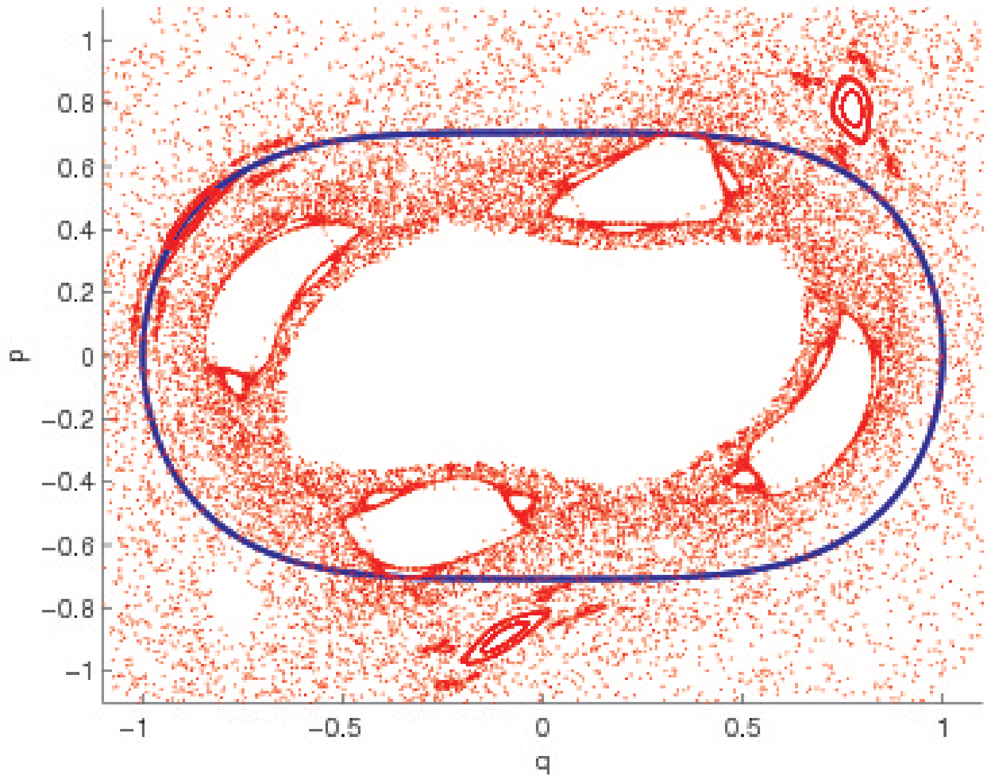
Indeed, the phase portrait shows the generic picture: islands of nonlinear stability surrounded by chaotic sea of the unstable regime: The latter case is illustrated in the next three figures. The growth of the average energy seems to be linear. Further numerical calculations show that at about $n \approx 700$ a dramatic acceleration takes place, where the energy starts to grow exponentially.



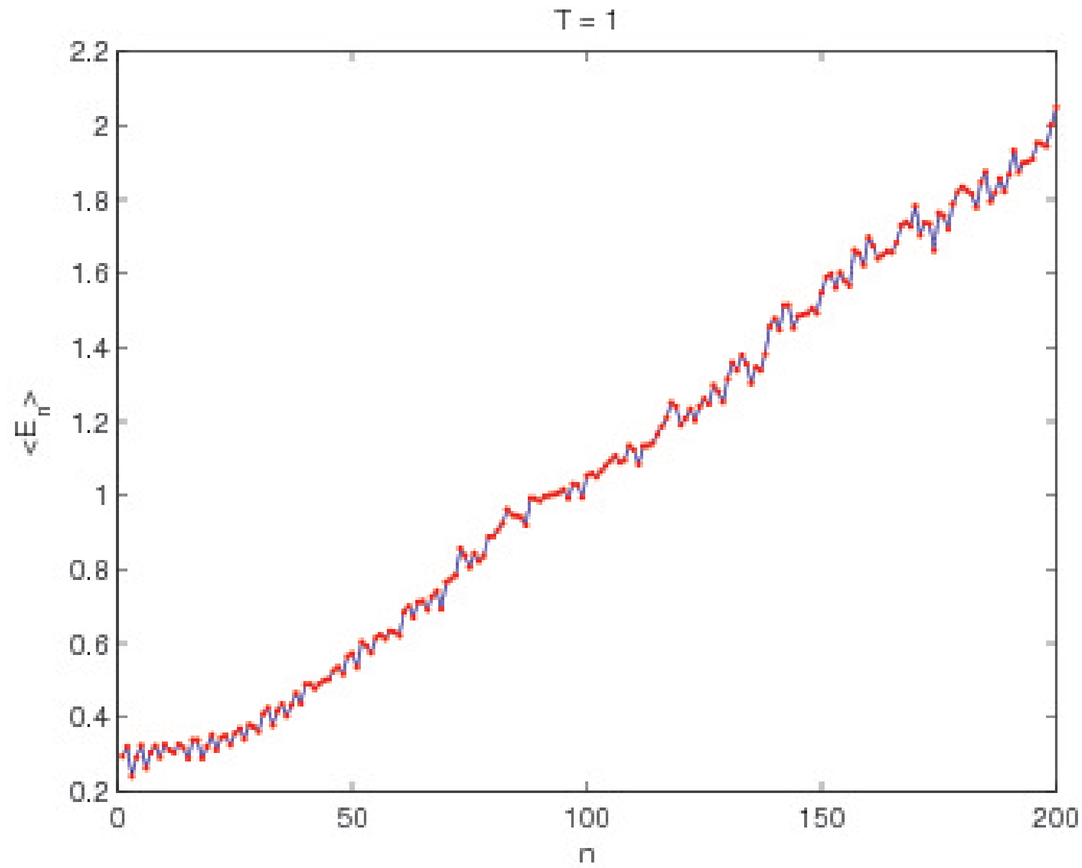
The phase portrait of the phase space map Φ_p for the periodically kicked quartic obtained by iterating chosen initial conditions for a certain n (number of periods of the parameter a). The parameter values are $a_0 = 1,\ a_1 = 2,\ T = 1$.



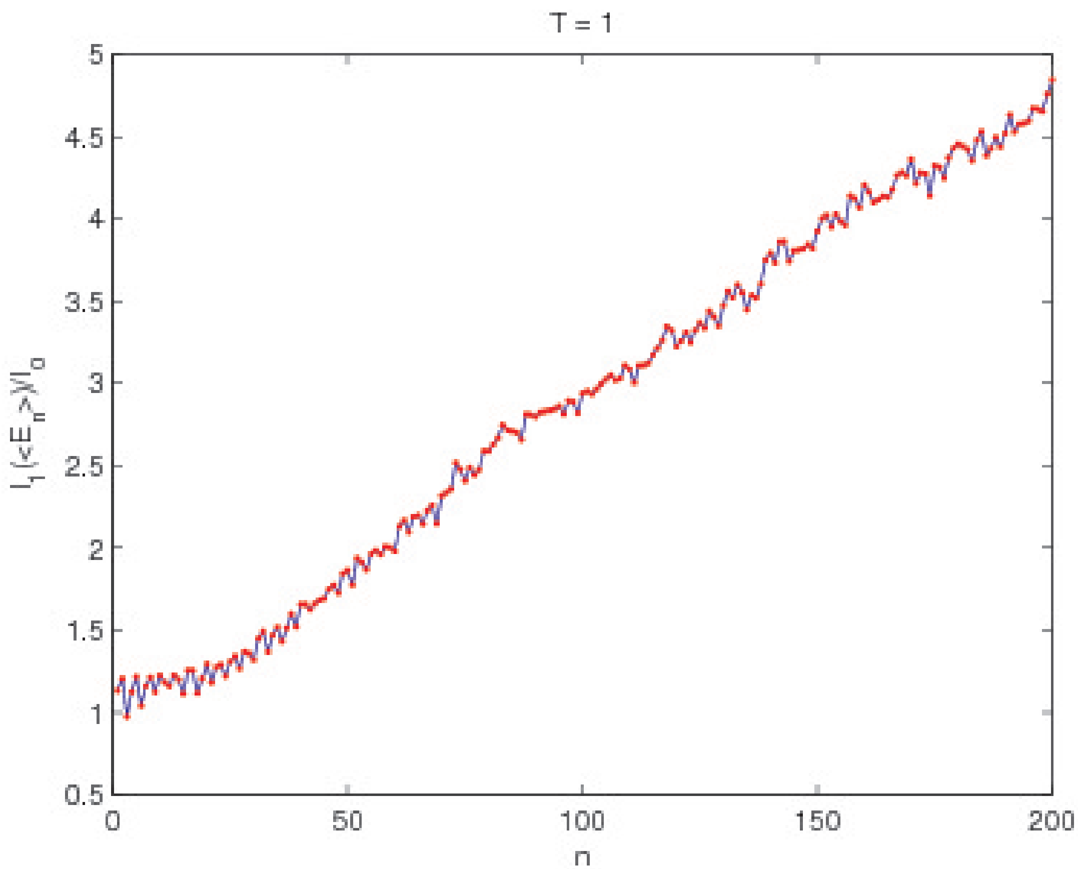
The zoom-in of the inner phase portrait of the central regular island of phase space map Φ_p for the periodically kicked quartic oscillator obtained by iterating chosen initial conditions for a certain n (number of periods of the parameter a). The parameter values are $a_0 = 1,\ a_1 = 2,\ T = 1$.



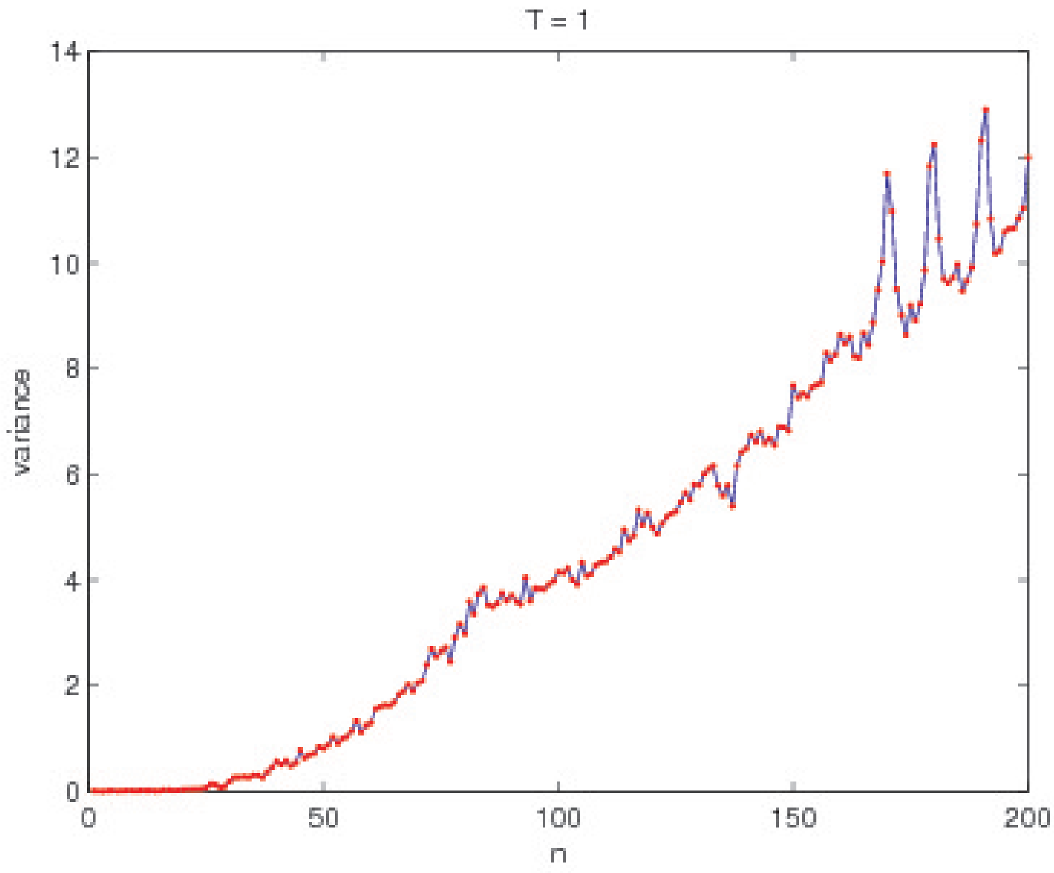
The phase portrait of the phase space map Φ_p for the periodically kicked quartic oscillator with the display of the microcanonical initial conditions with $E_0 = 1/4$, for the parameter values $a_0 = 1,\ a_1 = 2,\ T = 1$.



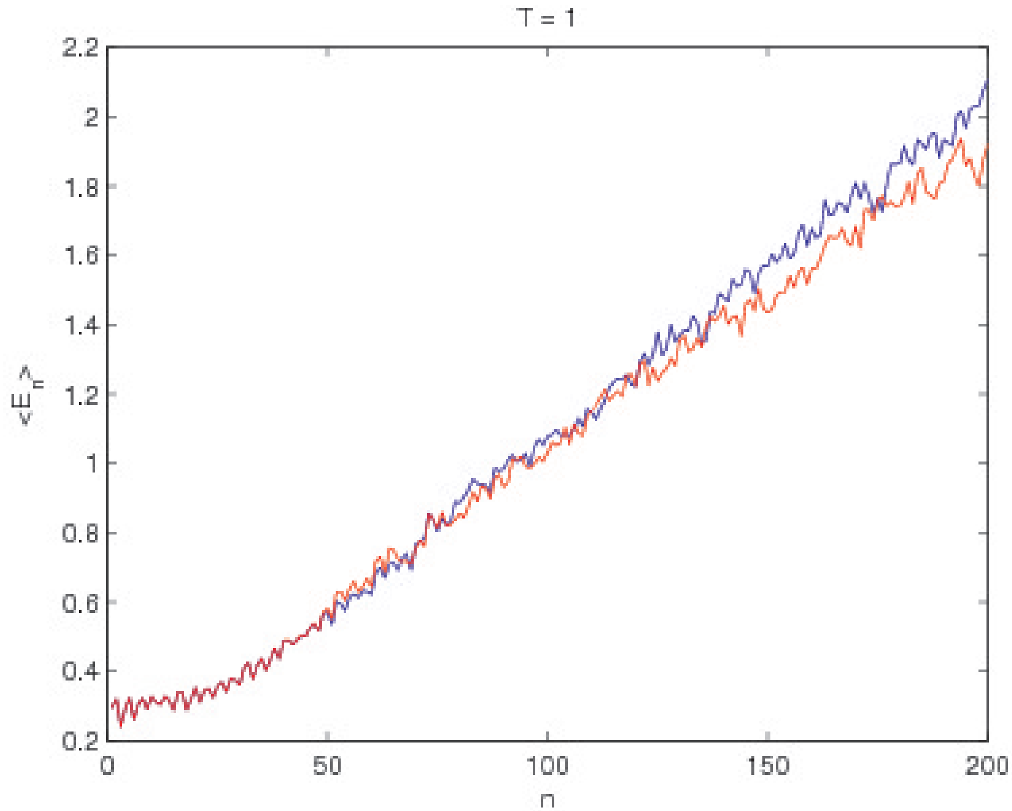
The average energy $\bar{E}_1$ for the periodically kicked quartic oscillator for 1000 ensemble initial conditions at energy $E_0 = 1/4$, as a function of discrete time n (number of periods of the parameter a). The parameter values are $a_0 = 1$, $a_1 = 2$ and $T = 1$. The data are dots and the line connecting them is just to guide the eye.



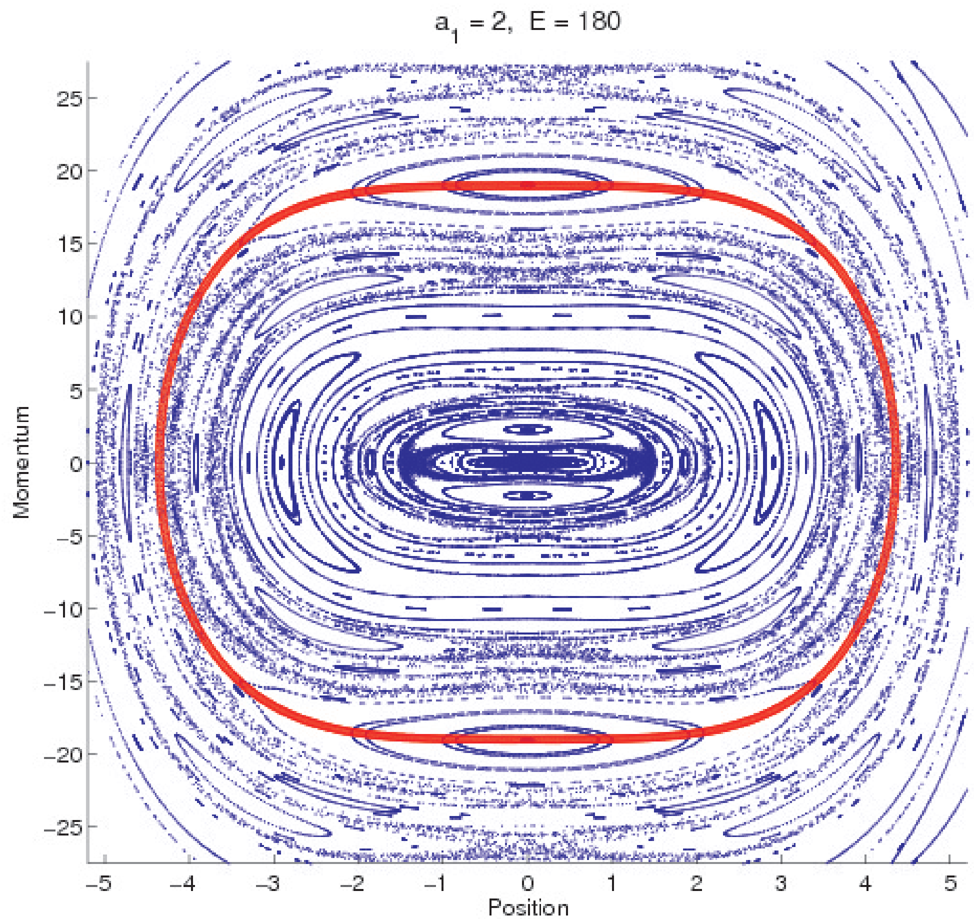
The action ratio $I_1(\bar{E}_1)/I_0$ for the periodically kicked quartic oscillator for 1000 ensemble initial conditions at energy $E_0 = 1/4$, as a function of discrete time n (number of periods of a). The parameter values are $a_0 = 1$, $a_1 = 2$, $T = 1$. We observe oscillations below 1 for small $n = 3$, showing that in nonlinear oscillators the action of the mean final energy can decrease. The data are dots, the line connecting them is just to guide the eye.



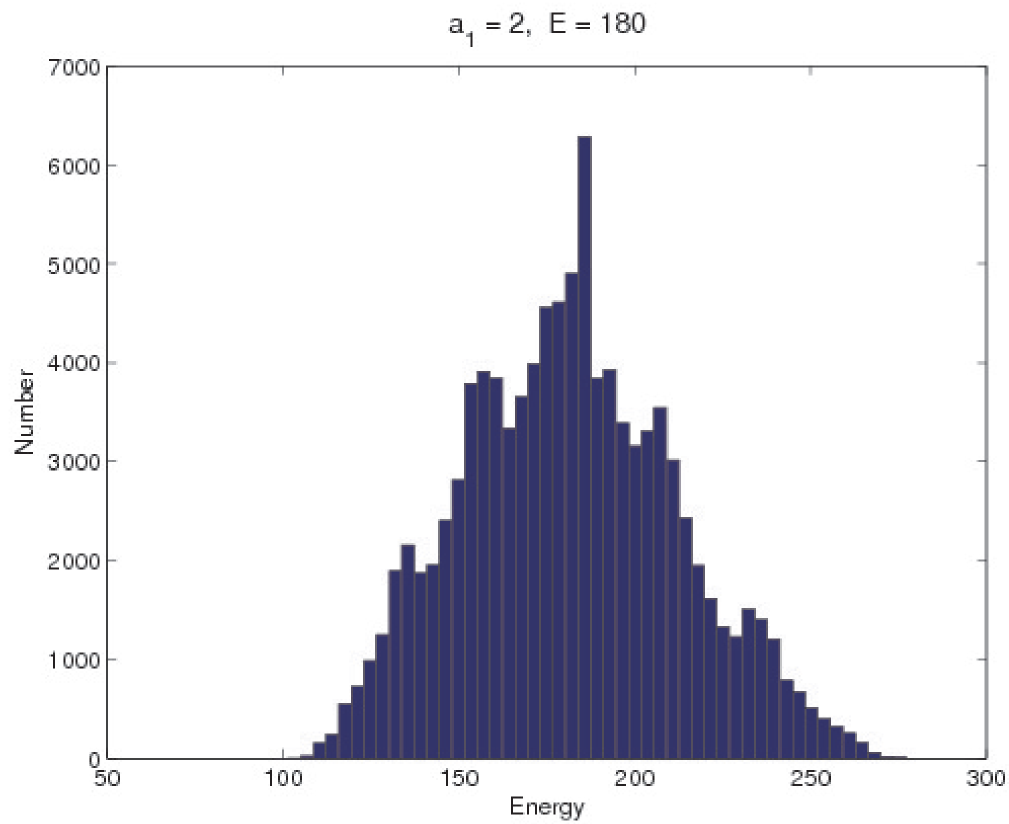
The variance μ^2 for the periodically kicked quartic oscillator for 1000 ensemble initial conditions at energy $E_0 = 1/4$, as a function of discrete time n (number of periods of parameter a). The parameter values are $a_0 = 1,\ a_1 = 2\ T = 1$. The data are dots and the line connecting them is just to guide the eye.



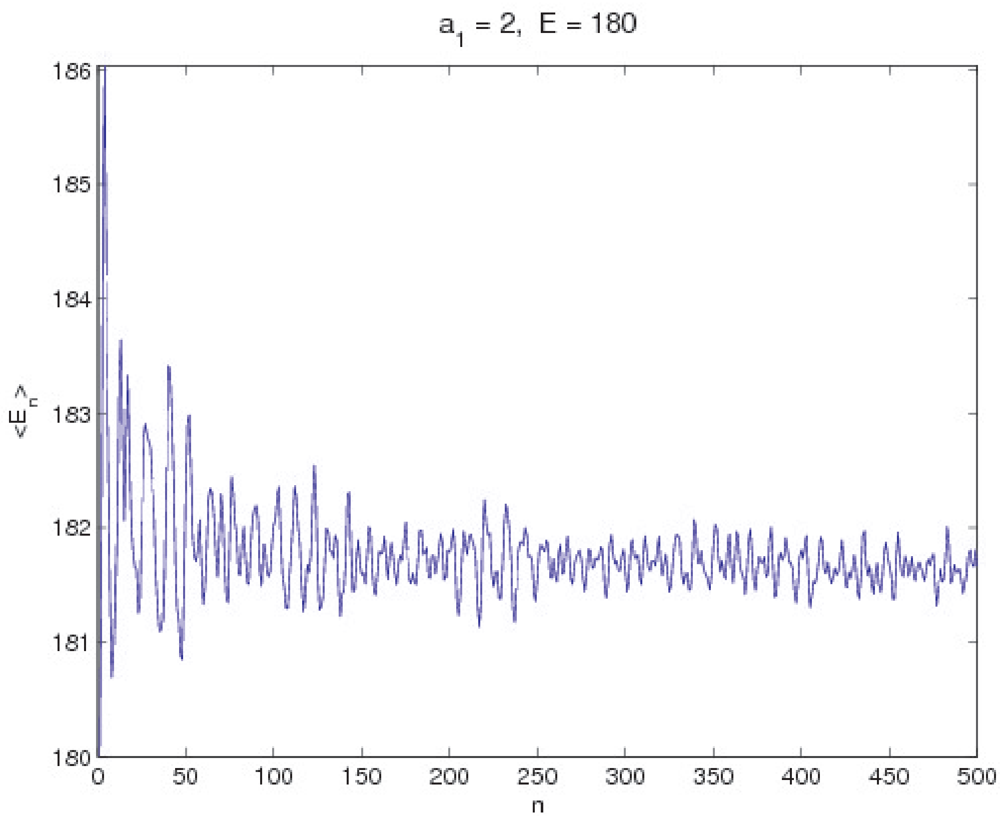
The average energy for the periodically kicked quartic oscillator for two sets of microcanonical initial conditions with $E_0 = 1/4$ for the parameter values $a_0 = 1,\ a_1 = 2,\ T = 1$, differing in q_0 by only 10^{-12}.



The sawtooth modulated quartic oscillator: The phase space at $a_1 = 2$ with the contour of 4000 initial conditions at $E = 180$.



The sawtooth modulated quartic oscillator: The histogram (distribution) of the final energies of the microcanonical ensemble after 100 iterations (periods of length $2T$).



The sawtooth modulated quartic oscillator: The average energy of the initial microcanonical ensemble as a function of the number of iterations n, for 1000 initial conditions.

7. Discussion and conclusions

• We have studied the time evolution of the energy in a general time-dependent 1D harmonic oscillator in a rigorous way, and then also calculated the final energy distribution $P(E_1)$ for a microcanonical ensemble of initial conditions at energy E_0.

• $P(E_1)$ is universal and does not depend on the details of $\omega(t)$:

$P(E_1) = \frac{1}{\pi\sqrt{2\mu^2 - x^2}}$, where $x = E_1 - \bar{E}_1$

• The analysis clearly shows when the adiabatic invariant $I(t) = E(t)/\omega(t)$ is conserved or not. In the adiabatic limit $T \to \infty$ it is conserved. If it is not conserved, when T is finite, we calculate $\mu^2 \neq 0$ using WKB method analytically in closed form.

• The action of the final average energy always increases except when $\epsilon = 0$ when the equality holds.

• We have also studied three specific solvable models and shown that the leading WKB term well describes the oscillatory but in the mean power law behaviour of μ^2 when $\epsilon = 1/T$ goes to zero.

• If $\omega(t)$ is periodic, $\bar{E}_1$ can grow exponentially, and so does the variance μ^2, in which case $I(t) = E(t)/\omega(t)$ is not conserved, but we can describe the system.

• In the nonlinear oscillators the action of the final average energy generally does not increase, but does so nevertheless for larger values of ϵ, in particular for power law potentials, which we have shown by direct exact calculation. The general proof is difficult.

• The case of a kick and anti-kick: in the linear oscillator we have periodic μ^2, including vanishing μ^2 at certain phases, which means restoring the microcanonical distribution.

• The case of a kick and anti-kick: in the nonlinear oscillator we have nonperiodic oscillatory behaviour of μ^2, but it never comes back to the microcan. ens. $\mu^2 = 0$.

• The periodic kicking: in the linear oscillator we have stable and unstable regime, depending on the trace B. In the unstable regime the energy grows exponentially: **parametric resonance**, now understanding the statistical behaviour.

• The periodic kicking: in the nonlinear oscillator we see also stable and unstable regime: in the latter one first the energy grows linearly, then exponentially: **parametric resonance**. The distribution of the final energies can be just anything.

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This work has been done in collaboration with Professor Valery Romanovski and Georgios Papamikos, CAMTP.

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