The Physics of Finance: Collective Dynamics in a Complex World

Lisa Borland
Integral Dev. Corp., CA, USA

Synergetics Symposium in Honor of Professor Haken
On the Occasion of his 85th Birthday
Delmenhorst, November 2012
The price formation process: multitude of complex global interactions
Not possible to run experiments: just one single history, one realization
The price formation process: multitude of complex global interactions
Not possible to run experiments: just one single history, one realization

Tsallis (q = 1.4)
• For $q = 1.4$, a **10-sigma event** has probability $p = 0.028\%$ (about 1 in a decade)

• This is equally probable as a **3.45-sigma event** based on a Gaussian distribution

• For a Gaussian, a **10-sigma event** has probability $p = 0\%$
Exploring the joint stochastic process

What do we know about volatility?

- Across time for a given stock  Part 1

- Across stocks at a given time  Part 2
Some more properties of financial time-series

Universal features: Across time, Across the globe

For stocks (IBM), Commodities (Oil), Currencies (EUR/USD) etc

- Power Law distributions, persistent
- Slow decay to Gaussian
- Volatility clustering
- Close-to log-normal distribution of volatility
- Time-reversal asymmetry
- Leverage effect
  (Skew: Negative returns --> Higher volatility)
Across Time:

Vast number of anomalous statistics (stylized facts)

Some Stylized Facts

\( \log P \)

\( Y(t+\tau) - Y(t) \)

Kurtosis decays as \( \tau \)

Leverage effect (skew)

SP500

\(-0.3\)

\(-1\)
Challenge:

A somewhat realistic model of price variations

Important for real-world reasons:

- Risk control
- Hedging
- Development of trading strategies
- Option pricing
- Pricing of credit risk
Century long quest:

First random walk model by Bachelier in 1900

Gaussian model, basis for celebrated 1974 Black-Scholes option pricing formula

\[ dS = \mu S dt + \sigma S d\omega \]

- \( S \) Price
- \( \mu \) Rate of return
- \( \sigma \) Volatility
- \( \omega \) Gaussian noise
Real log-returns not Gaussian

10 NASDAQ stocks, 1 min, 2001

100 SP stocks, daily

- Empirical
- --- Gaussian
- q=1.43 Tsallis Distribution
The ideal model:

- Intuitive
- Analytically tractable
- Parsimonious

Popular models:
- Stochastic Volatility (Heston 1993)
- Levy noise
- GARCH
- Multi-fractal model (Bacry, Delour Muzy PRE 64 2001)
Exploring the Joint Stochastic Process:

Across time ....

1. A non-Gaussian model of returns

2. Options pricing incorporating fat-tails

3. Applications
Non-Gaussian model based on statistical feedback

\[ dS = \mu S dt + \sigma S d\Omega \]
Non-Gaussian model based on statistical feedback

\[ dS = \mu S dt + \sigma S d\Omega \]

\[ d\Omega = P(\Omega)^{\frac{1-q}{2}} \ d\omega \]

For stocks \( q = 1.4 \)

Mathematical details: the non-Gaussian model

\[ dS = \mu S dt + \sigma S d\Omega \]
\[ d\Omega = P(\Omega)^{\frac{1}{2}} d\omega \]
\[ \frac{dP}{dt} = \frac{1}{2} \frac{d^2 P^{2-q}}{d\Omega^2} \]
\[ P = \frac{1}{Z(t)} (1 - (1 - q)\beta(t)\Omega^2)^{\frac{1}{1-q}} \]

Nonlinear Fokker-Planck

Tsallis Distribution (Student-t)
In other words: After inserting the expression for $P \rightarrow$

a state dependent deterministic model

$$d\Omega_t = \left[ a_t + (q - 1)b_t \left( \Omega_t - \Omega_0^2 \right) \right]^2 d\omega_t$$

Work with

$$\Omega = \Omega(S)$$

as a computational tool allowing us to find the solution

Generalization: M. Vellekoop and H. Nieuwenhuis, QF, 2007

Averaging and conditioning w.r.t variable $\Omega_0$
Not a perfect model of returns:

Well-defined starting price and time

Nevertheless:

Reproduces fat-tails and volatility clustering

Closed form option-pricing formulae

Success for options and credit (CDS) pricing
SP500

q = 1.5

Reproduces fat-tails and volatility clustering
With a model of stock behavior, we want to study more complicated derivative markets

Some derivatives:

- Options

- Credit default swaps

Like stocks, these are traded in large volumes, across the world

How to price them?

The price must be related to the underlying instrument, the stock
Options

The right to buy or sell a stock at a certain price (the strike) at a certain time in the future.

Greek mathematician Thales used call options on olives to make a huge profit when he foresaw a good harvest.

In Holland in the 1600s, tulip options were traded quite a bit by speculators prior to the famous tulip bubble.

1973 - the Nobel-prize winning Black-Scholes option pricing formula.
Black-Scholes Option Pricing Paradigm

\( S \) the risky stock

\( F(S) \) the risky option on \( S \)

Risk-less portfolio: \( \Pi = S + nF \)

Return on \( \Pi \) must be risk-free rate \( r \), due to no-arbitrage

This leads to the famous

Black-Scholes PDE for solving \( F \) (1973, Nobel prize 1997)

\[
\frac{\partial F}{\partial t} + rS \frac{\partial F}{\partial S} + \frac{1}{2} \frac{\partial^2 F}{\partial S^2} \sigma^2 S^2 - rF = 0
\]
Black-Scholes Option Pricing Paradigm

Equivalently:

Price options as expectations with respect to a (risk-neutral) equivalent martingale measure

\[ F = \left\langle e^{-rT} h(S) \right\rangle_Q \]

where \( h \) = the payoff of the option

(Whole field of mathematical finance based on these notions)
Example European Call Option (F=C): the right to buy at strike K at time T

\[ c = \left\langle e^{-rT} \max[S(T) - K, 0] \right\rangle_Q \]

The price of the call option depends on \( S(0), K, T, r \) and \( \sigma \).

Problem: Black-Scholes model must use a different \( \sigma \) for each K.

Volatility Smile: the plot of \( \sigma \) versus K.
Options: Real-time pricing and hedging

<table>
<thead>
<tr>
<th>Series</th>
<th>cLast</th>
<th>cBid</th>
<th>cAsk</th>
<th>cThv</th>
<th>AIVol</th>
<th>pLast</th>
<th>pBid</th>
<th>pAsk</th>
<th>pThv</th>
</tr>
</thead>
<tbody>
<tr>
<td>5SEP30A</td>
<td>16.60</td>
<td>17.40</td>
<td>18.90</td>
<td>18.10</td>
<td>149.89</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.00</td>
</tr>
<tr>
<td>5SEP32A</td>
<td>14.10</td>
<td>14.90</td>
<td>16.40</td>
<td>15.61</td>
<td>199.92</td>
<td>0.50</td>
<td>0.50</td>
<td>0.50</td>
<td>0.00</td>
</tr>
<tr>
<td>5SEP35A</td>
<td>11.60</td>
<td>12.40</td>
<td>13.90</td>
<td>13.11</td>
<td>169.68</td>
<td>0.50</td>
<td>0.50</td>
<td>0.50</td>
<td>0.00</td>
</tr>
<tr>
<td>5SEP37A</td>
<td>9.40</td>
<td>10.20</td>
<td>11.10</td>
<td>10.61</td>
<td>96.67</td>
<td>0.10</td>
<td>0.50</td>
<td>0.50</td>
<td>0.00</td>
</tr>
<tr>
<td>5SEP40A</td>
<td>7.00</td>
<td>7.70</td>
<td>8.60</td>
<td>8.11</td>
<td>75.47</td>
<td>0.10</td>
<td>0.50</td>
<td>0.50</td>
<td>0.00</td>
</tr>
<tr>
<td>5SEP42A</td>
<td>4.60</td>
<td>5.30</td>
<td>6.20</td>
<td>5.64</td>
<td>60.22</td>
<td>0.50</td>
<td>0.15</td>
<td>0.15</td>
<td>0.02</td>
</tr>
<tr>
<td>5SEP45A</td>
<td>3.20</td>
<td>3.00</td>
<td>3.70</td>
<td>3.32</td>
<td>49.04</td>
<td>0.25</td>
<td>0.15</td>
<td>0.50</td>
<td>0.21</td>
</tr>
<tr>
<td>5SEP47A</td>
<td>1.50</td>
<td>1.05</td>
<td>1.55</td>
<td>1.52</td>
<td>33.29</td>
<td>0.65</td>
<td>0.40</td>
<td>0.90</td>
<td>0.90</td>
</tr>
<tr>
<td>5SEP50A</td>
<td>0.30</td>
<td>0.20</td>
<td>0.30</td>
<td>0.50</td>
<td>30.82</td>
<td>2.40</td>
<td>1.80</td>
<td>2.40</td>
<td>2.38</td>
</tr>
<tr>
<td>5SEP52A</td>
<td>0.05</td>
<td>0.20</td>
<td>0.11</td>
<td>35.04</td>
<td>4.90</td>
<td>4.00</td>
<td>4.80</td>
<td>4.49</td>
<td>4.49</td>
</tr>
<tr>
<td>5SEP55A</td>
<td>0.50</td>
<td>0.05</td>
<td>0.50</td>
<td>0.02</td>
<td>86.30</td>
<td>7.30</td>
<td>6.50</td>
<td>7.40</td>
<td>6.92</td>
</tr>
<tr>
<td>5SEP57A</td>
<td>0.05</td>
<td>0.05</td>
<td>0.00</td>
<td>62.44</td>
<td>9.80</td>
<td>9.00</td>
<td>9.90</td>
<td>9.42</td>
<td>9.42</td>
</tr>
<tr>
<td>5SEP60A</td>
<td>0.05</td>
<td>0.05</td>
<td>0.00</td>
<td>74.48</td>
<td>12.00</td>
<td>11.20</td>
<td>1...</td>
<td>11.92</td>
<td>11.92</td>
</tr>
<tr>
<td>5SEP65A</td>
<td>0.05</td>
<td>0.05</td>
<td>0.00</td>
<td>96.27</td>
<td>17.00</td>
<td>16.20</td>
<td>1...</td>
<td>16.92</td>
<td>16.92</td>
</tr>
<tr>
<td>5SEP70A</td>
<td>0.05</td>
<td>0.05</td>
<td>0.00</td>
<td>115.69</td>
<td>21.70</td>
<td>20.90</td>
<td>2...</td>
<td>21.92</td>
<td>21.92</td>
</tr>
<tr>
<td>5SEP75A</td>
<td>0.05</td>
<td>0.05</td>
<td>0.00</td>
<td>133.26</td>
<td>26.70</td>
<td>25.90</td>
<td>2...</td>
<td>26.92</td>
<td>26.92</td>
</tr>
<tr>
<td>5OCT35A</td>
<td>11.90</td>
<td>12.60</td>
<td>14.10</td>
<td>13.26</td>
<td>73.83</td>
<td>0.50</td>
<td>0.50</td>
<td>0.03</td>
<td>0.03</td>
</tr>
<tr>
<td>5OCT40A</td>
<td>7.60</td>
<td>8.30</td>
<td>9.20</td>
<td>8.54</td>
<td>51.02</td>
<td>0.45</td>
<td>0.30</td>
<td>0.80</td>
<td>0.28</td>
</tr>
<tr>
<td>5OCT45A</td>
<td>3.80</td>
<td>4.30</td>
<td>5.10</td>
<td>4.62</td>
<td>43.02</td>
<td>1.35</td>
<td>1.10</td>
<td>1.70</td>
<td>1.34</td>
</tr>
</tbody>
</table>
Volatility smile in the example of previous slide

![Volatility Smile Diagram](image.png)

Traders intuitively correct for the tails by bumping up the Black-Scholes volatility at far strikes.
Part 1: Across time ....

1. A non-Gaussian model of returns

2. Options pricing incorporating fat-tails

3. Applications
Non-Gaussian option pricing with statistical feedback model:

There is a unique equivalent martingale measure \( \Phi \) can obtain closed-form option pricing formulae

Note: This is not the case for Levy noise or stochastic volatility

Parsimony:

Because the model incorporates tails for \( q = 1.4 \), we only need one value of sigma to capture all option prices across strikes
1) Exploit PDE’s implied by arbitrage-free portfolios

Solve PDE to get option price

Generalization: M. Vellekoop and H. Nieuwenhuis, 2007

2) Convert prices of assets into martingales

Take expectations to get option price

\[ c = \left< e^{-rT} h(S(T)) \right>_Q \]
Use non-Gaussian model as basis for stock price dynamics

\[
S(T) = S(0) \exp \left\{ \Omega_T + rT - \frac{\sigma^2}{2} \int_0^T P(\Omega_t)^{1-q} \, dt \right\}
\]

Integrate using generalized Feynman-Kac

\[
\approx \gamma(T) \Omega_T^2
\]

- Approximation allows for closed-form solutions

- Due to the approximation:
  
  Martingale “breaks down” but valid for \( \sigma^2 T \ll 1 \)
European Call

Stock Price

\[ c = \left\{ e^{-rT} \max[S(T) - K, 0] \right\}_Q \]

\[ S(T) = S(0) \exp \left\{ \sigma \Omega_T + rT - \frac{\sigma^2}{2} \gamma_T - (1 - q) g_T \Omega_T^2 \right\} \]

Payoff if \( \{ S(T) > K \} \quad \Rightarrow \quad d_1 \leq \Omega_T \leq d_2 \)

\[ c = e^{-rT} \int_{d_1}^{d_2} (S(T) - K) P_q(\Omega_T) d\Omega_T \]

q = 1: P is Gaussian \quad q > 1: P is fat tailed Tsallis dist.
Call Price Difference

\[ C(q = 1.5) - C(q = 1) \]

S(0) = $50, \quad r = 6\%, \quad \sigma = 0.3
This is very similar to the volatilities that traders use.

Need only one sigma across all strikes if $q = 1.4$.
Example (Japanese Yen futures)
Extension to include skew (asymmetry) in the model [Borland and Bouchaud, 2004]

Price crashes induce higher volatility than price rallies

Leverage effect

Fluctuations modeled as

$$\sigma S^{\alpha} P^{2} d\omega$$

$$\alpha \leq 1$$

$q$ Controls the tail

$\alpha$ Controls the skew

$\sigma$ Volatility parameter

We obtain closed-form formulae for call options

Pade expansion, Feynman-Kac
SP500 OX

q=1.4, with skew

T=.03

T=0.1

T=0.2

T=0.3

T=0.55

Implied Volatility

30.00 45.00 60.00

Strike K

0.200 0.300 0.400 0.500

Strike K
Part 1: Across time ....

1. A non-Gaussian model of returns

2. Options pricing incorporating fat-tails

3. Applications
Can fit the model to historical distribution \( \rightarrow \) calculate option prices

Can imply the parameters from the option market

Results for stocks (top 100 stocks in S&P index)

In both stock and option markets:

- \( q = 1.4 \) converging slowly to Gaussian
- \( \alpha \approx 0.3 \)
- \( \sigma \approx 30\% \)
Back test of trading strategy (with A. Christian Silva)

Exploit slight deviations which should converge

Live returns 2009

Cumulative Profit

Margin Cost

1999 - 2005
Options look good ... what about pricing credit?

Credit Default Swaps:

Credit Default Swaps:

Payment if the company defaults

Protection Buyer

→

Protection Seller

Premium (CDS price)

CDS price depends on probability to default.

We get pricing formula from the non-Gaussian model.

We implied $q$, $\alpha$, $\sigma$ from empirical CDS prices:

$$q = 1.2 - 1.4$$

$$\alpha = 0.2 - 0.5$$

$$\sigma = 30\%$$
Non-Gaussian model well describes many features of:

- Stock Markets
- Option Markets
- Debt and Credit Markets

• Key ingredient of success for statistical feedback model: long-range memory (with respect to a single time)

• Random increments are uncorrelated but not independent
Motivation of statistical feedback model:

Intuition - traders react to extreme events

Generalization: Multi timescale statistical feedback model:

Intuition -

different classes of traders react on different time-scales

For example: HARCH (Muller et al)
FIGARCH (Baillie et al)
Multi-timescale statistical feedback (Borland and Bouchaud)
A multi-time scale non-Gaussian model of stock return [Borland and Bouchaud 2012]

\[
\Delta y_t = \sigma_t \Delta \omega_t
\]

\[
\sigma_t^2 = \sigma_0^2 \left[ 1 + g \sum_{l=1}^{\infty} \frac{1}{l^\alpha} (y_t - y_{(t-l)})^2 \right]
\]

Reproduces “all” stylized facts across time
Motivation: Traders act on all different time horizons

A multi-time scale non-Gaussian model of stock return [Borland and Bouchaud 2012]

\[ y = \log(\text{Stock Price}) \]

\[ \omega_t \quad \text{Gaussian noise uncorrelated in time} \]

\[ \sigma_t \quad \text{Volatility} \]

\[ \Delta y_t = \sigma_t \Delta \omega_t \]

\[ \sigma_t^2 = \sigma_0^2 \left[ 1 + g \sum_{l=1}^{\infty} \frac{1}{l^\alpha} \left( y_t - y_{(t-l)} \right)^2 \right] \]

Reproduces “all” stylized facts across time

\[ g \quad \text{controls the feedback} \]

\[ \alpha \quad \text{controls the memory} \]

For \( t-l = 0 \) recovers Statistical feedback model

GARCH

HARCH (Muller et al)

FIGARCH (Baillie et al)
In our study:
- Simulations
- Some analytic calculations
- Calibration to stylized facts

Other typical calibration methods:
- Method of moment
- Maximum likelihood
15.1, 85.0 = \frac{z}{\gamma} = 1.15

Squares: \gamma = 1.15
Diamonds: \gamma = 1.3
Triangles: US stocks

Calibration to stylized facts \quad z = 0.85, \gamma = 1.15
Interesting analogy between multi-timescale models and multi-fractal (cascade) models

For finance, multi-timescale makes more sense because of time-reversal asymmetry
So far, we have studied models of stock prices across *time*

Now let us look at the dynamics across *stocks* [Correlations]
What do we know about volatility?

- Across time for a given stock  Part 1

- Across stocks at a given time  Part 2
Across stocks ....

1. Cross-sectional dispersion
2. Cross-sectional kurtosis
3. Correlations
4. Do markets exhibit a phase-transition in times of panic?
5. A simple model

Cross-Sectional Kurtosis
Even by eye you can see the very high negative correlation between cross-sectional dispersion and kurtosis.

\[ \langle \text{Disp}(t) \text{ Kurt}(t) \rangle = -30\% \]
In times of market panic:
c-s kurtosis LOW but c-s dispersion HIGH

Some panic periods: Fall 2008, Sep 08 - April 09, 2002
This implies that the distribution of cross-sectional returns is different in panic or normal states.
Next thing to look at are correlations

- Perform rolling Principal Components Analysis

Look at percentage of variance captured by first eigenvector

Larger percentage -> More of a “market model”
-> Higher cross-sectional co-movement of stocks
Distribution becomes more Gaussian in panic times

Two possible explanations:

1) Distribution of individual stock volatilities narrows
2) Correlations among random stock moves increases
Distribution becomes more Gaussian in panic times

Two possible explanations:

1) Distribution of individual stock volatilities narrows
2) Correlations among random stock moves increases
Define on each day the quantity

\[ S = \frac{S^{\uparrow} - S^{\downarrow}}{S^{\uparrow} + S^{\downarrow}} \]

Where

\( S^{\uparrow} \) is the number of stocks that had positive moves

\( S^{\downarrow} \) is the number of stocks that had negative moves
Correlations:

Define on each day the quantity

\[ S = \frac{S^\uparrow - S^\downarrow}{S^\uparrow + S^\downarrow} \]

Where \( S^\uparrow \) is the number of stocks that had positive moves
\( S^\downarrow \) is the number of stocks that had negative moves

If \( s = 0 \) no correlation, if \( s \neq 0 \) correlation among stocks

s can be seen as an Order Parameter
Analogy:

\( s \) is the Order Parameter

\( \sigma \) (external volatility perception) is the Control Parameter

\[ V(s) = a s^2 + b s^4 \]

If \( \sigma < \sigma_c \) \( s \) is in the disordered state - low correlation across stocks

If \( \sigma > \sigma_c \) \( s \) is in the ordered state - high correlation

Symmetry breaking (can be positive or negative)

\[ s = 0 \]

\[ s = \pm \sqrt{-\frac{a}{b}} \]
If this is plausible we expect to see

- unimodal distribution of $s$ in normal times
- bimodal distribution of $s$ in panic times
In times of panic, correlations are high

\[ S^\uparrow \] is the number of stocks that had positive moves

\[ S^\downarrow \] is the number of stocks that had negative moves

\[ S = \frac{S^\uparrow - S^\downarrow}{S^\uparrow + S^\downarrow} \]
A simple phase transition model:
Statistical signatures in times of panic  [L. Bprland. Quantitative Finance 2011]

- Log returns follow multi timescale model across time

- Log returns correlated across stocks with corr = |s|
  (disorder to order)

- The transition is triggered by a volatility shock

We get:

Kurtosis down
Dispersion up
Distribution of s bimodal
Putting it all together

For each stock \( i \) across time \( t \) \( y = \log(\text{stock price}) \)

\[
dy_t^i = \sigma_t^i d\omega_t^i
\]

\[
\sigma_t^2 = \sigma^2 \left[ 1 + g \sum_{l=1}^{\infty} \frac{1}{l^\alpha} (y_t^i - y_{t-l}^i)^2 \right]
\]

Across stocks for each time

If \( a > 0 \Rightarrow s = 0 \) : disordered state, correlations \( \langle \omega_t^i \omega_t^j \rangle_{i \neq j} = 0 \)

If \( a < 0 \Rightarrow s \neq 0 \) : ordered state, correlations \( \langle \omega_t^i \omega_t^j \rangle_{i \neq j} = |s| \)

\[
\frac{ds}{dt} = -as - bs^3 + F_t
\]

\( a = \sigma_c - \sigma_M \) \( \sigma_c = \text{critical volatility} \)

Market volatility can increase due to:

i) Exogenous jumps (news)

ii) Endogenous jumps (multi time-scale dynamics)
SIMULATIONS:

\[ dy^i_t = \sigma_t^i d\omega_t^i \]

\[
\sigma_t^2 = \sigma_0^2 \left[ 1 + \sum_{l=1}^{N} \frac{g_l}{\sigma^2_l} (y_t^i - y_{t-l}^i)^2 + \kappa \sum_{l=1}^{N} \frac{g_l}{\sigma \sqrt{l}} (y_t^i - y_{t-l}^i) \right]
\]

\[ \alpha = \sigma_c - \sigma_M \quad \text{Panic: } \alpha < 0 \]

\[ \sigma_c = 2 \text{ standard deviations of recent market returns} \]

\[ \sigma_M = \text{Current market volatility} \]

[Extension to include skew [Borland and Hasaad 2010]
RESULTS

Histogram of $s$: Normal Period

Histogram of $s$: Panic Period
Summary and Open Questions

• A realistic model of the joint stochastic process of stock prices

• Dynamics across time reproduced

• Market volatility drives correlations across stocks

• In times of panic, fluctuations become more correlated

• In times of panic, cross-sectional distribution becomes more Gaussian. All stocks experience higher volatility.

• All our models have been extended to incorporate the asymmetry that negative returns lead to more panic than positive ones.

• Calibration?

• Prediction of critical sigma?

• Analytic solutions (e.g. for option pricing, trading strategies)?
References


