Many-particle theory of anyons in 1D

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School and workshop on anyons in ultracold atom gases

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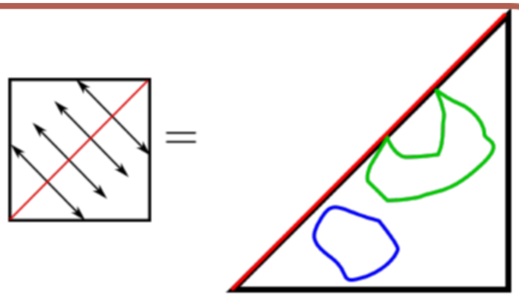
Second quantization of Leinaas-Myrheim anyons in one dimension and their relation to the Lieb-Liniger model

Thore Posske,[1] Björn Trauzettel,[2] and Michael Thorwart[1]

Motivation

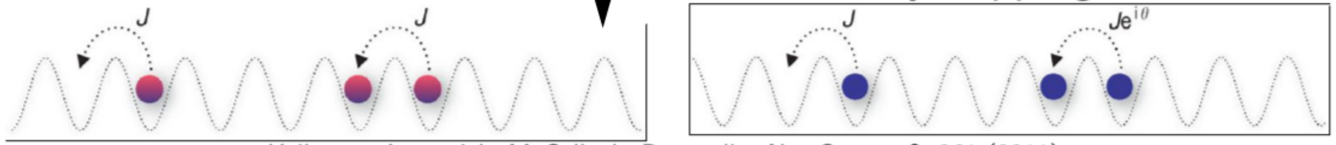
- How to bring together differently seeming anyon models?

$$\mathcal{C}^{\text{ind.}}_{2,1D} = \{\{x_1, x_2\} \,|\, x_i \in \mathbb{R}, x_1 \neq x_2\} = $$



$\updownarrow$?

Anyon-Hubbard model　　　conditionally hopping bosons



Keilmann, Lanzmich, McCulloch, Roncaglia: Nat Comm. **2**, 361 (2011)

$$\mathcal{H}^a = -t\sum_{j=1}^{L}\left(a_j^\dagger a_{j+1} + h.c.\right) + U/2\sum_{j=1}^{L} n_j(n_j - 1)$$

$$a_j a_k - e^{i\theta\,\mathrm{sgn}(j-k)} a_k a_j = 0, \quad a_j a_k^\dagger - e^{-i\theta\,\mathrm{sgn}(j-k)} a_k^\dagger a_j = \delta_{k,j}$$

?

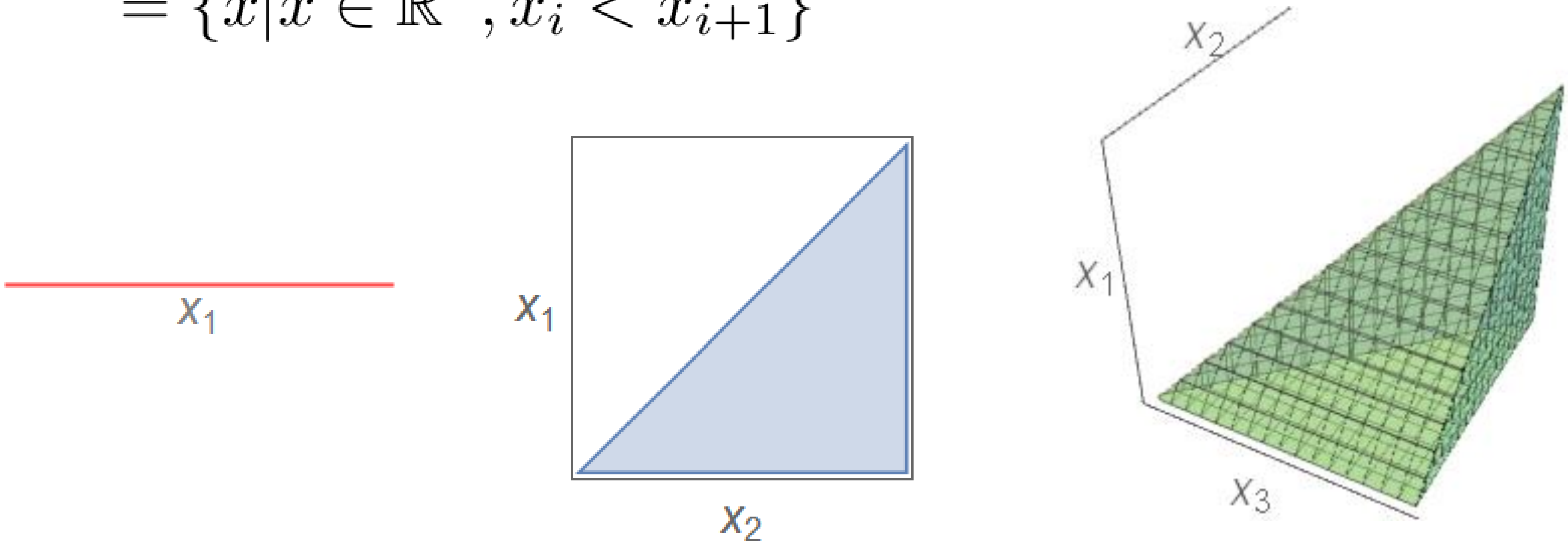
Outline

- The **connection** between the 1D continuum anyon Hubbard model and the theory of Leinaas and Myrheim
- **Second quantization of Leinaas-Myrheim anyons**
- Bethe ansatz equations for particles in a box
- Physical properties (energy levels, momentum distr. and density distr.)
- **Anyonic interpretation**

Leinaas-Myrheim anyons in 1D (1977)

- Configuration space

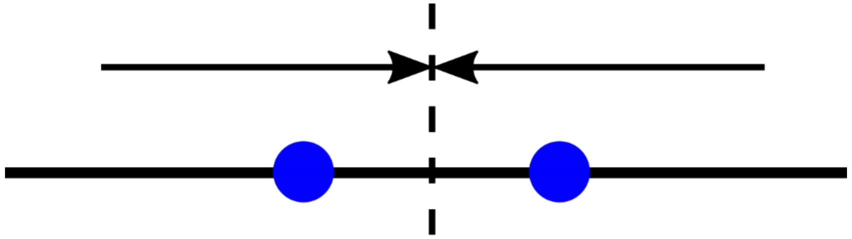
$$\mathcal{C}_{n,1D}^{\text{ind.}} = \{\{x_1, x_2, \ldots, x_n\}\,|x_i \in \mathbb{R}, x_i \neq x_j\}$$
$$= \{\vec{x}|\vec{x} \in \mathbb{R}^n, x_i < x_{i+1}\}$$



Boundary conditions

- $\mathcal{H} = \frac{-\hbar^2}{2m}\Delta$ Hamiltonian needs to be hermitian
- $\partial_\perp \Psi \mid_{\partial \mathcal{C}} = \eta \Psi \mid_{\partial \mathcal{C}}$ ensured by Robin boundary condition
- $\Psi \mid_{\partial \mathcal{C}} = 0$ Dirac $\partial_\perp \Psi \mid_{\partial \mathcal{C}} = 0$ Neumann

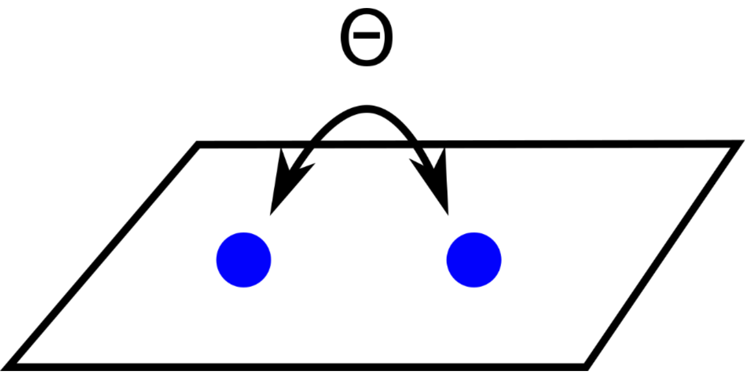
$$\Rightarrow \vec{j}^{\perp} = 0$$

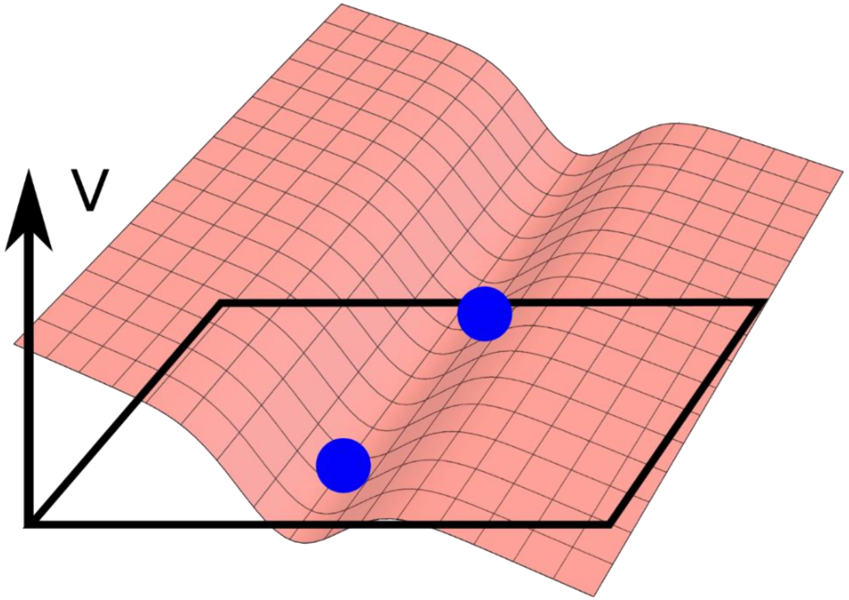


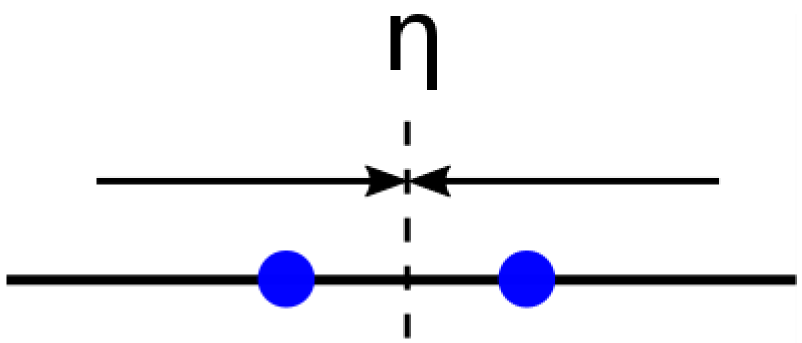
Exercise sheet online
https://hp.physnet.uni-hamburg.de/tposske
→Lectures →Many-particle theory for 1D anyons

2D anyons → 1D anyons

- Hansson 1992: Confine 2D anyons to 1D by local potential







$$\eta^{-1} \approx 0.538 d \cot^2\left(\frac{\theta}{2}\right)$$

Hansson, Leinaas, Myrheim: Nuclear Physics B **384**, 559-580

Lieb-Liniger model (1963)

- Locally interacting bosons in 1D

$$\mathcal{H}_{LL} = -\frac{\hbar^2}{2m}\sum_{j=1}^{n}\partial_{x_j}^2 + 2c\sum_{i\neq j}\delta\left(x_i - x_j\right)$$

- Implement δ-interaction into wave function in real-space representation (like δ-scatterer in QM1)

$$\left(\partial_{x_j} - \partial_{x_{j+1}}\right)\Psi(\vec{x})\,|_{x_j=x_{j+1}} = c\,\Psi(\vec{x})\,|_{x_j=x_{j+1}}$$

- Reduction to non-crossing space

$$x_1 \leq x_2 \leq .. \leq x_n$$

Comparing Leinaas-Myrheim to Lieb-Liniger

- Different observables
  - Leinaas-Myrheim anyons

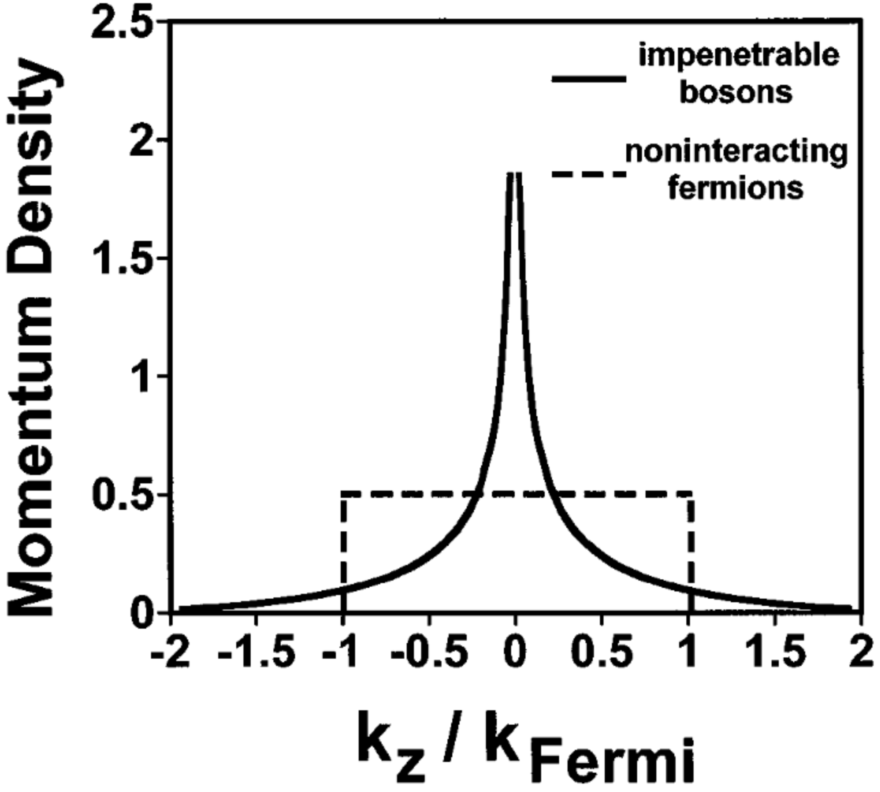
$$\langle A\rangle_{\mathrm{LM}} = \int_{x_{n-1}}^{\infty} dx_n \ldots \int_{x_2}^{x_3} dx_2 \int_{-\infty}^{x_2} dx_1 \Psi^*(\vec{x}) A \Psi(\vec{x})$$

  - Lieb-Liniger bosons

$$\langle A\rangle_{\mathrm{LL}} = \int_{-\infty}^{\infty} d^n\vec{x}\ \Psi^*(\vec{x}) A \Psi(\vec{x})$$

  - With continued wave function

$$\Psi(\vec{x}) := \Psi(P(\vec{x}))$$



M. Olshanii: PRL **81**, 938 (1998)

Kundu's model: PRL **83** 1275 (1999)

- Double delta interacting 1D bosons

$$H_N = -\sum_k^N \partial_{x_k}^2 + \sum_{\langle k,l \rangle} \delta(x_k - x_l)[c + i\kappa(\partial_{x_k} + \partial_{x_l})]$$

$$+ \gamma_1 \sum_{\langle j,k,l \rangle} \delta(x_j - x_k)\delta(x_l - x_k) + \gamma_2 \sum_{\langle k,l \rangle} [\delta(x_k - x_l)]^2$$

$$H = \int dx \{:[(\psi_x^\dagger \psi_x + c\rho^2 + i\kappa\rho(\psi^\dagger \psi_x - \psi_x^\dagger \psi)]:$$

$$+ \kappa^2(\psi^\dagger \rho^2 \psi)\}, \qquad \rho \equiv (\psi^\dagger \psi), \qquad (2)$$

Kundu's model: PRL **83** 1275 (1999)

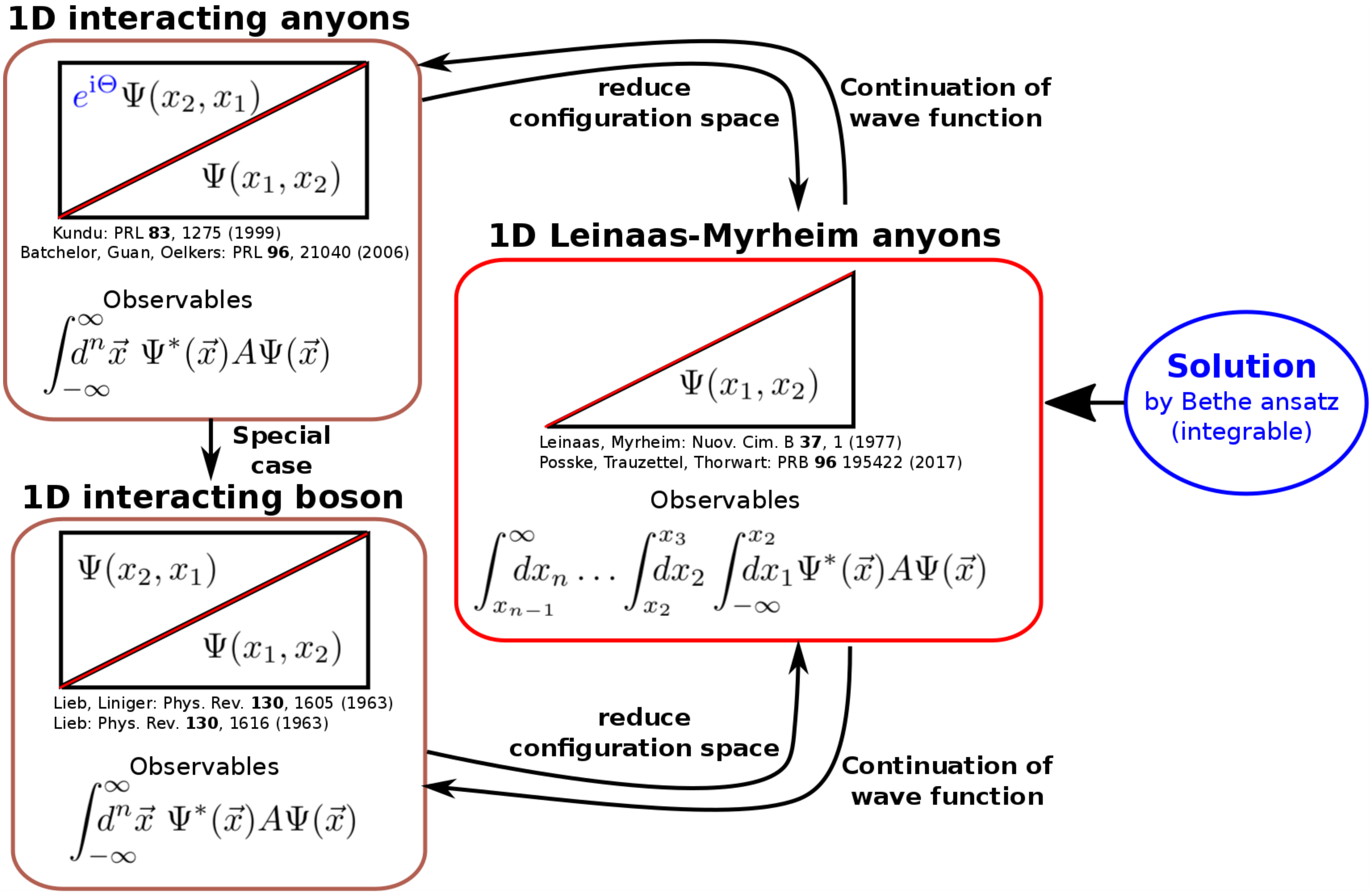
- Tranforms to anyonic fields $\tilde{\psi}(x) = e^{-i\kappa \int_{-\infty}^{x} \psi^\dagger(x')\psi(x')\,dx'} \psi(x)$

$$\tilde{H} = \int dx \,\vdots[\tilde{\psi}_x^\dagger \tilde{\psi}_x + c(\tilde{\psi}^\dagger \tilde{\psi})^2]\vdots\,. \qquad (7)$$

Note, however, that, in spite of the same form as NLSE, (7) is not the same as the known model, since the fields involved are no longer bosonic operators but exhibit *anyon*like properties

$$\tilde{\psi}^\dagger(x_1)\tilde{\psi}^\dagger(x_2) = e^{i\kappa\epsilon(x_1-x_2)}\tilde{\psi}^\dagger(x_2)\tilde{\psi}^\dagger(x_1),$$

$$\tilde{\psi}(x_1)\tilde{\psi}^\dagger(x_2) = e^{-i\kappa\epsilon(x_1-x_2)}\tilde{\psi}^\dagger(x_2)\tilde{\psi}(x_1) + \delta(x_1 - x_2), \qquad (8)$$



1D interacting anyons
$e^{i\Theta}\Psi(x_2, x_1)$
$\Psi(x_1, x_2)$
Kundu: PRL **83**, 1275 (1999)
Batchelor, Guan, Oelkers: PRL **96**, 21040 (2006)
Observables
$\int_{-\infty}^{\infty} d^n\vec{x}\ \Psi^*(\vec{x})A\Psi(\vec{x})$
reduce configuration space
Continuation of wave function
Special case
1D interacting boson
$\Psi(x_2, x_1)$
$\Psi(x_1, x_2)$
Lieb, Liniger: Phys. Rev. **130**, 1605 (1963)
Lieb: Phys. Rev. **130**, 1616 (1963)
Observables
$\int_{-\infty}^{\infty} d^n\vec{x}\ \Psi^*(\vec{x})A\Psi(\vec{x})$
1D Leinaas-Myrheim anyons
$\Psi(x_1, x_2)$
Leinaas, Myrheim: Nuov. Cim. B **37**, 1 (1977)
Posske, Trauzettel, Thorwart: PRB **96** 195422 (2017)
Observables
$\int_{x_{n-1}}^{\infty} dx_n \dots \int_{x_2}^{x_3} dx_2 \int_{-\infty}^{x_2} dx_1 \Psi^*(\vec{x})A\Psi(\vec{x})$
Solution
by Bethe ansatz
(integrable)
reduce configuration space
Continuation of wave function

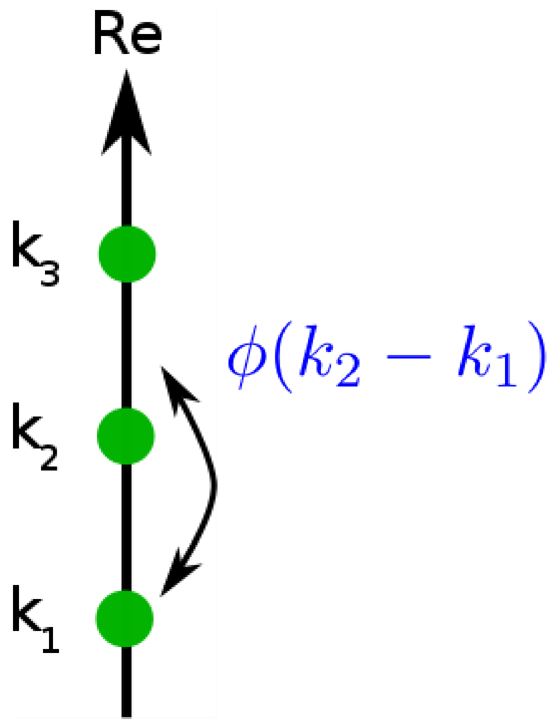
Solution – Bethe ansatz

$$\left(\partial_{x_{j+1}} - \partial_{x_j}\right) \Psi(\vec{x}) \,|_{x_j \to x_{j+1}} = \eta \; \Psi(\vec{x}) \,|_{x_j \to x_{j+1}}$$

- Ansatz $\Psi\left(\vec{x}\right) = \int_{\vec{k}\in\mathbb{C}^n} d^n k \; \alpha\left(\vec{k}\right) e^{\mathrm{i}\vec{k}\vec{x}}$ (complex momenta!)

$$\alpha\left(\vec{k}\right) = e^{-\mathrm{i}\phi_\eta(k_{j+1}-k_j)} \alpha\left(\sigma_j \vec{k}\right) \quad \text{if } k_{j+1} - k_j \neq \mathrm{i}\eta$$

$$\alpha\left(\vec{k}\right) = 0 \quad \text{if } k_{j+1} - k_j = \mathrm{i}\eta$$



$$\phi_\eta(k_{j+1} - k_j) = 2\arctan\left[\eta/(k_{j+1} - k_j)\right]$$

Momentum space exchange phase

Wave functions - precursor

$$\alpha(\vec{k}) = e^{\mathrm{i}\phi_\eta^P(\vec{k})}\alpha(P\vec{k}) \qquad P = \sigma_{j_1}\dots\sigma_{j_r}$$

$$\phi_\eta^P\left(\vec{k}\right) = \sum_{i=1}^r \phi_\eta\left[\left(\sigma_{j_1}\dots\sigma_{j_i}\vec{k}\right)_{j_i} - \left(\sigma_{j_1}\dots\sigma_{j_i}\vec{k}\right)_{j_i+1}\right]$$

$$\Psi_{\boldsymbol{k}}\left(\boldsymbol{x}\right) \propto \sum_{P\in S_n} e^{\mathrm{i}\phi_\eta^P(\boldsymbol{k})} e^{\mathrm{i}(P\boldsymbol{k})\boldsymbol{x}}$$

Clusters

	D_1	D_2	D_n
$\mathrm{Re}(\vec{k})$	(K_1)	(K_2, K_2)	$(K_n, K_n, \ldots, K_n)$
$\mathrm{Im}(\vec{k})$	(0)	$-\frac{\eta}{2}(-1, 1)$	$-\eta\left(\frac{1-n}{2}, \frac{3-n}{2}, \ldots, \frac{n-1}{2}\right)$

- Ordering essential (smallest real part, then smallest imaginary part)

$$\mathcal{O}\left(K_2 - \mathrm{i}\eta/2, K_2 + \mathrm{i}\eta, K_1\right) = \left(K_1, K_2 + \mathrm{i}\eta/2, K_2 - \mathrm{i}\eta/2\right)$$

- Ordered parts of the wave function do not diverge

Wave functions

- Wave functions naturally build up by clusters

$$\vec{k} = D_1 \oplus D_2 \cdots \oplus D_n$$

$$\Psi_{\mathcal{O}(\vec{k})}(\vec{x}) = N_{\vec{k}} \sum_{P \in S_n} e^{\mathrm{i}\phi_\eta^P(\vec{k})} e^{\mathrm{i}(P\vec{k})\vec{x}}$$

Second quantization

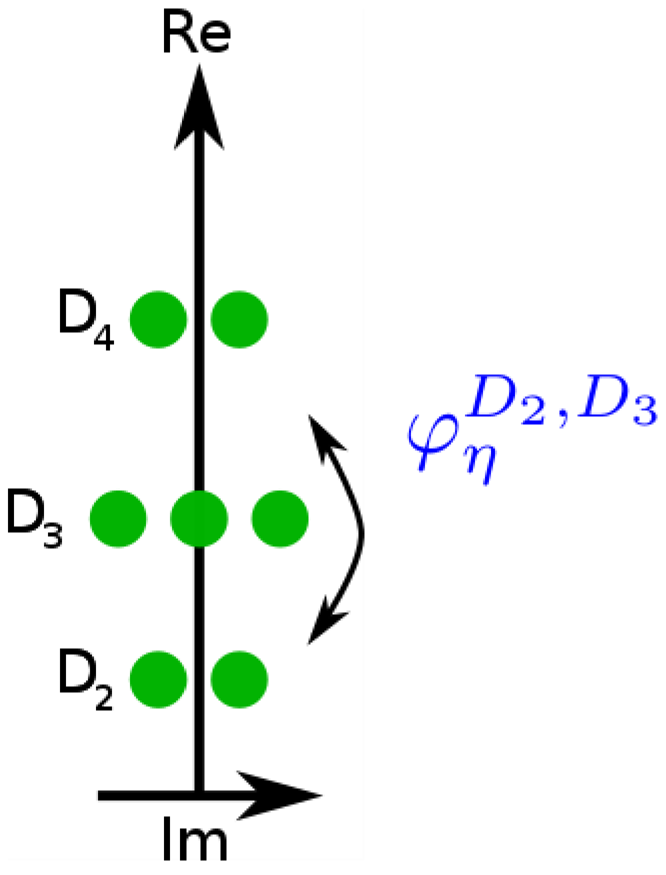
- Define creation operators between orthogonal base wavefunctions

$$a_D^\dagger \Psi_{\vec{k}} = \sqrt{M(D)+1}\; e^{\mathrm{i}\Phi_\eta^D(\vec{k})} \Psi_{\mathcal{O}(\vec{k}\oplus D)}(\vec{x})$$

$$\Phi_\eta^D\left(\vec{k}\right) = \sum_{(\tilde{D}\in\vec{k})<D} \varphi_\eta^{\tilde{D},D}$$

- Cluster exchange phase

$$\varphi_\eta^{\tilde{D},D} = \sum_{k\in D} \sum_{\tilde{k}\in\tilde{D}} \phi_\eta(\tilde{k}-k)$$



Second quantization – momentum algebra

- Cluster algebra

$$a^\dagger_{D_1} a^\dagger_{D_2} = e^{\mathrm{i}\varphi_\eta^{D_1,D_2}} a^\dagger_{D_2} a^\dagger_{D_1}$$

$$a_{D_1} a^\dagger_{D_2} = e^{\mathrm{i}\varphi_\eta^{D_1,D_2}} a^\dagger_{D_2} a_{D_1} + \delta\left(K_1 - K_2\right)$$

- Single anyon algebra

$$a^\dagger_p a^\dagger_q = e^{\mathrm{i}\phi_\eta(p-q)} a^\dagger_q a^\dagger_p$$

$$a_p a^\dagger_q = e^{-\mathrm{i}\phi_\eta(p-q)} a^\dagger_q a_p + \delta(p-q)$$

$$\phi_\eta(k_{j+1} - k_j) = 2\arctan\left[\eta/(k_{j+1} - k_j)\right]$$

Second quantization – real space algebra

$$\{\Psi(x), \Psi^\dagger(y)\} = \delta(x-y) + \int_0^\infty dz \frac{2e^{-\frac{z}{|\eta|}}}{|\eta|}\Psi^\dagger(y-z)\Psi(x-z)$$

$$\{\Psi^\dagger(x), \Psi^\dagger(y)\} = \int_0^\infty dz \frac{2e^{-\frac{z}{|\eta|}}}{|\eta|}\Psi^\dagger(y-z)\Psi^\dagger(x+z),$$

Generalized real space Pauli principle

Thanks, Axel

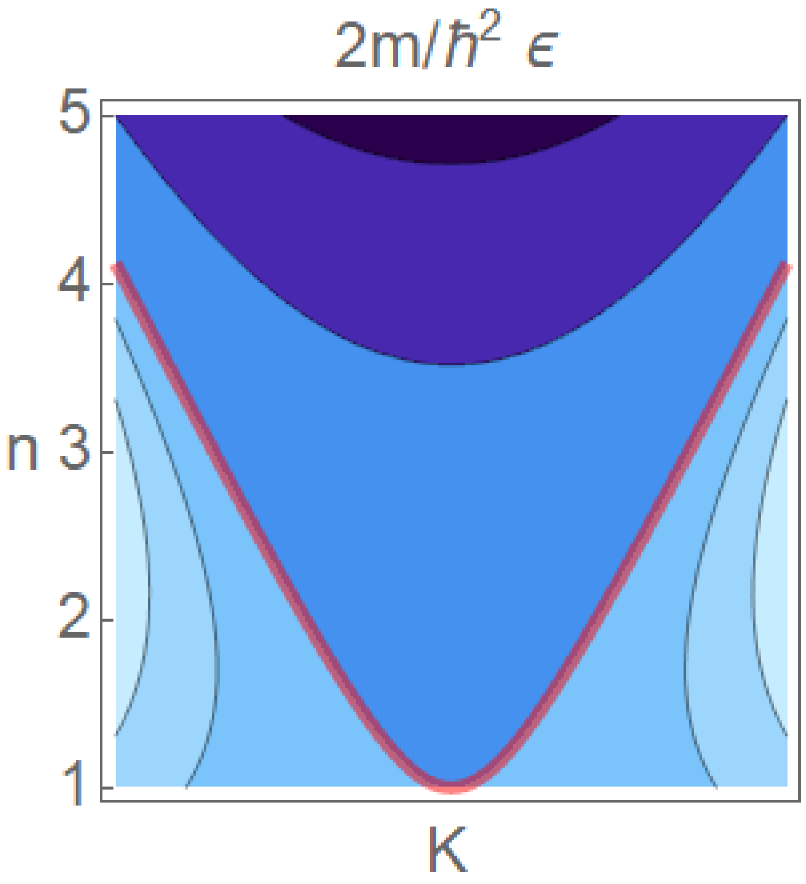
$$\left[\Psi^\dagger(x)\right]^2 = \int_0^\infty dz \frac{e^{-z/|\eta|}}{|\eta|}\Psi^\dagger(x-z)\Psi^\dagger(x+z)$$

Second quantization – Hamiltonian

- Free Hamiltonian in second quantization
  Boundary conditions encoded in algebra

$$\mathcal{H} = \sum_D \epsilon_D a_D^\dagger a_D$$

$$\epsilon_D = \frac{\hbar^2 n}{2m}\left(K^2 - \frac{1}{12}\eta^2(n^2-1)\right)$$



Generalized Jordan-Wigner transformation

- Transforming 1D Leinaas-Myrheim anyons to bosons

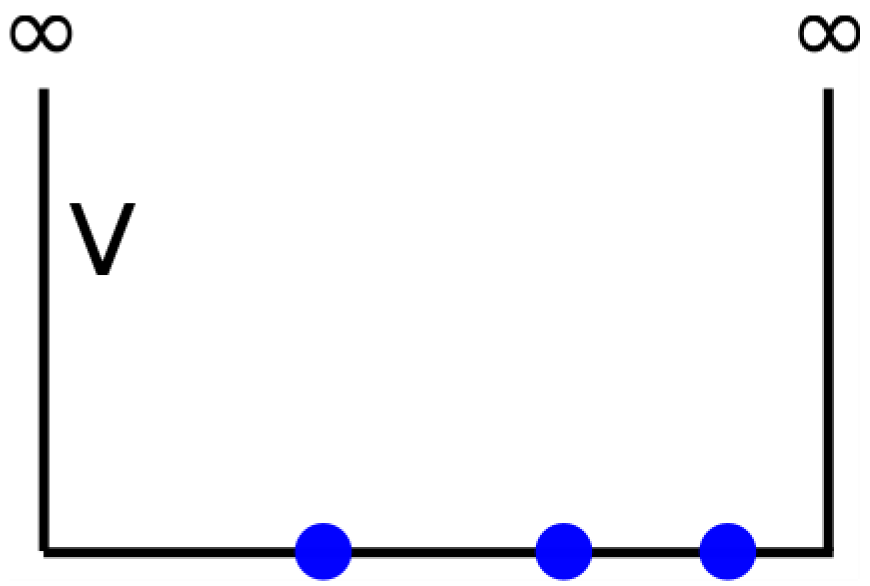
$$\tilde{a}(j) = \lim_{\epsilon \to 0^+} e^{\mathrm{i} \int_{-\infty}^{j-\epsilon} dk \; b_k^\dagger b_k \phi_\eta(k-j)} b(j)$$

- Attention: Bosonic operators alone do not respect boundary conditions

Anyons in a box

- Extra Dirichlet boundary conditions

$$\Psi\left(0, x_2, \ldots, x_n\right) = 0 \qquad \Psi\left(x_1, \ldots, x_{n-1}, L\right) = 0$$



Bethe ansatz equations

- Constraints on momentum space coefficients of wave function

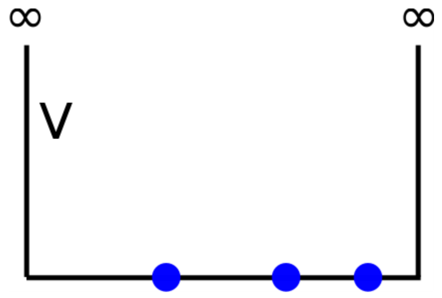
$$\alpha\left(-k_1,\ldots,k_n\right) = -\,\alpha\left(\vec{k}\right) \qquad \alpha\left(k_1,\ldots,-k_n\right) = -\,e^{2\mathrm{i}k_nL}\alpha\left(\vec{k}\right)$$

$$\alpha\left(\vec{k}\right) = e^{-\mathrm{i}\phi_\eta\left(k_{j+1}-k_j\right)}\alpha\left(\sigma_j\vec{k}\right) \qquad \text{if } k_{j+1}-k_j \neq \mathrm{i}\eta$$

$$\alpha\left(\vec{k}\right) = 0 \qquad \text{if } k_{j+1}-k_j = \mathrm{i}\eta$$

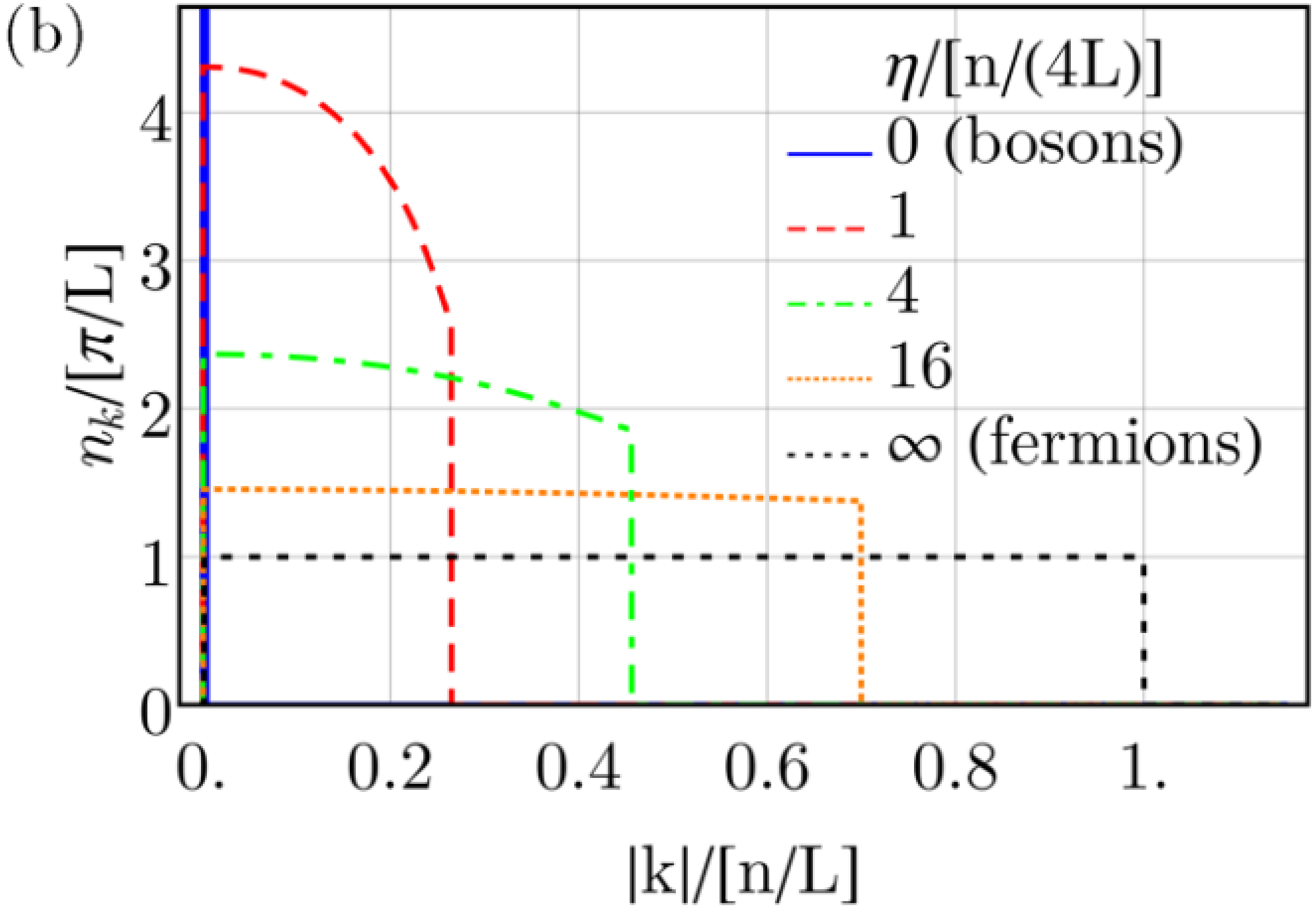
- Bethe ansatz equations

$$Lk_j + \frac{1}{2}\sum_{i\neq j}\phi_\eta\left(k_i-k_j\right) - \phi_\eta\left(k_i+k_j\right) = \pi z_j$$



Physical properties of Leinaas-Myrheim anyons numerical results

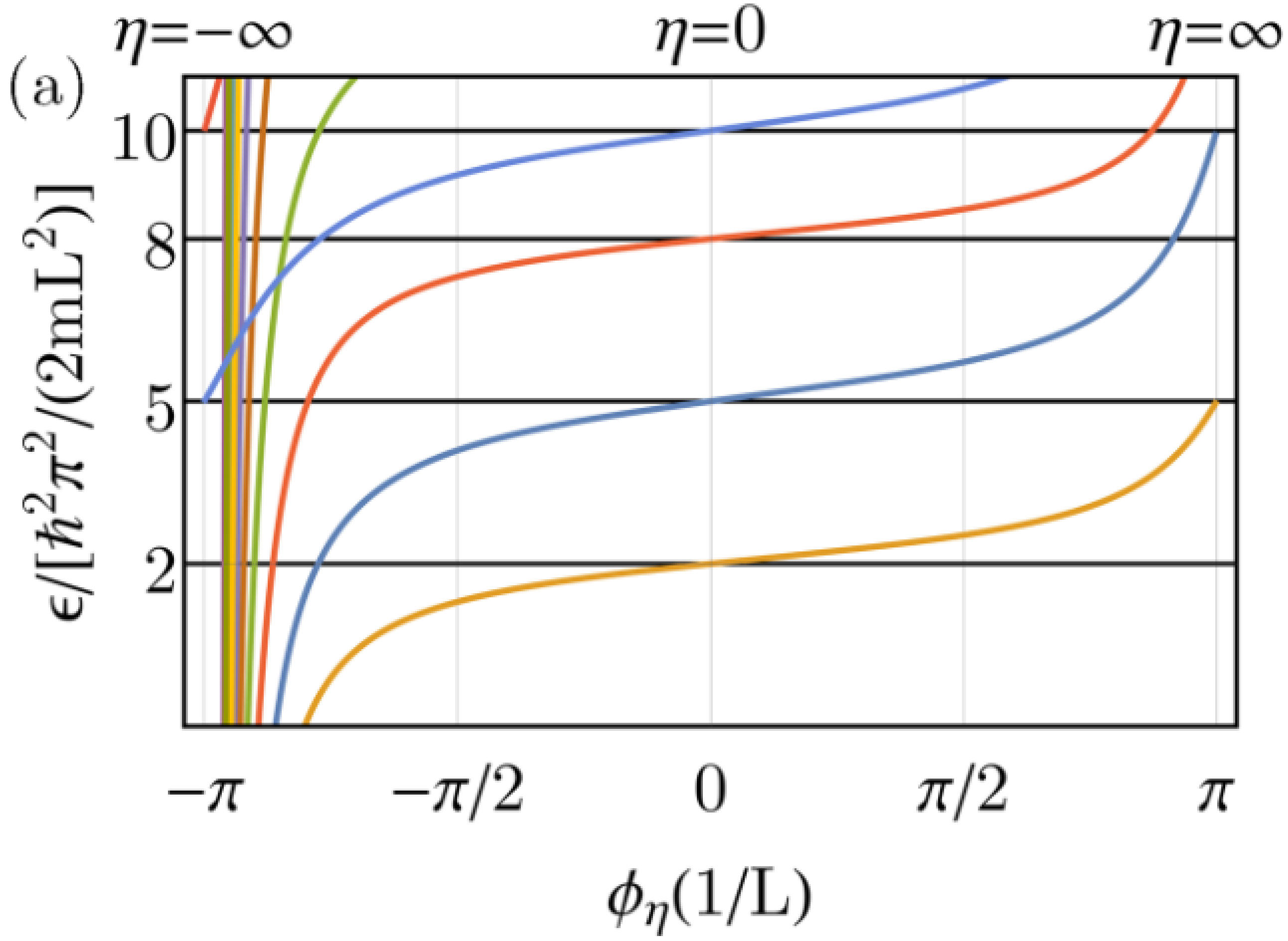
Momentum statistics



n_k gives the number of anyons with momentum between k_1 and k_2 by $\int_{k_1}^{k_2} dk\, n_k$

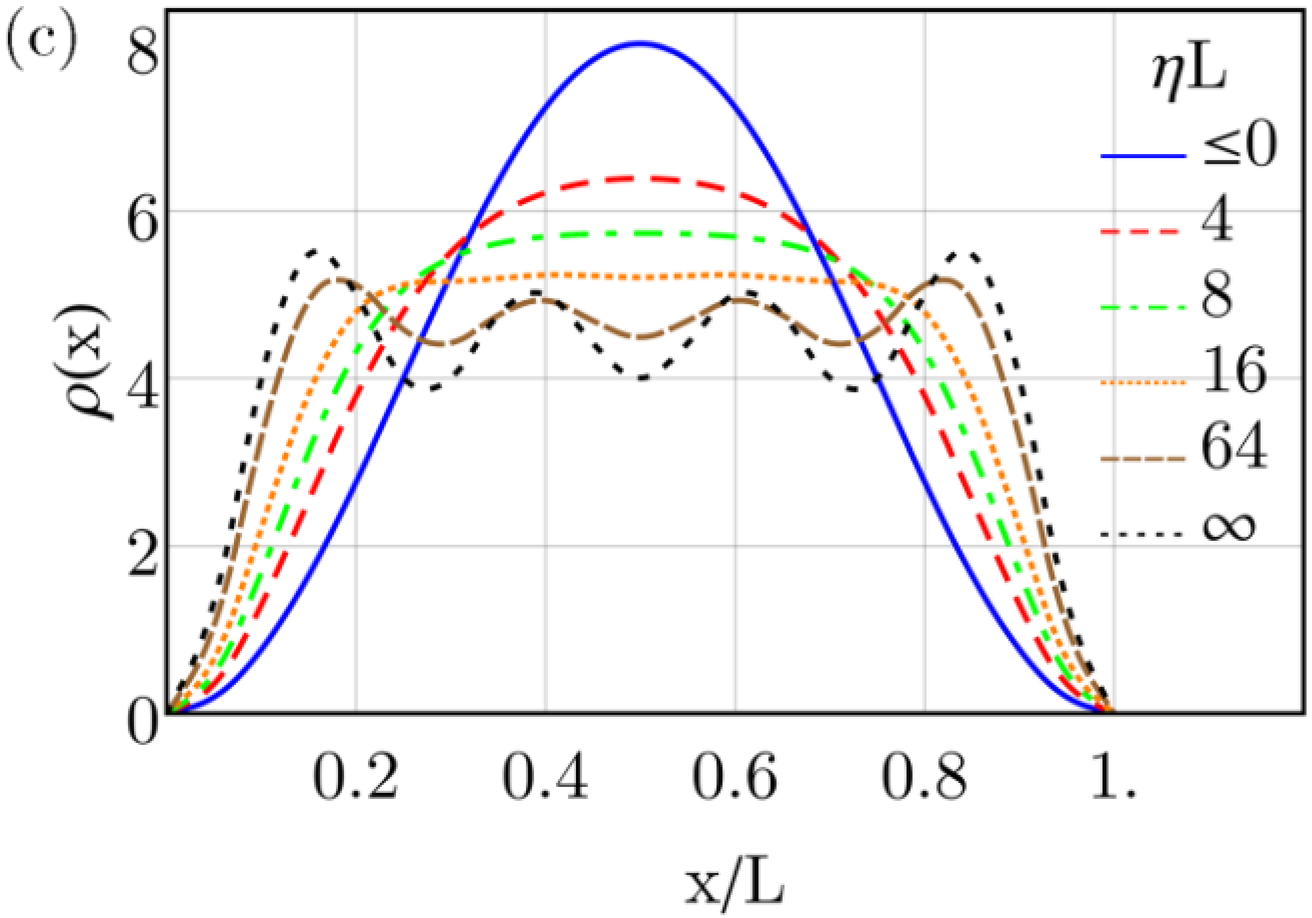
Physical properties of Leinaas-Myrheim anyons numerical results

2 particles in a box



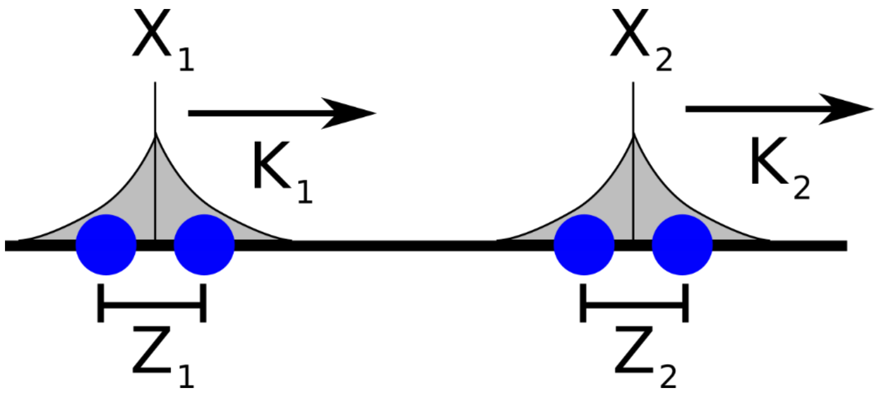
Physical properties of Leinaas-Myrheim anyons numerical results

4 particles in a box



Composite anyon picture

- Fusion-like behavior of anyonic clusters



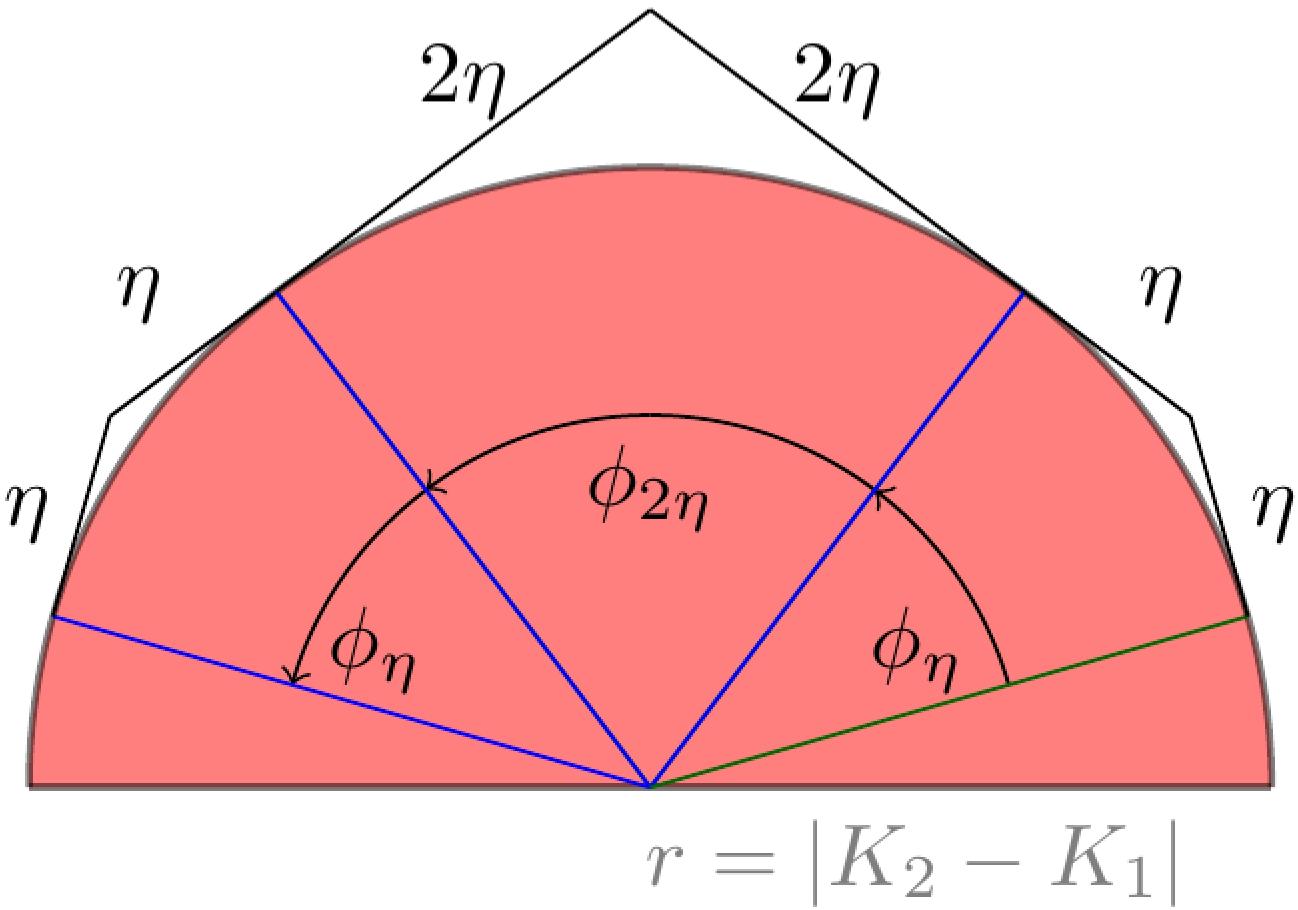
$$D_1 = (K_1 + \mathrm{i}\eta/2, K_1 - \mathrm{i}\eta/2)$$
$$D_2 = (K_2 + \mathrm{i}\eta/2, K_2 - \mathrm{i}\eta/2)$$

$$\Psi_{(D_1,D_2)}(X_1, X_2, Z_1, Z_2) = \frac{1}{N}$$
$$\left[e^{2\mathrm{i}(K_1X_1+K_2X_2)} + e^{\mathrm{i}\Omega_\eta^{D_1,D_2}} e^{2\mathrm{i}(K_2X_1+K_1X_2)}\right] e^{\eta(Z_1+Z_2)}$$

$$\Omega_\eta^{D_1,D_2} = 2\phi_\eta(K_2 - K_1) + \phi_{2\eta}(K_2 - K_1)$$

Composite anyon picture

$$\Omega_\eta^{D_1,D_2} = 2\phi_\eta(K_2 - K_1) + \phi_{2\eta}(K_2 - K_1)$$

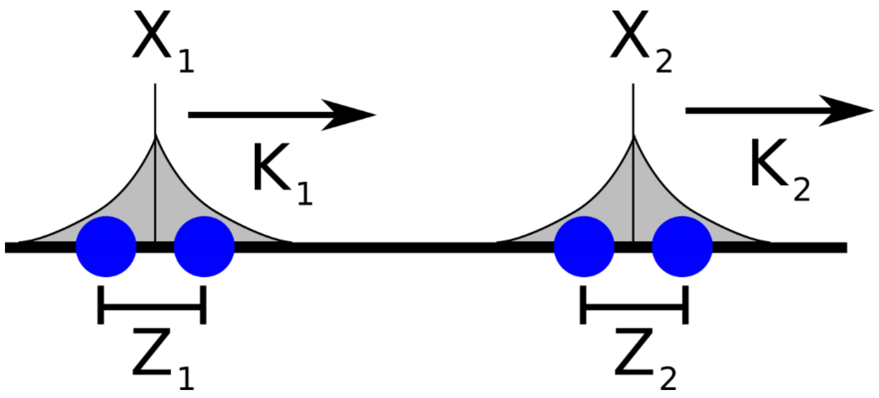


Interesting questions

- Free Dirac particles (linear dispersion) and consistency to Luttinger liquid
- Statistics of 1D anyons as implicit analytical formula
- Comparison of continuum to lattice model

Conclusion

- Leinaas Myrheim anyons formally equivalent to Lieb-Liniger bosons
- Still different phenomenology
- Solution by the Bethe ansatz
- Mutual benefits by interpretations:
    - Second quantization
    - Composite anyons picture
- Hopefully directly observable in cold atom systems



$$Lk_j + \frac{1}{2}\sum_{i\neq j}\phi_\eta\left(k_i - k_j\right) - \phi_\eta\left(k_i + k_j\right) = \pi z_j$$

Thank you

Universität Hamburg

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Anyon Physics of Ultracold Atomic Gases

Technische Universität Kaiserslautern (Germany), December 10 - 14, 2018

Additional information

- In k-space, the Pauli principle is valid for all statistics but Fermi statistics

$$\left(a_D^\dagger\right)^2 \overset{\eta\neq\pm\infty}{=} (a_D)^2 \overset{\eta\neq\pm\infty}{=} 0$$