Additive noise tunes stability in nonlinear systems

Axel Hutt

Team CORTEX
INRIA Nancy
France

Cooperations

- Jérémie Lefebvre
  (U Geneva)
- André Longtin
  (U Ottawa)

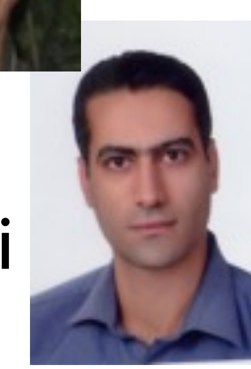
MATHANA-team

Laure Buhry

Eric Nichols

Pedro Gonzalez

Meysam Hashemi



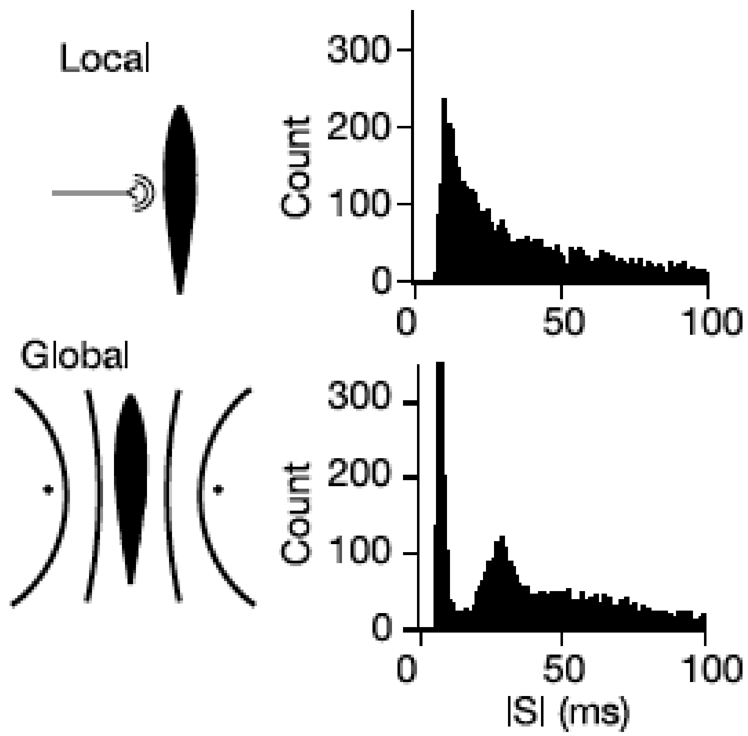
Work supported by
ERC Starting Grant **MATHANA**:
MATH*ematical modelling of* ***ANA****esthesia*

Outline

- a linear neural system
- *everything starts with the slaving principle*
- additive noise in non-delayed nonlinear systems
- additive noise in delayed nonlinear systems

Linear response

feedback system in weakly electric fish



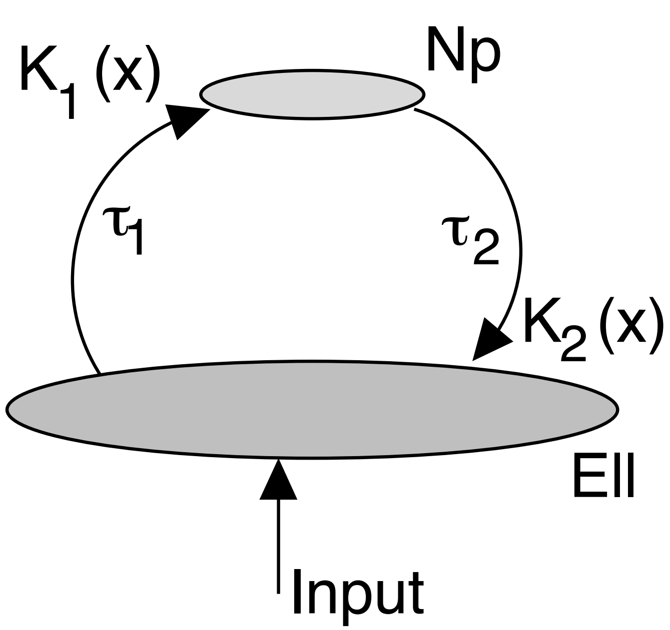
experimental results obtained
in vivo from electric fish

(Doiron et al., Nature (2003))

topological feedback model

with spatially correlated input

(Hutt, Sutherland and Longtin, Phys.Rev.E 78, 021911(2008))



nonlinear evolution:

$$\tau_e \frac{\partial E(x,t)}{\partial t} = -E(x,t) - \int_{\Omega_n} dy\ K_{en}(x-y) S_n[N(y,t-\tau_2)] + I(x,t)$$

$$\tau_n \frac{\partial N(y,t)}{\partial t} = -N(x,t) - \int_{\Omega_e} dx\ K_{ne}(y-x) S_e[E(x,t-\tau_1)]$$

stationary constant state: $E_0 = \kappa_{en} S_n\left(\kappa_{ne} S_e\left(E_0\right)\right)\ ,\ N_0 = \kappa_{ne} S_n\left(E_0\right)$

linearization with *u(x,t)=E(x,t)-E0*:

$$\boxed{\hat{L}u(x,t) = -\frac{g}{\tau_e \tau_n} \int_{\Omega_e} dx'\ F(x-x') u(x', t-\tau_0) + I(x,t)}$$

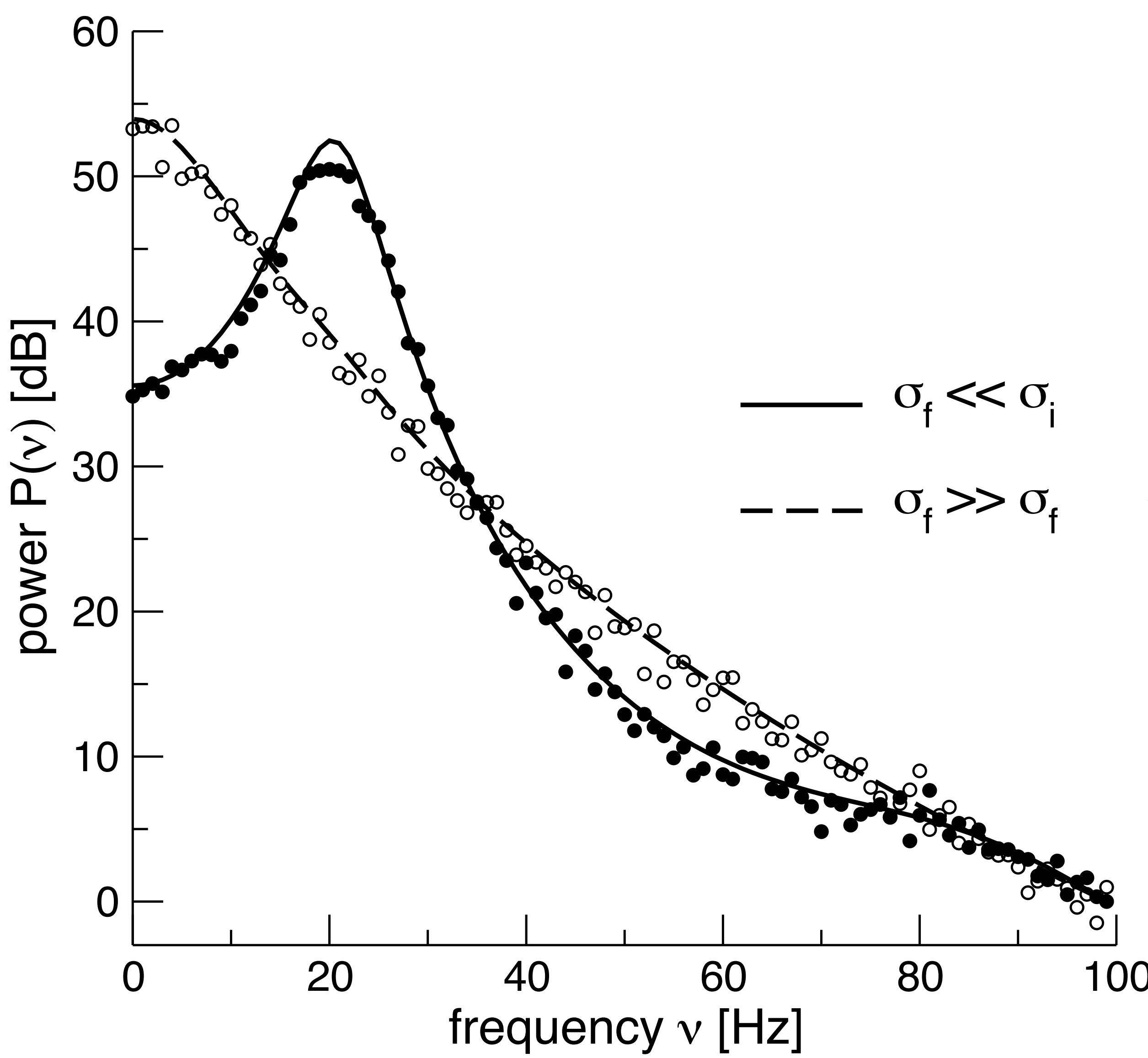
noisy external input: $\langle I(x,t) I(y,T)\rangle = Q\delta(t-T)C(x-y)$

input correlation: $C(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2\sigma_i^2}$

spatial feedback: $F(z) = \frac{1}{\sqrt{2\pi}\sigma_f} e^{-z^2/2\sigma_f^2}$

(Hutt, Sutherland and Longtin, Phys.Rev.E 78, 021911(2008)) power spectrum

$$P(\nu,\eta) = Q\int_{-\infty}^{\infty} dl \frac{\tilde{C}(l)}{A(\nu) + B(\nu)\tilde{F}(l) + D\tilde{F}^2(l)}$$



$\sigma_f \ll \sigma_i$ strong input correlation

$\sigma_f \gg \sigma_f$ weak input correlation

$\tau_d = 6ms$

(Hutt, Sutherland and Longtin, Phys.Rev.E 78, 021911(2008))

linear approximation ....

1. ... assumes a stimulus-independent stationary state

2. ... assumes small deviations from the stationary state (low noise level)

3. ... assumes stimulus-independent time scales

but:

what happens if assumptions do not hold ?

Z. Phys. B – Condensed Matter 63, 493–504 (1986)

Condensed Matter
Zeitschrift für Physik B

The Slaving Principle for Stratonovich Stochastic Differential Equations

G. Schöner and H. Haken
Institut für Theoretische Physik, Universität Stuttgart,
Federal Republic of Germany

Received January 24, 1986

We treat nonlinear Stratonovich stochastic differential equations (including multiplicative noise). We assume that the variables can be grouped into the linearly damped (slaved) variables $\underset{\sim}{s}$ and linearly undamped variables (order parameters) $\underset{\sim}{u}$. We present a systematic and constructive procedure to eliminate the slaved variables. A family of processes $\underset{\sim}{Z}_t^{(\nu)}$ ($\nu \geqq 2$) is introduced to represent the explicit chance dependence of the slaved variables. The stochastic properties of the Z-processes are discussed. An example serves to illustrate the elimination procedure. The adiabatic approximation is defined as a partial summation of the systematic elimination procedure and an equivalent stochastic differential equation (the stochastic generalization of $\dot{s}=0$) is derived. We illustrate the adiabatic approximation by an example. The relation between the present approach and other elimination procedures for stochastic systems is discussed briefly.

major results

fast variable depends on slow variable (slaving principle):

$$
\begin{aligned}
\underset{\sim}{s}_t &= \underset{\sim}{s}(\text{time, chance}) \\
&= \underset{\sim}{s}(\underset{\sim}{u}_t, t, \underset{\sim}{Z}_t^{(v)}\ (v = 2, 3, \ldots)). \qquad (2.1)
\end{aligned}
$$

application to Haken-Zwanzig model:

$$d u_t = (\alpha u_t - a u_t s_t)\, dt + F_u\, d W_t^{(1)} \qquad (5.1)$$

$$d s_t = (-\beta s_t + b u_t^2) dt + F_s d W_t^{(2)}. \qquad (5.2)$$

resulting order parameter equation (lowest order):

$$du_t = \left(\alpha u_t - \frac{ab}{\beta} u_t^3 - \alpha F_s u_t Z_t^{(2)}\right) dt + F_u dW_t^{(1)} \qquad (5.14)$$

$$dZ_t^{(2)} = -\beta Z_t^{(2)} dt + d W_t^{(2)} \qquad (5.12)$$

1996:

Xu and Roberts show similar results based on a stochastic centre manifold approach

Haken-Zwanzig:

$$dx = (\alpha x - axy)\,dt + F_x dW_1,$$

$$dy = (-\beta y + bx^2)\,dt + F_y dW_2,$$

ansatz:

$$x = s + \xi(t, s),$$

$$y = \eta(t, s),$$

order parameter equation (up to 5th order):

$$ds \sim \left(\alpha s - \frac{ab}{\beta} s^3 + \frac{2\alpha ab}{\beta^2} s^3 - \frac{2a^2b^2}{\beta^3} s^5\right) dt + \left(1 + \frac{2ab}{\beta^2} s^2\right) F_x dW_1 - \left(\frac{a}{\beta} s + \frac{4a^2 b}{\beta^3} s^3\right) F_y dW_2 - \frac{a}{\beta} F_x F_y Z^{(2)} dW_1 + \frac{2ab}{\beta^2} s F_x^2 Z^{(3)} dW_1.$$

noise-induced shift

$$Z^{(3)} = e^{-\beta t} \star dW_1$$

nonlocal neural fields

$$\frac{\partial V(x,t)}{\partial t} = -V(x,t) + \int_\Omega dy\, K(x-y) S[V(y,t)] + I(x,t)$$

stationary homogeneous state: $$V_0 = \int_\Omega dy\, K(y) S[V_0] + I_0$$

global random fluctuations:
(Gaussian i.i.d.)

$$I(x,t) = I_0 + \eta\Gamma(t)$$

$$\langle \Gamma(t) \rangle = 0 \quad , \quad \langle \Gamma(t)\Gamma(T) \rangle = 2\delta(t-T)$$

$$dU(x,t) \approx \eta dW(t)$$

$$+ \left(\int_\Omega dy\, K_1(x-y)\; U(y,t) + K_2(x-y)\; U^2(y,t) + K_3(x-y)\; U^3(y,t) \right) dt$$

(Hutt and Atay, Physica D 2000; Hutt aet al., PhysicaD 2008))

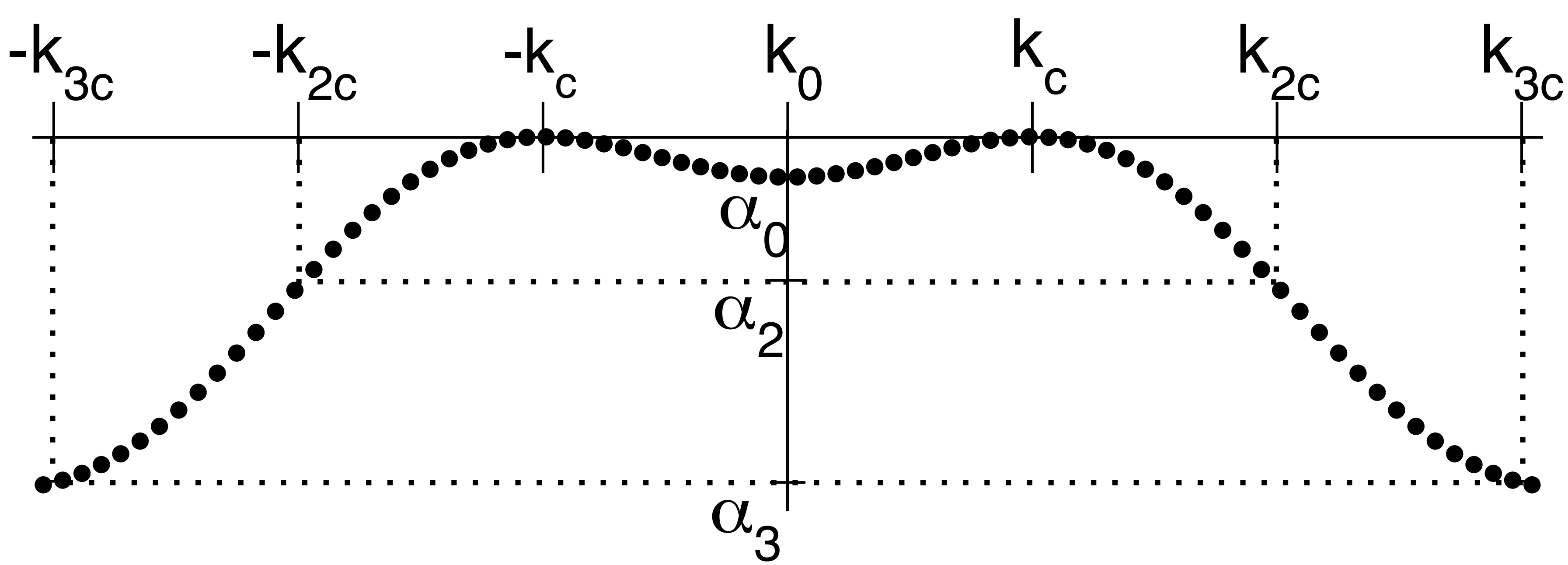
analytical treatment

projection to Fourier space:

$$U(x,t) = \frac{1}{\sqrt{|\Omega|}} \sum_{n=-\infty}^{\infty} u_n(t) e^{ik_n x}$$

$\Longrightarrow$

$$\boxed{du_n(t) = \delta_{n,0}\eta dW(t) + \left(\alpha_n u_n(t) + \beta_n \sum_l u_l(t) u_{n-l}(t) + \gamma_n \sum_{l,m} u_l(t) u_m(t) u_{n-l-m}(t) \right) dt}$$



deterministic centre manifold theorem says:

$$\alpha_c = 0 \rightarrow u_n = u_n(u_c), n \neq c$$

Boxler (1989, 1991) : it exists a stochastic centre manifold (**proof of stochastic slaving principle**).

$$u_0 = h_0(u_c, t) = \sum_{n=2}^{\infty} h_0^{(n)}(u_c, t) \quad , \quad h_0^{(n)}(u_c, t) \sim O(\varepsilon^{n/2})$$

$$\Rightarrow \qquad du_0 = \frac{\partial h_0}{\partial u_c} du_c + \frac{\partial h_0}{\partial t} dt$$

K(x) is Mexican hat kernel:

$$du_c = (\alpha_c + b u_0 u_c + 2\gamma_c u_c^3 + 3\gamma_c u_c u_0^2)dt$$

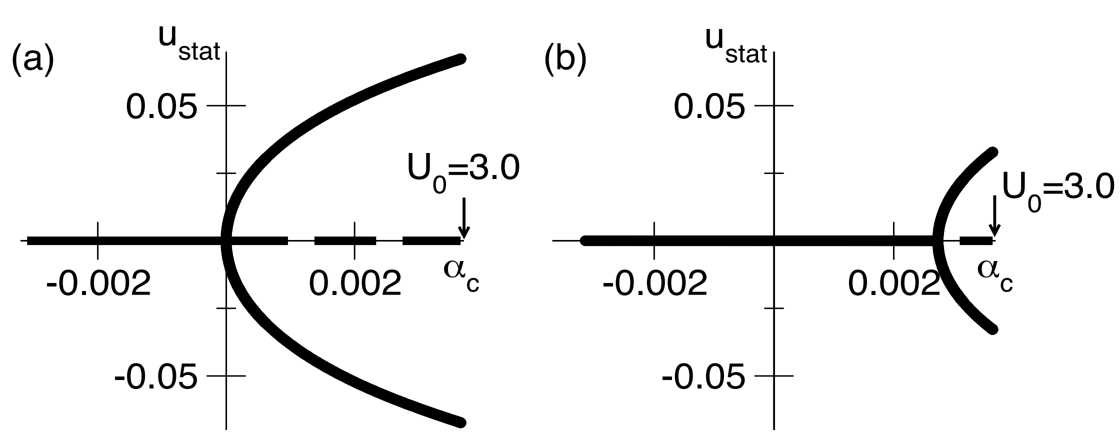
$$du_0 = (\alpha_0 + 4\beta_0 u_c^2 + 2\gamma_c u_c^3 + 3\gamma_c u_c u_0^2)dt + \eta dW(t)$$

center manifold reduction

$$\dot{u}_c = (\alpha_c - \alpha_{th}(\eta))\, u_c + C u_c^3 + D u_c^5$$ (5th order)

noise-induced shift of bifurcation point by

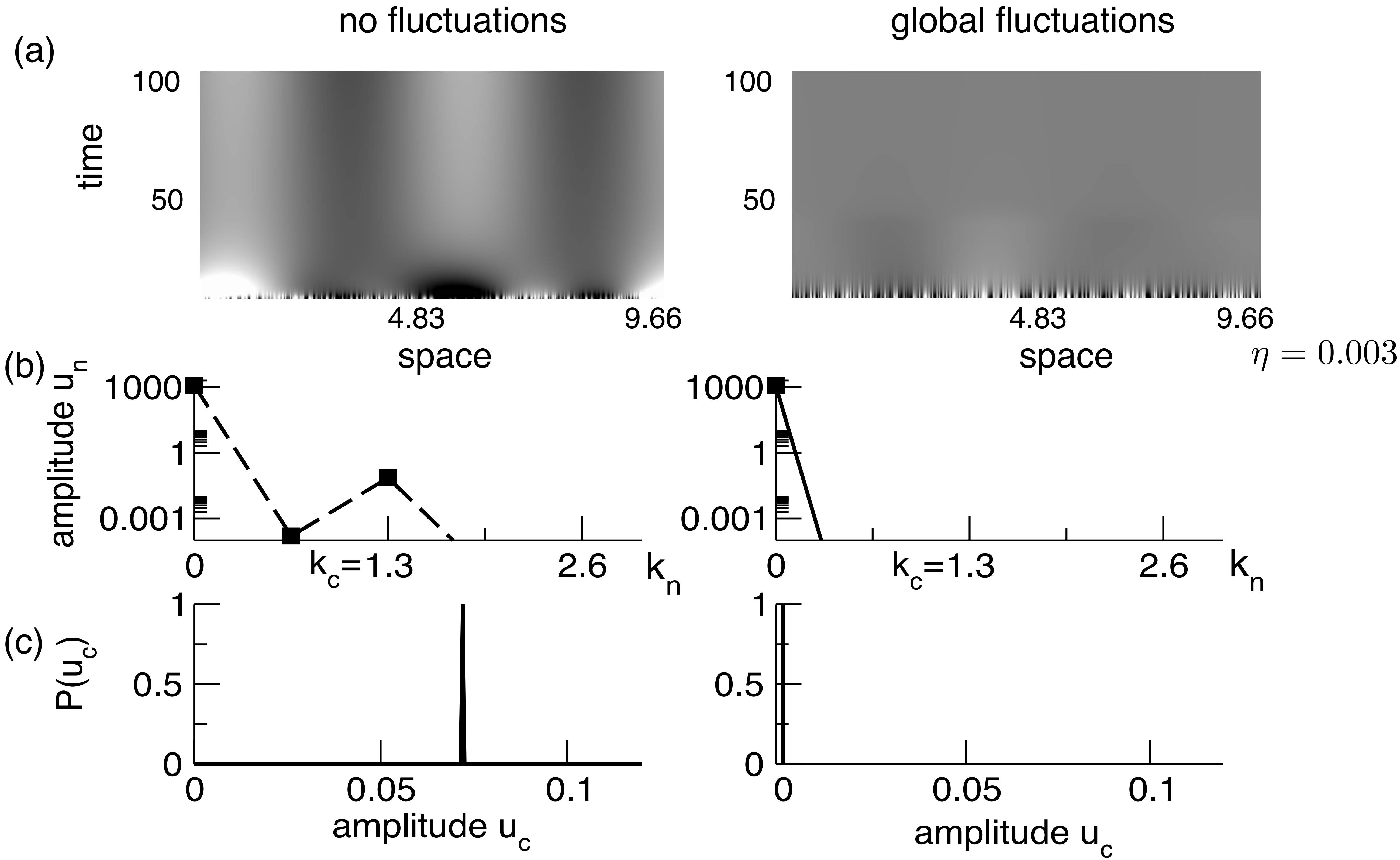
$$\alpha_{th}(\eta) = \eta^2 \left(\frac{\beta_0 b}{\alpha_0^2} - 3\frac{\gamma_c}{|\alpha_0|} \right)$$



no noise   with noise

(Hutt, et al., Phys.Rev.Lett. (2007))   (Hutt et al.,, Physica D 2008)

additive noise destroys instability



other example: extended Swift-Hohenberg equation

$$\frac{\partial U(x,t)}{\partial t} = aU(x,t) + bU^3(x,t) - cU^5(x,t) - \left(1 + \frac{\partial^2}{\partial x^2}\right)^2 U(x,t) + \eta\Gamma(t)$$

projection onto spatial Fourier modes:

$$\mathrm{d}x = (\alpha_c x + \gamma_c(xy^2 + x^3) - \mu_c x^5)\mathrm{d}t,$$
$$\mathrm{d}y = (\alpha_0 y + \gamma_0 x^2 y)\mathrm{d}t + \eta \mathrm{d}W(t),$$

(Hutt and Atay, Physica D(2005); Hutt, Phys.Rev.E 75, 026214 (2007)) after stochastic centre manifold reduction:

shift term

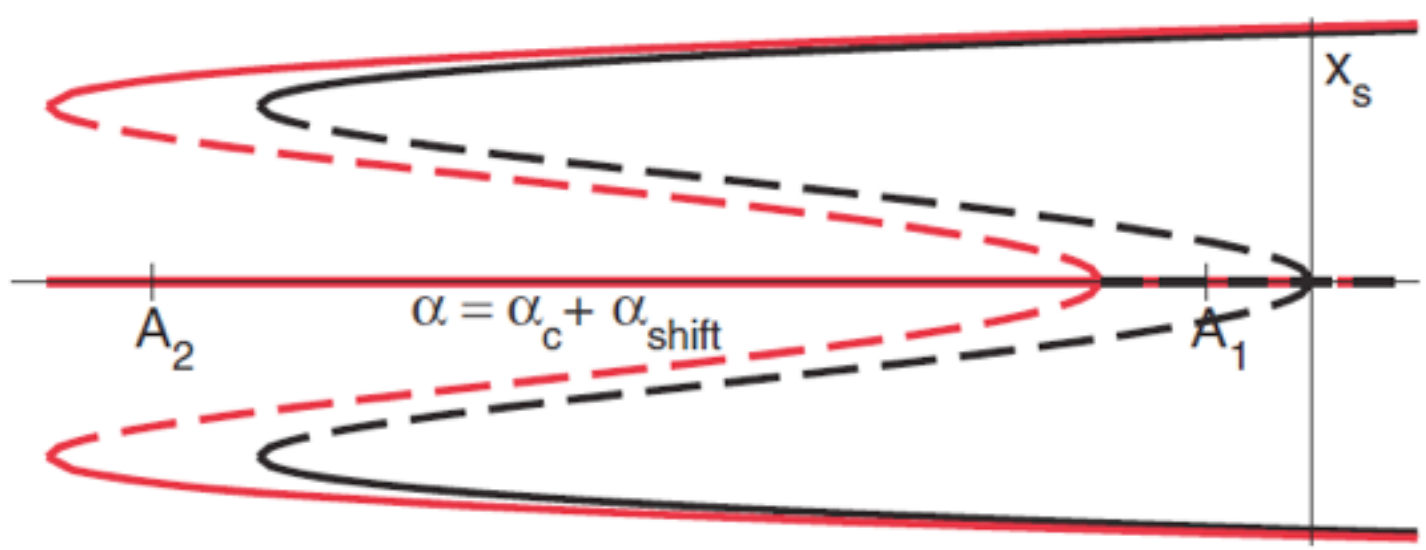
$$\mathrm{d}x = \left((\alpha_c + \eta^2\gamma_c Z^2(t))x + \gamma_c x^3 - \mu_c x^5\right)\mathrm{d}t$$
$$\mathrm{d}Z = \alpha_0 Z\mathrm{d}t + \mathrm{d}W(t).$$

$$Z^2(t) \approx \frac{1}{|\alpha_0|} + \psi(t), \quad \langle\psi(t)\psi(\tau)\rangle = \frac{1}{\alpha_0^2}e^{\alpha_0|t-\tau|}$$

$$\boxed{\alpha_{shift} = \eta^2\gamma_c/|\alpha_0| > 0}$$

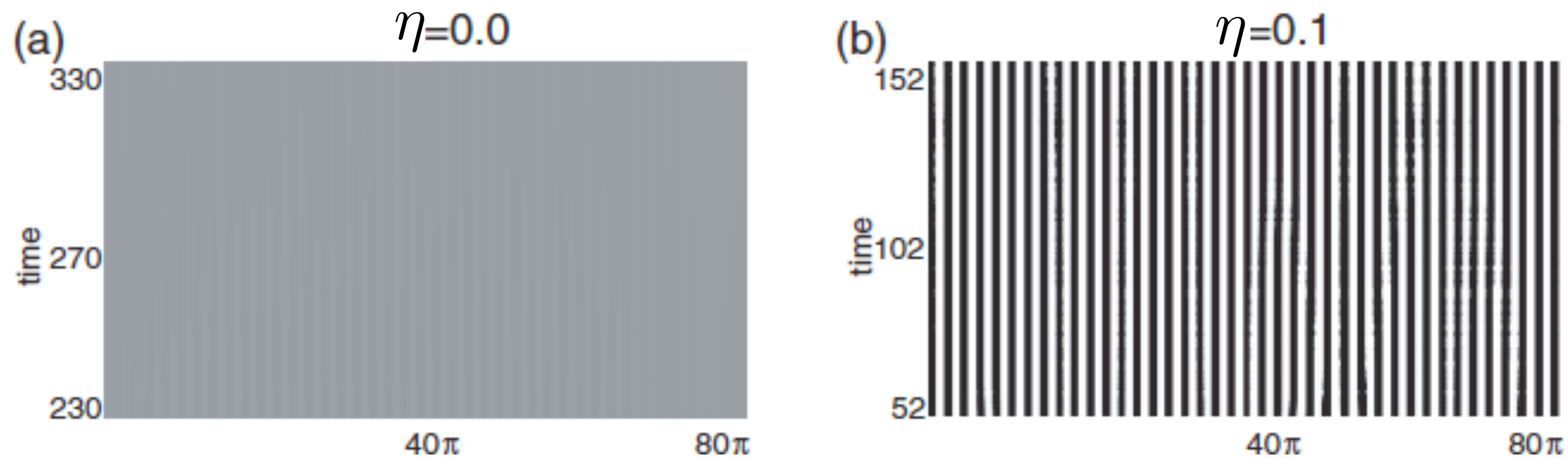
$$\boxed{\dot{x} = (\alpha_c + \alpha_{shift})x + \gamma_c x^3 - \mu_c x^5 + \eta^2\gamma_c\psi(t)x,}$$

schematic bifurcation diagram:

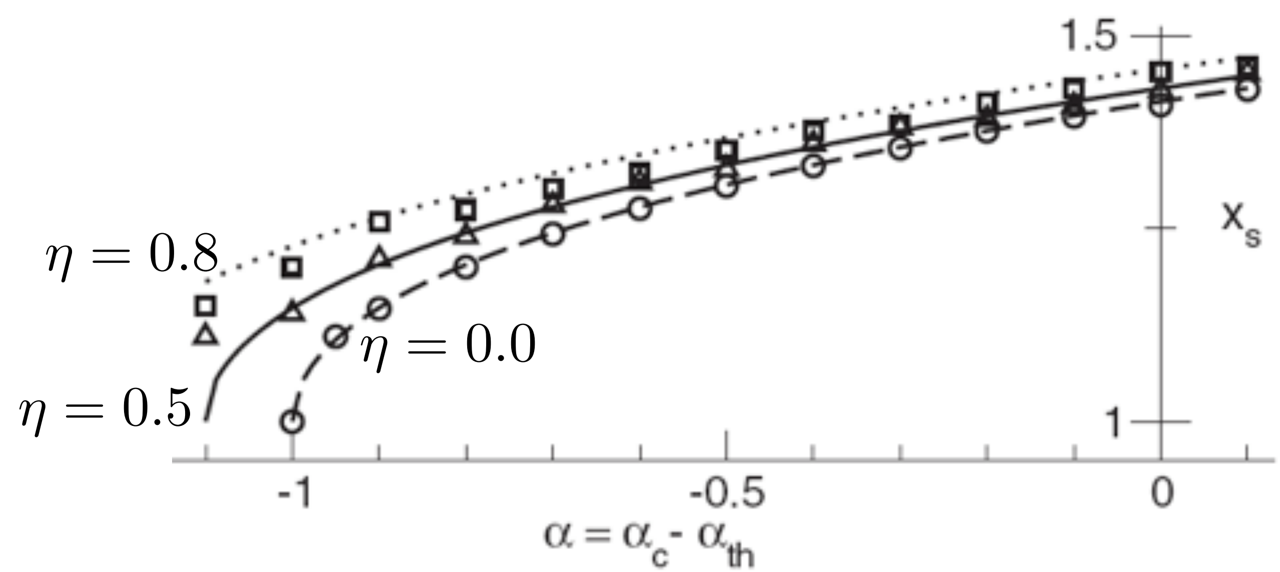


now: additive noise induces instability

noise-induced phase transition:



numerical test:



(Hutt, Europhys.Lett. (2008))

conditions that additive noise changes stability:

- presence of slow and fast mode

e.g.
$$\mathrm{d}x = (\alpha_c x + \gamma_c(xy^2 + x^3) - \mu_c x^5)\mathrm{d}t,$$
$$\mathrm{d}y = (\alpha_0 y + \gamma_0 x^2 y)\mathrm{d}t + \eta \mathrm{d}W(t),$$

- specific nonlinear coupling of slow and fast mode

- easy to understand if fast mode is noisy

- found in two- and infinite dimensional (spatial) systems

<u>question</u>: does it occur in delayed systems as well ?

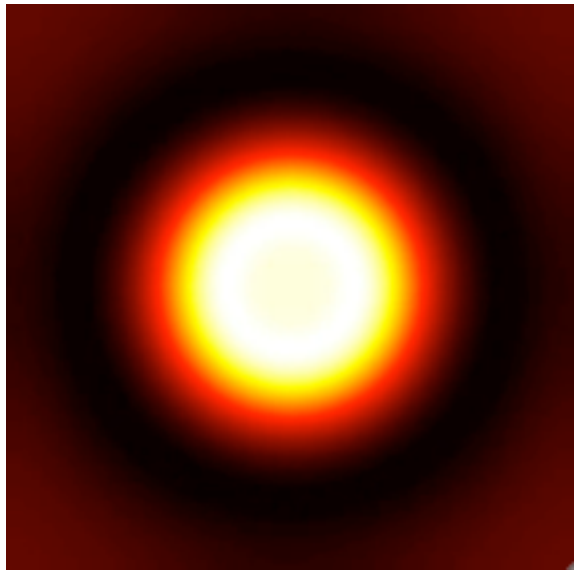
additive noise affects neural breathers

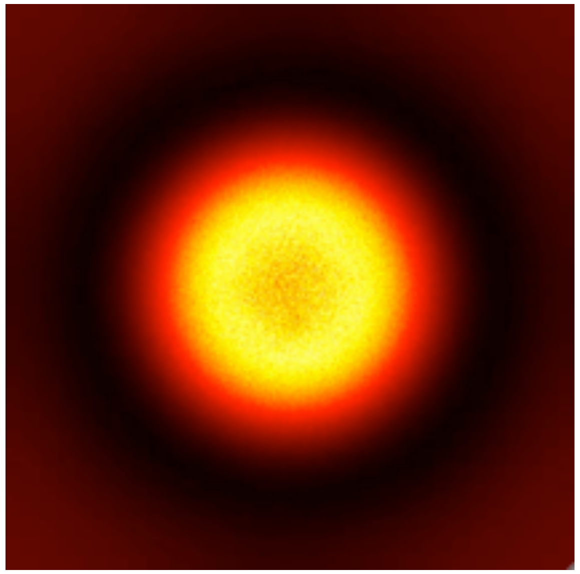
$$\tau \frac{\partial}{\partial t} V(\mathbf{x},t) = -V(\mathbf{x},t) + \int_\Omega d^2y K(|\mathbf{x}-\mathbf{y}|) H\left[V\left(\mathbf{y}, t - \frac{|\mathbf{x}-\mathbf{y}|}{c}\right)\right] + I(\mathbf{x},t)$$

$$K(r) = 0.176e^{-r/3} - 0.045e^{r/7}$$ (local excitation, lateral inhibition)

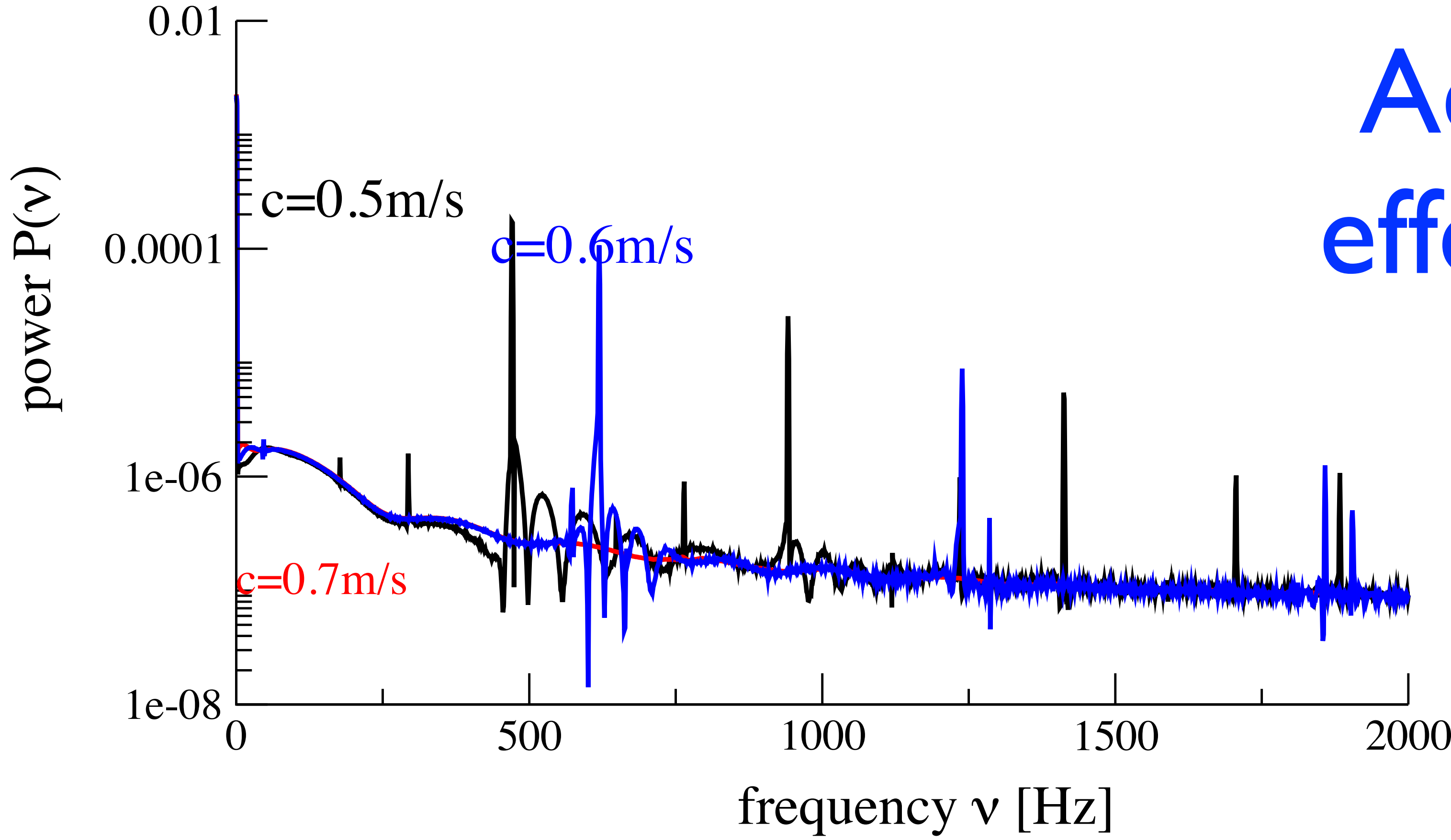
$$I(\mathbf{x},t) = (I_0 + \xi(\mathbf{x},t))e^{-\mathbf{x}^2/2\sigma}$$

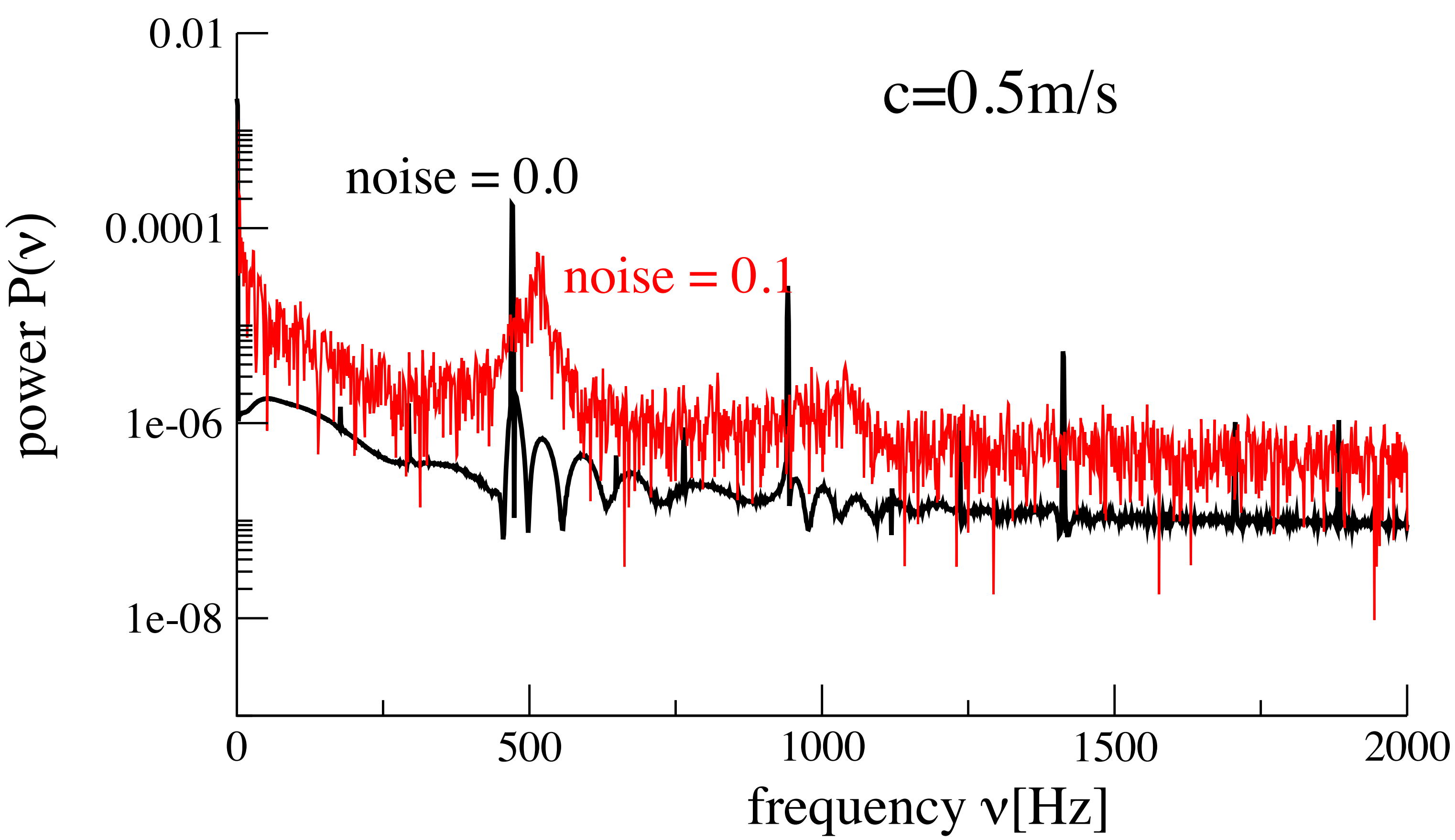
$$\langle \xi(\mathbf{x},t) \rangle = 0, \langle \xi(\mathbf{x},t)\xi(\mathbf{y},\tau) \rangle = 2\kappa^2 \delta(\mathbf{x}-\mathbf{y})\delta(t-\tau)$$





Activity at origin and effect of additive noise



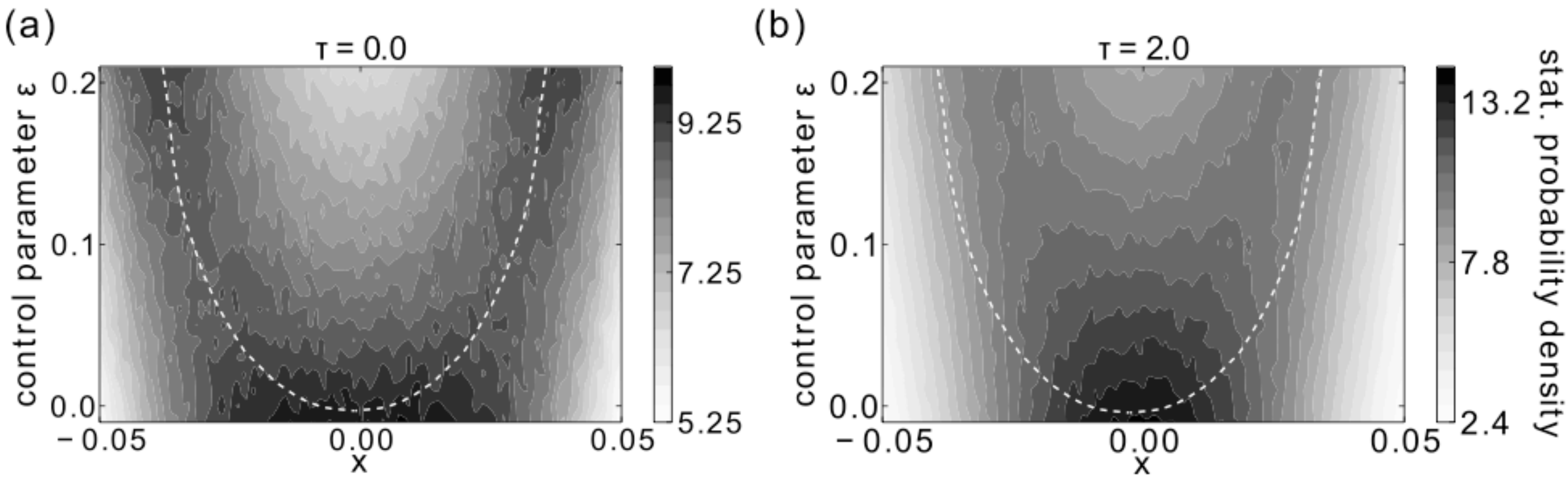


$l = 3mm, \tau = 100ms$

simple neural delayed system:

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} = -\alpha U(t) + K_0 S[U(t-\tau_0)] + E_0 + \xi(t).$$

$$S(V) = S_0/(1+\exp(-c(V-V_{thr})))$$



Hutt et al., Europhys. Lett. (2012)

$$\varepsilon = \beta - 1$$

stationary state: $x_0 = 0,\ x_{1,2} = \pm\sqrt{\varepsilon/\eta(\varepsilon+1)^3}$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -x(t) + x(t-\tau) + F(x,\varepsilon,t), \quad \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = 0$$

$$F(x,\varepsilon,t) = \varepsilon x(t-\tau) - \eta(1+\varepsilon)^3 x^3(t-\tau) + \kappa\xi(t)$$

characteristic equation: $\lambda = -1 + \exp(-\lambda\tau)$ (only critical value $\lambda=0$)

new variable: $z_t(\theta) = (x(t+\theta),\varepsilon)^T$, $z_t(\theta) = (u(t),\varepsilon)^T + s_t(\theta)$.

projection on stable and unstable eigenvector:

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = (1+\tau)^{-1} F[u+s_t,\varepsilon,t], \quad \frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t} = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} s_t(\theta) = \mathcal{A}(s_t) + \left(X_o - \frac{1}{1+\tau}\right) F[u+s_t,\varepsilon,t].$$

centre manifold ansatz:

$$s_t(\theta) = h_{det}(u, \varepsilon, \theta) + h_t(\theta, t)$$

result:

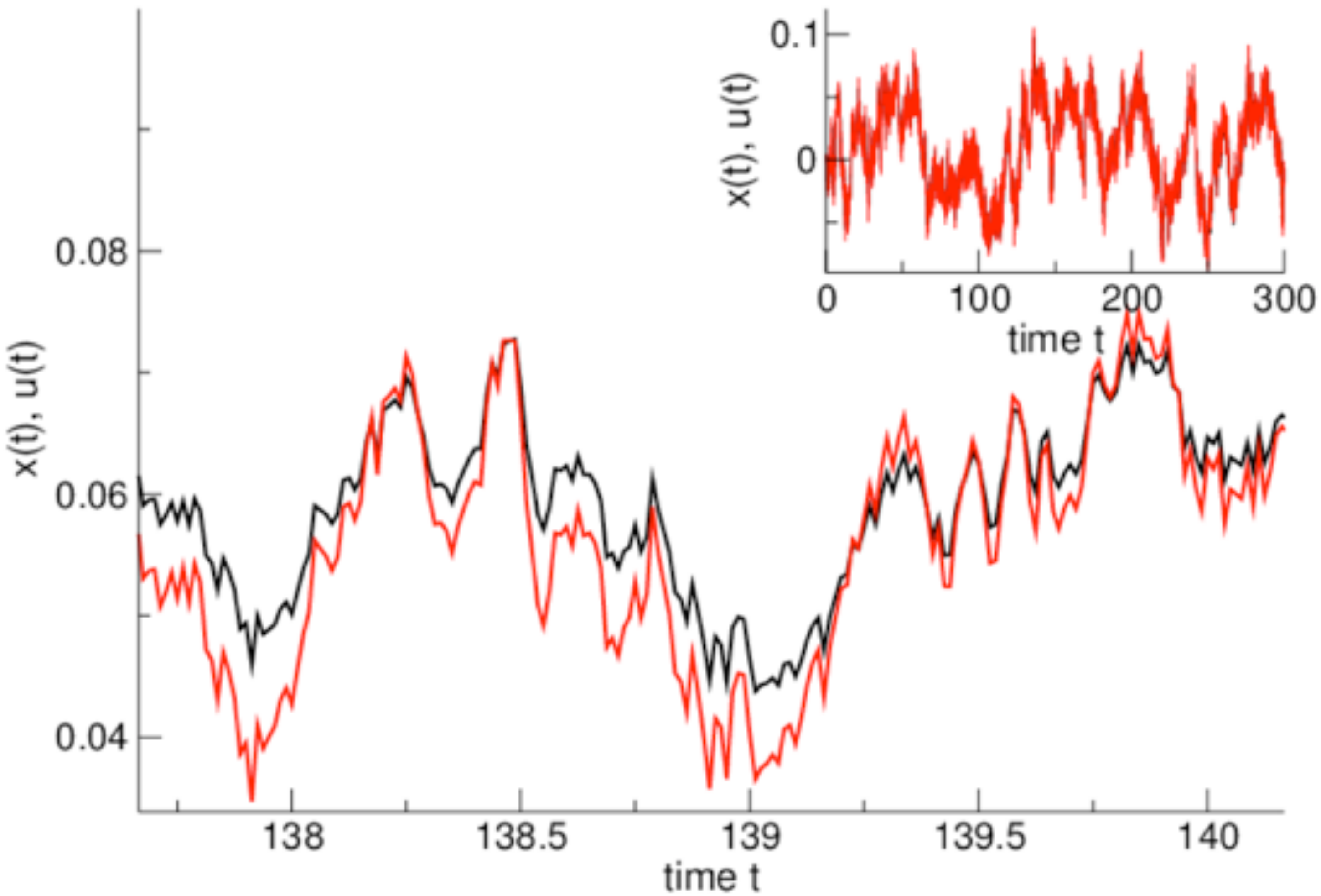
$$\begin{aligned} h_t(\theta, t) = \kappa & \int_0^{t+\theta} H_0(t+\theta-s)\mathrm{d}W(s) \\ & - \frac{\kappa}{1+\tau}\int_0^t H_0(t-s)\mathrm{d}W(s) \\ & - \frac{\kappa}{1+\tau}\int_{-\tau}^0 \int_0^{t+s} H_0(t+s-r)\mathrm{d}W(r)\mathrm{d}s \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}u(t)}{\mathrm{d}t} = & \frac{\kappa}{1+\tau}\xi(t) + A_0 Z(t) + B_0 Z^3(t) \\ & + (A_1 + B_1 Z(t) + C Z^2(t))u + (A_3 + B_3 Z^2(t))u^3 \\ & + A_5 u^5 + A_7 u^7 + A_9 u^9, \qquad (13) \end{aligned}$$

noise-induced shift

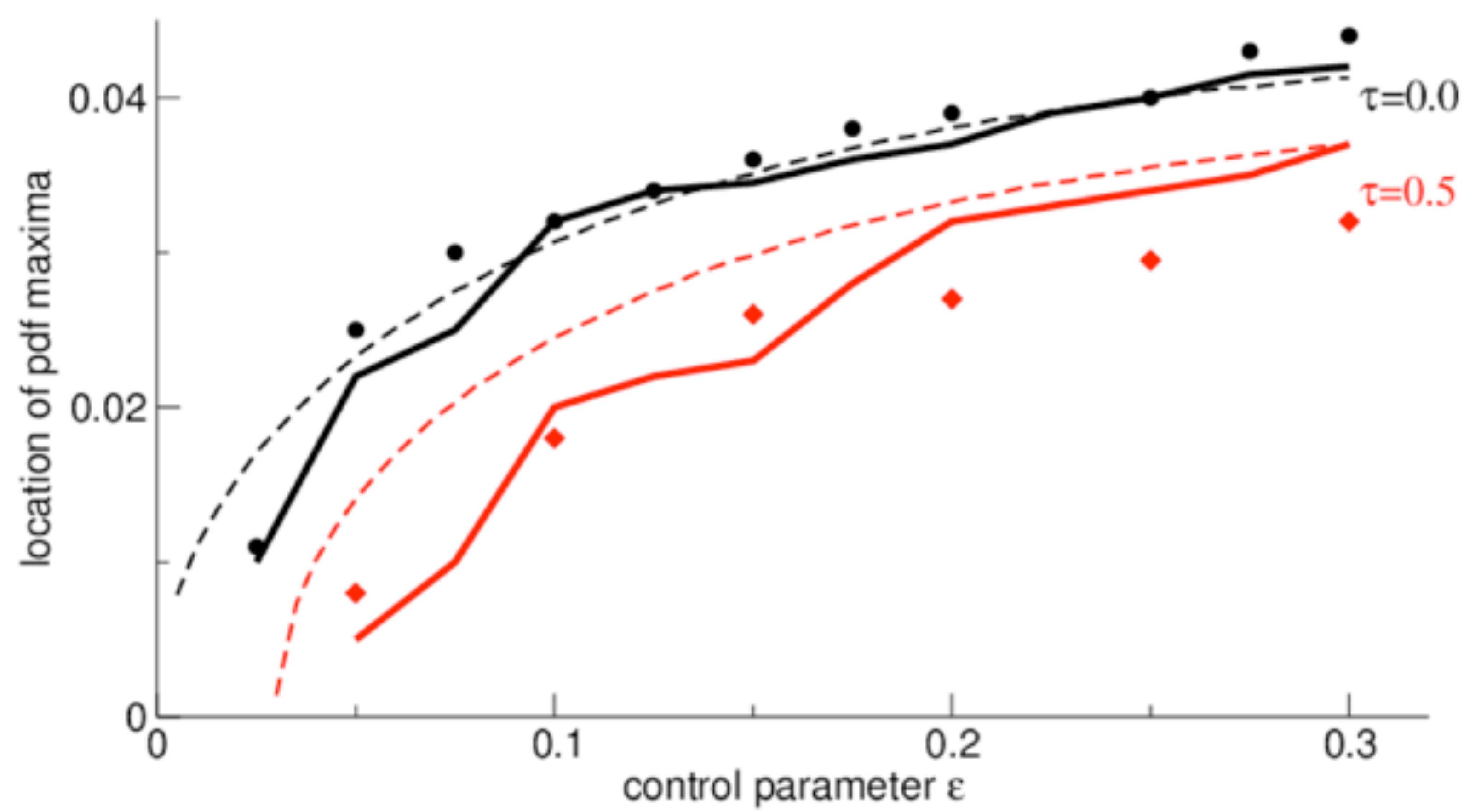
$Z(t) = h_t(-\tau, t)$

Hutt et al., Europhys. Lett. (2012)



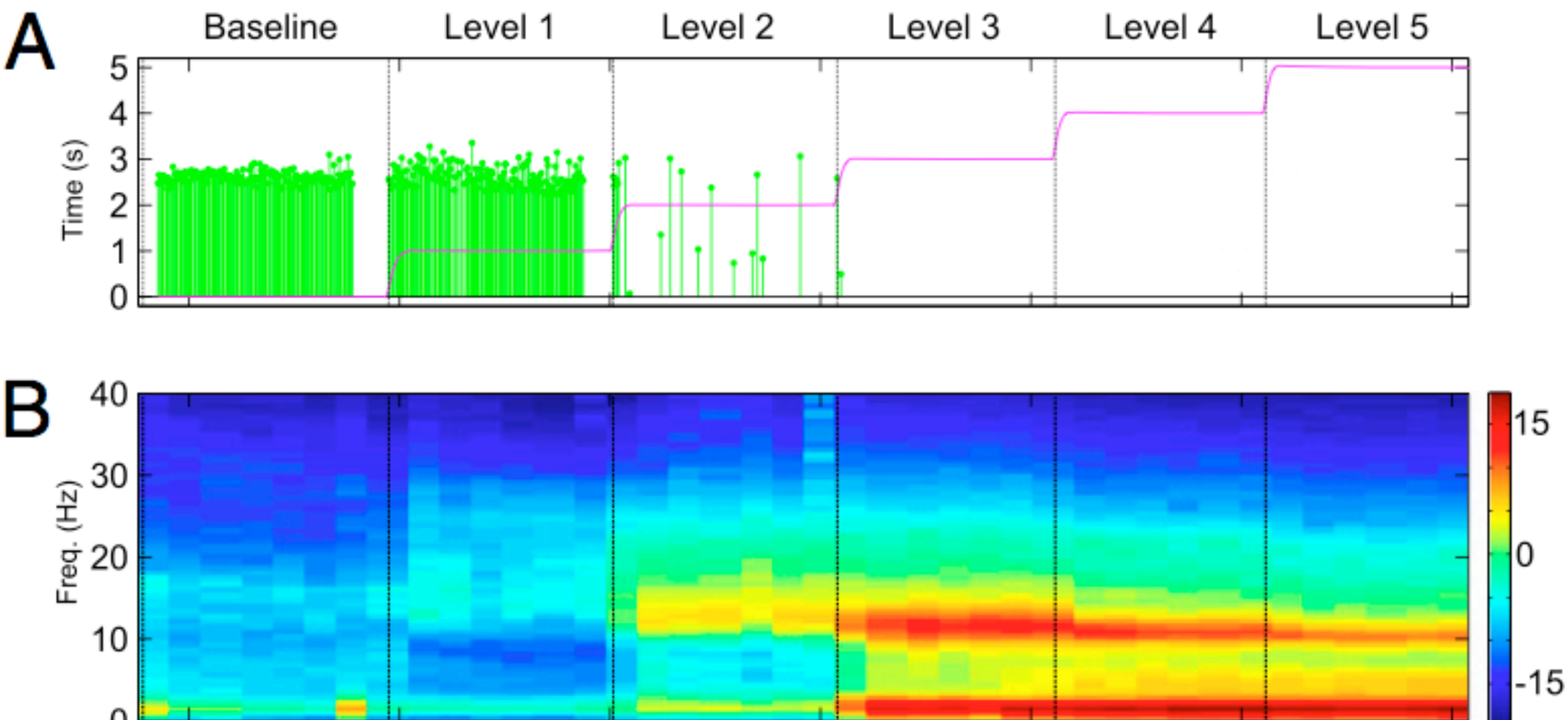
good pathwise reduction

delay necessary for effect



Hutt et al., Europhys. Lett. (2012)

effect of propofol on EEG



Ching et al. (2010)

$$\hat{L}_e\left(V_e(x,t) - V_E^r\right) = a_e K_E * S_E[V_e - V_i - \Theta_E]$$
$$\hat{L}_i\left(V_i(x,t) - V_E^r\right) = a_i f(p)\omega_0^2 K_I * S_I[V_e - V_i - \Theta_I]$$

$$K_N * S_N[V - \Theta_N] = \int_\Omega K_N(x-y) S_N\left[V(y, t - \frac{|x-y|}{v}) - \Theta_N\right] dy.$$