Quantum chaotic thermalization

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Hakenfest, Delmenhorst, 14.11.2012

to be exemplified for

Dicke model:
spin and oscillator coupled

$$\hat{H} = \hbar\left\{\omega_0 \hat{J}_z + \omega a^\dagger a + g\sqrt{\frac{2}{j}}(a + a^\dagger)\hat{J}_x\right\}$$

$$[a, a^\dagger] = 1 \qquad [J_x, J_y] = \mathrm{i}J_z \qquad \vec{J}^2 = j(j+1)$$

quantum phase transition at $g = g_c = \frac{\sqrt{\omega\omega_0}}{2}$

in superradiant phase: $*$ $\langle a \rangle \neq 0$ Hepp (1973) ...

$*$ chaos Graham (1984) ...

experiment with double condensate in optical resonator:

T. Esslinger 2010

phase transition observed

we propose extension for

chaos,

equilibration,

giant fluctuations

oscillator coherent states

$$|\alpha\rangle$$

complex α fixes mean displacement and momentum

$$\langle x \rangle \propto \alpha + \alpha^*, \qquad \langle p \rangle \propto \alpha - \alpha^*$$

with minimal uncertainty: one Planck cell

spin coherent states

$$|z\rangle$$

complex amplitude $z = \mathrm{e}^{\mathrm{i}\phi} \tan \frac{\theta}{2}$ fixes spin orientation as

$$\langle z|\hat{J}_x|z\rangle = j \sin\theta \cos\phi$$

$$\langle z|\hat{J}_y|z\rangle = j \sin\theta \sin\phi$$

$$\langle z|\hat{J}_z|z\rangle = j \cos\theta$$

with minimum uncertainty: single Planck cell

Glauber Q-function (Husimi)

$$Q(\alpha, z) \propto \langle \alpha, z|\rho|\alpha, z\rangle$$

exists for any ρ, real, non-negative

$$\langle a^m a^{\dagger n}\rangle = \int d^2\alpha d^2 z\, \alpha^m \alpha^{*n}\, Q$$

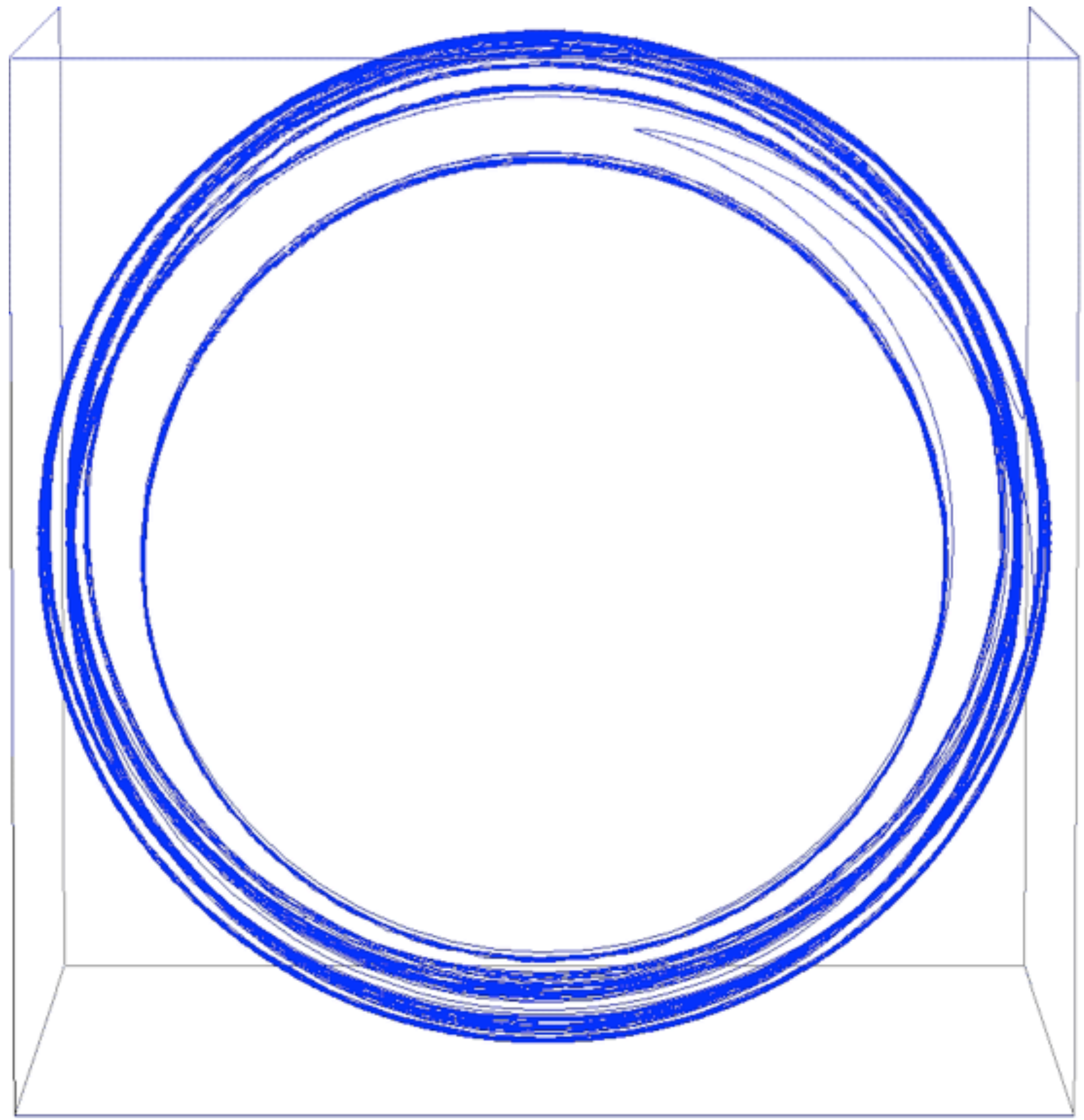
converges to classical phase-space density as $\hbar \to 0$

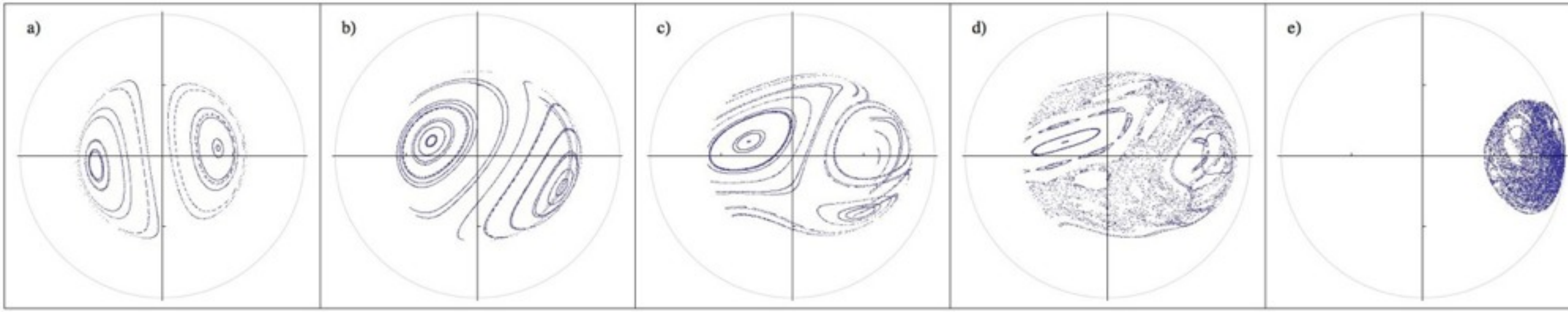
evolution equation is Fokker-Planck

$$\dot{Q} \quad = (\mathcal{L}_{\mathrm{drift}} + \mathcal{L}_{\mathrm{diff}})Q$$

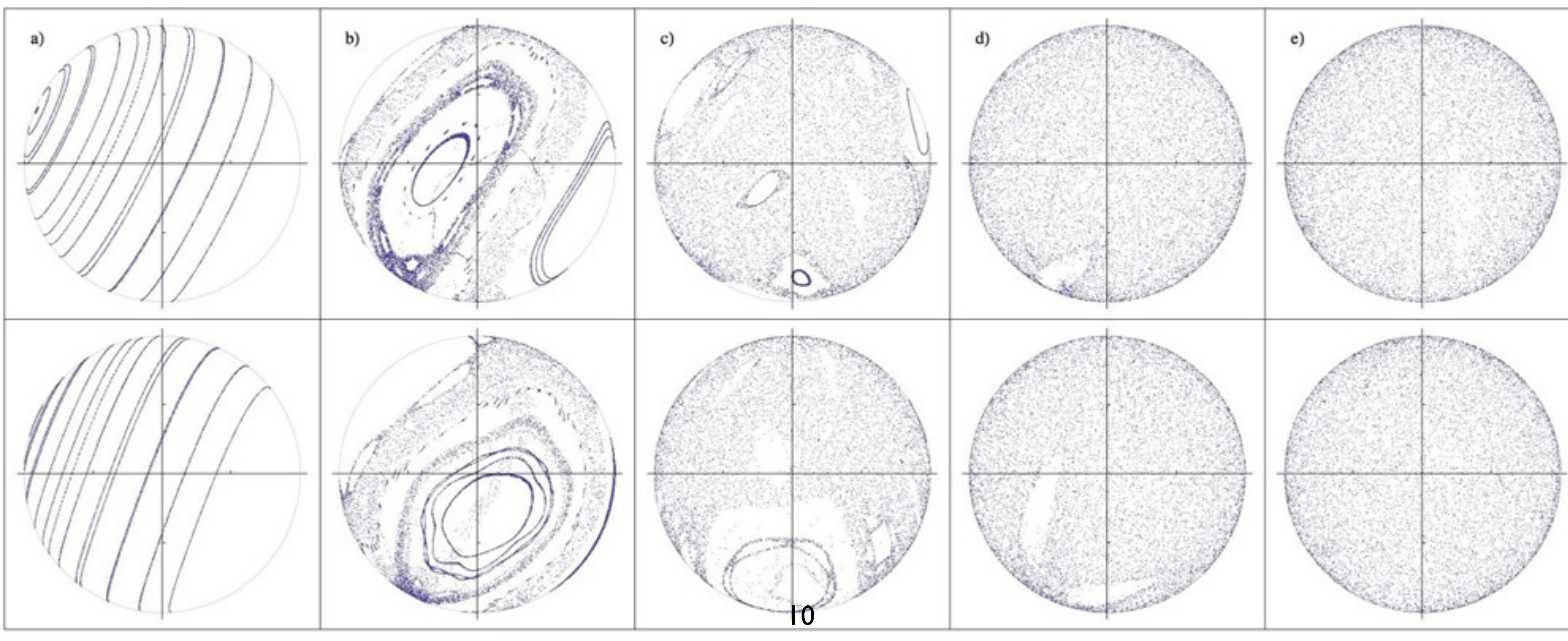
$$\begin{aligned}\mathcal{L}_{\mathrm{drift}} = \; & \mathrm{i}\partial_\alpha \left(\omega\alpha + g\sqrt{2j}\frac{z+z^*}{1+|z|^2} \right) \\ & + \mathrm{i}\partial_z \left(-\omega_0 z + \frac{g}{\sqrt{2j}}(1-z^2)(\alpha+\alpha^*) \right) + \mathrm{c.c.}\end{aligned}$$

$$\mathcal{L}_{\mathrm{diff}} = \frac{\mathrm{i}g}{\sqrt{2j}}\partial_\alpha \partial_z (1-z^2) + \mathrm{c.c.}$$





$$\frac{g}{g_c} = 0.2 \qquad 0.7 \qquad 0.9 \qquad 1.01 \qquad 1.5$$



drift of Q along classical trajectories (diffusion neglected):

assume initial coherent state
(with `tiny circular support´ in energy shell)

↕

swarm of initial points, gives rise to bundle of trajectories

initially circular support deforms while preserving `area´:

squeezes, stretches, bends, wriggles, without end,

visiting everywhere in the energy shell

until

until

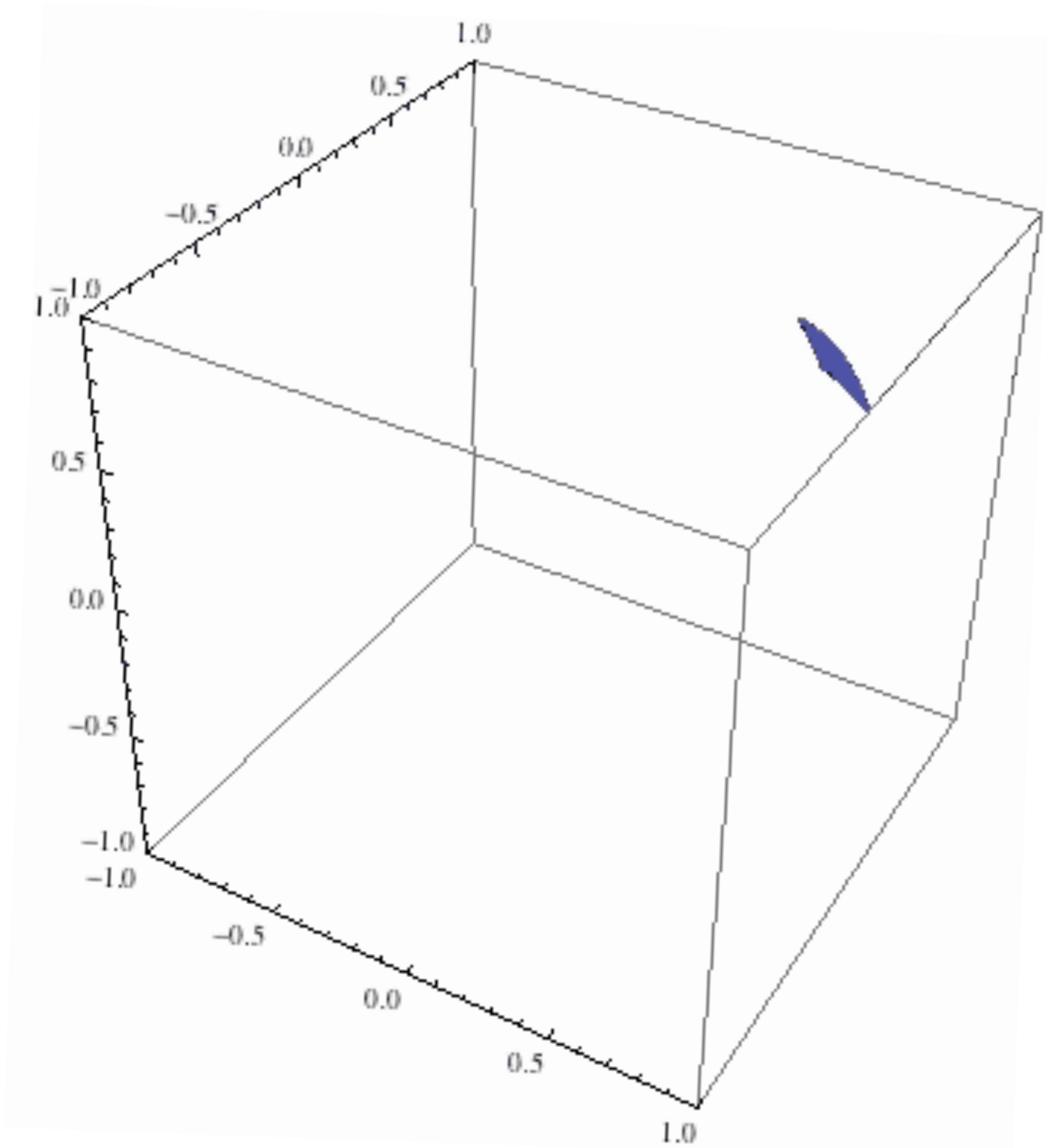
Q forms an `infinitely´ fissured landscape,

over its infinitely wriggling support,

finite resolution, however, suggests

constant Q over the energy shell,

microcanonical distribution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quantum mechanics

forbids

classical nonsense like infinitely fine structures

quantum diffusion washes out singular fissures in Q

so as to corroborate microcanonical distribution

but how?

quantum diffusion

$$\mathcal{L}_{\rm diff} = \frac{{\rm i}g}{\sqrt{2j}}\partial_\alpha\partial_z(1-z^2) + {\rm c.c.}$$

$$D = \begin{pmatrix} 0 & d \\ d^\dagger & 0 \end{pmatrix}$$ real symmetric, chiral

$$\lambda^4 - \lambda^2 \ {\rm tr}\, dd^\dagger + \det dd^\dagger = 0$$

eigenvalues come as two real $\pm$ pairs

2 eig‘vec‘s of D `expansive´	$\longleftrightarrow$	$\lambda > 0$	genuine diffusion
2 `contractive´	$\longleftrightarrow$	$\lambda < 0$	antidiffusion

classical chaotic drift also has 4 distinguished directions:

2 neutral (along flow and transverse to energy shell),

1 expansive (unstable), 1 contractive (stable) deterministic contraction and quantum diffusive expansion

balance at quantum scales $\propto \frac{1}{\sqrt{j}}$

antidiffusive shrinking cannot become dominant in the classically stable direction (Q exists!)

this is how classically favored singular fissures are quantum mechanically prevented

exponential growth in the classically unstable direction, quantum diffusive/antidiffusive corrections unimportant

projection onto a moving Poincaré section

quantitative check:

fix phase space point X_0 and choose deflection δX

write Fokker Planck eqn with δX as independent variables

$$\dot{Q} = \Big(\partial_{\delta X_i} d_i(X_0 + \delta X) + \partial_{\delta X_i}\partial_{\delta X_j} D_{ij}(X_0 + \delta X)\Big)Q$$

truncate wrt deflection, leading order

express deflection in terms of local coordinates along

classically stable, unstable, and neutral directions: s, u, ϵ, τ

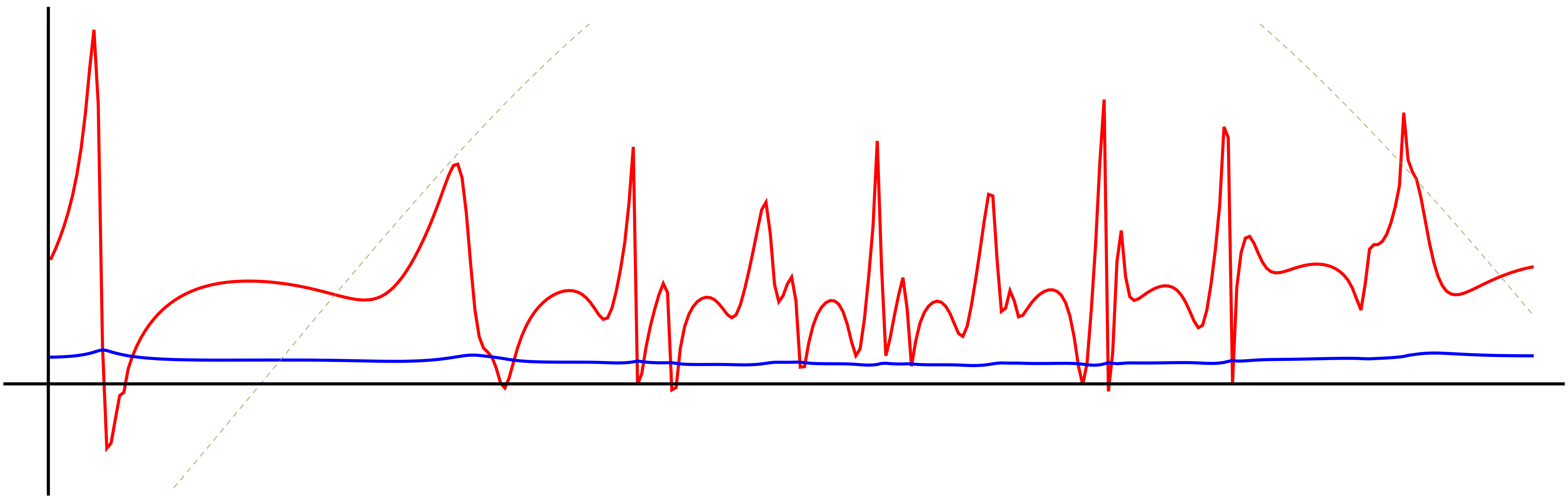
$$\dot{Q} = \left(\lambda \partial_s s - \lambda \partial_u u + \partial_s^2 D_{ss} + \partial_u^2 D_{uu} + 2\partial_s \partial_u D_{su}\right) Q$$

time dependent diffusion matrix

dynamics projected on comoving Poincaré section

$$\mathrm{v}ar_t(s) = \mathrm{e}^{-\lambda t}\mathrm{var}_0(s) + \int_0^t dt' \mathrm{e}^{-\lambda s} D_{ss}(t-t')$$

must be positive at all times since Q is



minimal scale for stable direction $\propto \frac{1}{\sqrt{j}}$

equilibration mechanism:

chaos provides endless stretching
and folding in unstable direction

→ equipartition in energy shell

quantum diffusion smoothes
in classically stable direction,
to minimal scale $\propto 1/\sqrt{j} \propto \sqrt{\hbar}$

→ equipartition in energy shell

how general?

Fokker-Planck equations often Q describe unitary dynamics

(kicked top, SU(3)-dynamics, Bose-Hubbard model)

given chaos, equilibration as above

in general, given chaos,

$$\dot{Q} = \mathcal{L}Q$$

$$\mathcal{L} = \sum_{n=1,2,\ldots} \hbar^{n-1} \partial_X^n f_n(X)$$

for classically stable coordinates, along classical trajectory

$$\partial_{\tilde{s}} \lambda \tilde{s} + \sum_{n=2,,\ldots} \partial_{\tilde{s}}^n \hbar^{(n-2)/2} f_n(X_t)$$

suggests asymptotic validity of Fokker-Planck eqn

summary

Q obeys Fokker-Planck equation

equilibration to microcanonical distribution,

due to classically chaotic drift and quantum diffusion

(stretching, folding, quantum smoothing)