Anyon models in 2D and 1D

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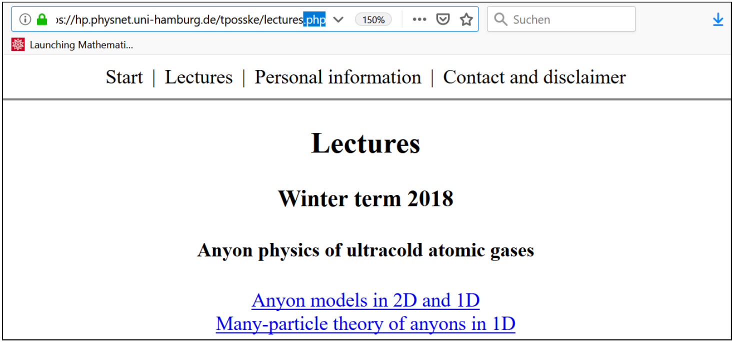
International School and Workshop
Anyon Physics of Ultracold Atomic Gases

TU Kaiserslautern
Monday, December 10, 2018

Online



https://hp.physnet.uni-hamburg.de/tposske $\rightarrow$ Lectures

- Slides
- All references

Funding for quantum computing with anyons

several 10^9\$ investment by Microsoft

Outline

1. History of anyons
2. Mathematical prerequisites
3. Anyons aux Leinaas and Myrheim
4. Anyonic systems and their applications
5. Remarks on statistical transmutation

Early history of anyons

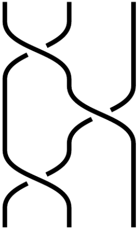
1925 **Artin's braid group**

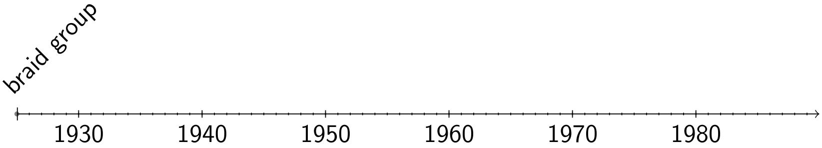
Artin's braid group

Mathematical formulation of braids (and knots)

Artin: Ann. Math., **48**, 1, 101-126 (1947)

$\sigma_1\sigma_2\sigma_1$





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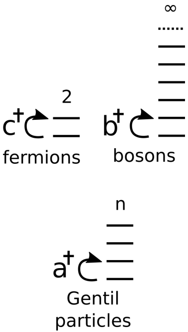
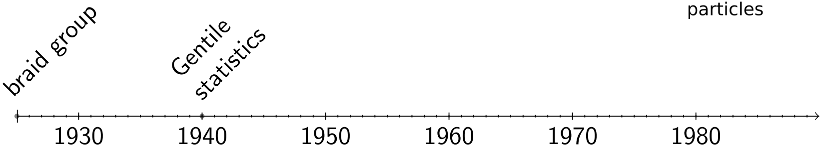
1940 Gentile statistics

Gentile statistics

Gentile: Nuovo Cimento, **17**, 493 (1940)





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1971 Laidlaw, De Witt path integrals for identical particles

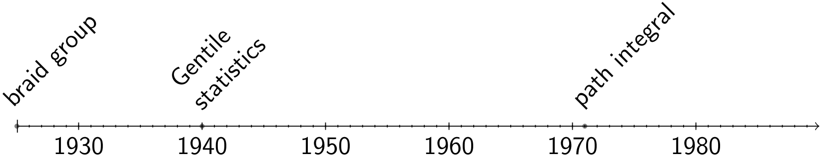
Path integral theory of identical particles

Laidlaw, DeWitt: PRD **3**, 1375 (1971)

$Y(n,m) = \{y = (\mathbf{x}_1, \ldots, \mathbf{x}_n);$
$\mathbf{x}_i = (x_i^1, \ldots, x^m)$ and $\mathbf{x}_i \neq \mathbf{x}_j\}$.

We observe that for $n \geq 2$,

$Y(n,1)$ is not connected,
$Y(n,2)$ is multiply connected,
$Y(n,m)$ is simply connected; $m \geq 3$.



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1971 Laidlaw, De Witt path integrals for identical particles

1977 **Leinaas' & Myrheim's theory of identical particles**

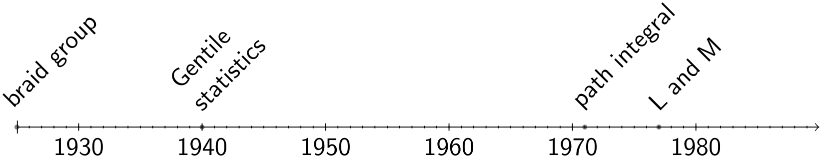
Leinaas and Myrheim "anyons"

Leinaas, Myrheim: Nuov. Cim. B **37**, 1 (1977)

First complete theory on anyons in 2D and 1D

$$\mathcal{C}_2^{2D} \cong \mathbb{R}^2 \backslash \{(0,0)\} \times \mathbb{R}^2$$

$$\pi_1\left(\mathcal{C}_2^{2D}\right) \cong \mathbb{Z}$$



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1981 Goldin, Menikoff, Sharp "anyons" by representations of current algebras

Goldin, Menikoff, Sharp

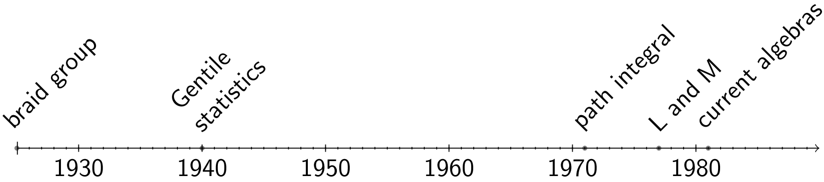
Goldin, Menikoff, Sharp: J. Math. Phys. **22**, 1664 (1981)

Representations of current algebras

$$[\rho(\mathbf{x}), \rho(\mathbf{y})] = 0,$$

$$[\rho(\mathbf{x}), J_k(\mathbf{y})] = -i\frac{\partial}{\partial x_k}[\rho(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})],$$

$$[J_j(\mathbf{x}), J_k(\mathbf{y})] = -i\frac{\partial}{\partial x_k}[J_j(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})] + i\frac{\partial}{\partial y_j}[J_k(\mathbf{y})\delta(\mathbf{x}-\mathbf{y})].$$



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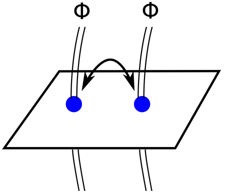
1981 Goldin, Menikoff, Sharp "anyons" by representations of current algebras

1982 **Wilczek flux-charge composites**

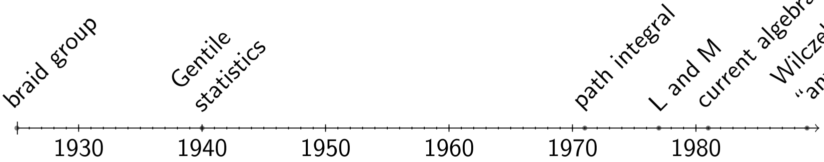
Wilczek "anyons"

Wilczek: PRL **49**, 957

flux-charge composite with any exchange phase: **anyons**

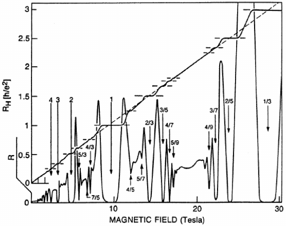


Since interchange of two of these particles can give *any* phase, I will call them generically anyons.



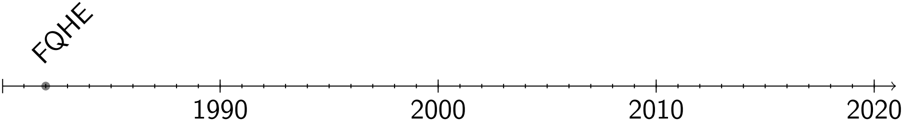
Experimental milestones

1982 Tsui and Störmer: **Fractional QHE**



Tsui, Stormer, Gossard: PRL **48**, 1559 (1982)

Tsui: Rev Mod Phys **71**, 891 (1999)



Experimental milestones

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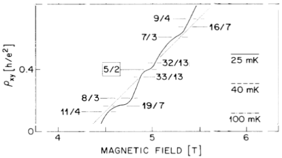
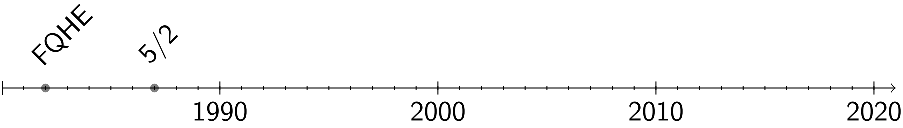
1987 Plateau at filling factor $\frac{5}{2}$ observed



Willett, Eisenstein, Störmer, et al.: PRL **59**, 1776 (1987)



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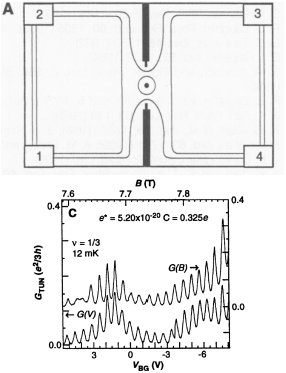
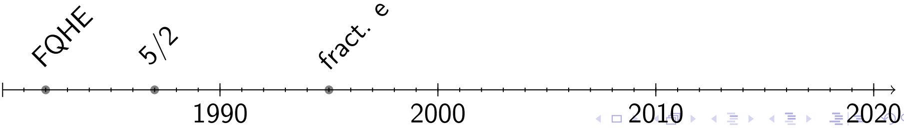


Goldman, Su: Science **17** 267 5200 (1995)



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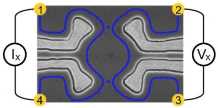
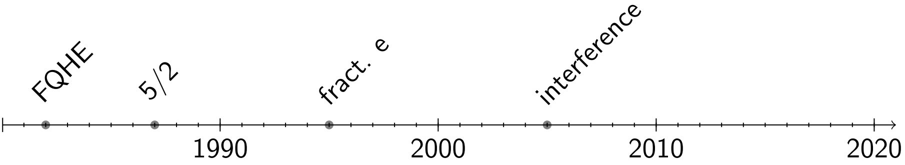
2005 Camino, Zhou, Goldman: Interference of anyons



Camino, Zhou, Goldman: PRB **72**, 075342 (2005)



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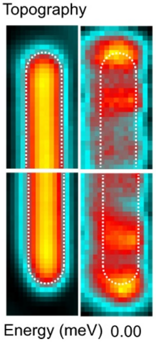
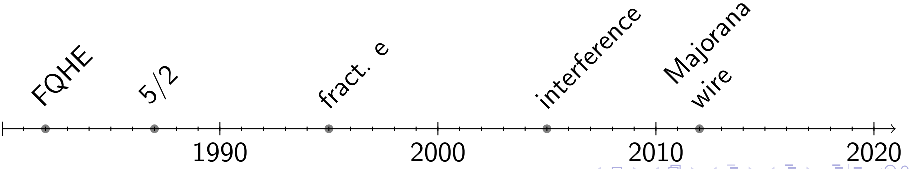
2012 **Majorana zero modes in 1D SC wires**



Kim, Palacio-Moralez, Posske, Thorwart, .., Wiesendanger: Science Adv. **4**, 5 (2018)



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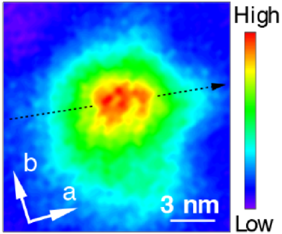
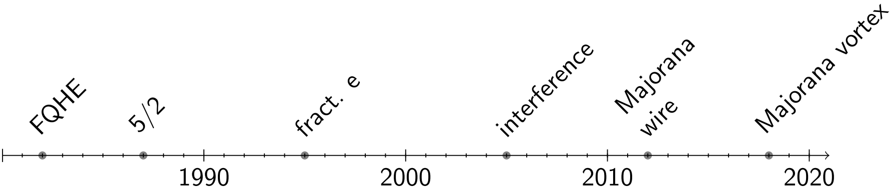
2018 Majorana zero modes in SC vortices



Ivanov: PRL **86**, 268 (2001)

Wang,.., Cao: Science **eaao1797** (2018)



Math prerequisites

1 History of anyons

2 Mathematical prerequisites
- Artin's braid group
- Linear representations
- Topologically equivalent paths
- Quantization of topologically nontrivial spaces

3 Anyons aux Leinaas and Myrheim

4 Anyonic systems and their applications

5 Remarks on statistical transmutation

Artin's braid group

Zopfgruppe

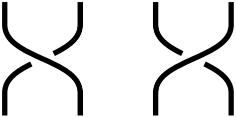
Group $\mathbb{B}_n$ generated by $n-1$ operators σ_i with the relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$

Describes all possible braids on n strands

σ_1 σ_3^{-1}



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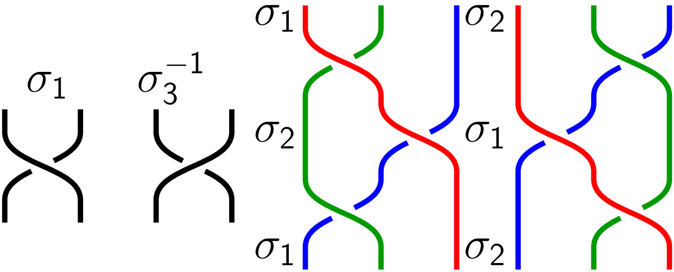

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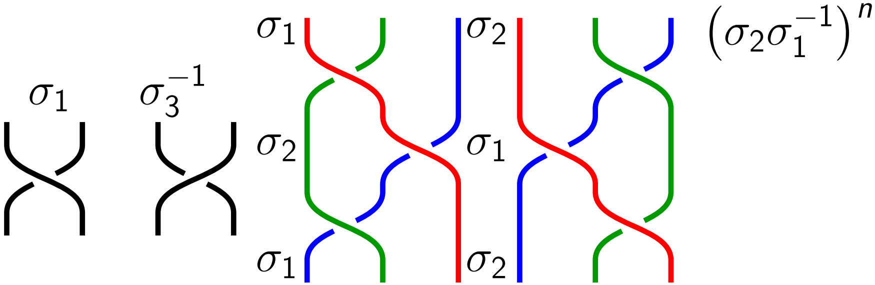

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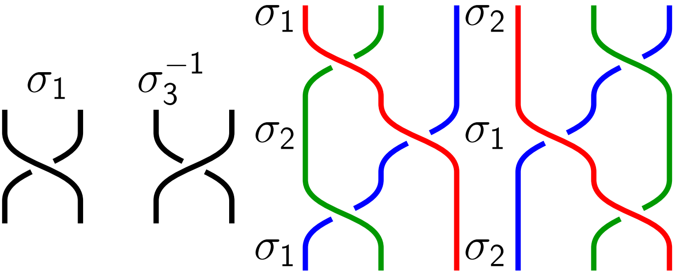


$\left(\sigma_2 \sigma_1^{-1}\right)^n$

Linear representations

(complex) linear representation

An n-dimensional *(complex) linear representation* $M : G \to \mathbb{C}^n$ of a group G is a map from G to the $n \times n$ dimensional complex matrices such that

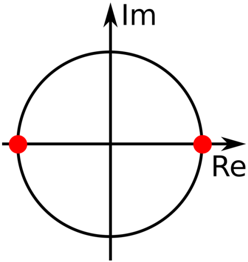
$$M(f \circ g) = M(f)M(g)$$

Examples

$$(\mathbb{Z}, +) \simeq \left\{ e^{\frac{2\pi m}{n}\mathrm{i}} \mid m \in \{1 \ldots n\} \right\}$$

infinitely many scalar representations of $\mathbb{Z}$

$$\{1\}, \{-1, +1\}, .., \left\{ e^{\frac{2\pi}{1000}\mathrm{i}} | m \in \{1 \ldots 1000\} \right\}, ..$$



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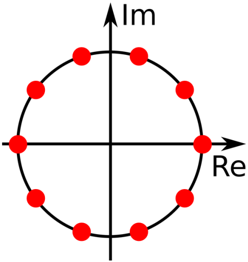

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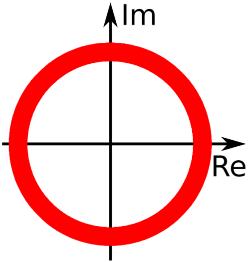

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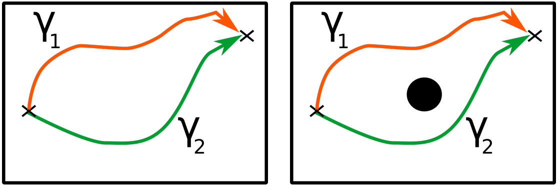
Examples

$$(\mathbb{Z}, +) \simeq \left\{ I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \right\}$$

$$\mathbb{B}_3 \simeq \left\{ \begin{pmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{pmatrix} \right\}$$

Topologically equivalent paths

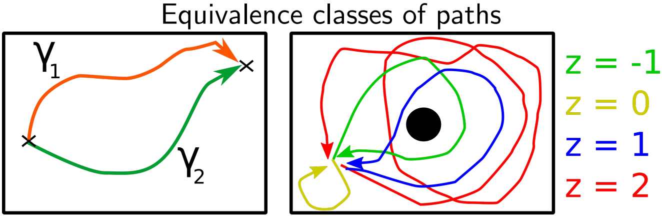
Equivalence classes of paths



Fundamental group

The fundamental group π_1 consists of all classes of loops that are not deformable into each other.

Topologically equivalent paths



Fundamental group

The fundamental group π_1 consists of all classes of loops that are not deformable into each other.

$$\pi_1\,(\text{punctured plane}\,) \cong \mathbb{Z}\;(\text{winding number})$$

path integral quantization

Propagator

$$\Psi(\vec{x_2}, t_2) = \int dx_1 \; K(\vec{x_2}, t_2, \vec{x_1}, t_1)\Psi(\vec{x_1}, t_1)$$

$$K(\vec{x_1}, t_1, \vec{x_2}, t_2) = \int D\gamma \, e^{(\mathrm{i}/\hbar)S(\gamma)}$$

- path $\gamma : [0,1] \to \mathcal{C}$
- Action $S(\gamma)$
- Integral over all paths γ from starting to final configuration

path integral quantization

Propagator

$$\Psi(\vec{x_2}, t_2) = \int dx_1 \; K(\vec{x_2}, t_2, \vec{x_1}, t_1)\Psi(\vec{x_1}, t_1)$$
$$K(\vec{x_1}, t_1, \vec{x_2}, t_2) = \int D\gamma \, \alpha(\gamma) e^{(i/\hbar)S(\gamma)}$$

- path $\gamma : [0,1] \to \mathcal{C}$
- Action $S(\gamma)$
- Integral over all paths γ from starting to final configuration
- $\alpha(\gamma)$: phase, from a representation of the fundamental group of $\mathcal{C}$.

path integral quantization

Propagator

$$\Psi_{\sigma,\sigma'}(\vec{x_2}, t_2) = \int dx_1 \; K(\vec{x_2}, t_2, \vec{x_1}, t_1)\Psi_{\sigma,\sigma'}(\vec{x_1}, t_1)$$

$$K_{\sigma,\sigma'}(\vec{x}_1, t_1, \vec{x}_2, t_2) = \int D\gamma \, \alpha_{\sigma,\sigma'}(\gamma) e^{(\mathrm{i}/\hbar)S(\gamma)}$$

- path $\gamma : [0, 1] \to \mathcal{C}$
- Action $S(\gamma)$
- Integral over all paths γ from starting to final configuration
- $\alpha_{\sigma,\sigma'}(\gamma)$: unitary matrix, from a representation of the fundamental group of $\mathcal{C}$.

Anyons aux Leinaas and Myrheim

1 History of anyons

2 Mathematical prerequisites

3 Anyons aux Leinaas and Myrheim
- Only bosons and fermions?
- Classical and quantum theory of identical particles

4 Anyonic systems and their applications

5 Remarks on statistical transmutation

Only bosons and fermions?

Wave function of two indiscernible particles at positions $\vec{x}_1$, $\vec{x}_2$.

$$
\begin{aligned}
|\Psi(\vec{x}_1, \vec{x}_2)|^2 = |\Psi(\vec{x}_2, \vec{x}_1)|^2 \Leftrightarrow \\
\Psi(\vec{x}_1, \vec{x}_2) = e^{\mathrm{i}\phi}\Psi(\vec{x}_2, \vec{x}_1)
\end{aligned} \tag{1}
$$

$$
\Psi(\vec{x}_1, \vec{x}_2) \overset{Eq.(1)}{=} e^{\mathrm{i}\phi}\Psi(\vec{x}_2, \vec{x}_1) \overset{Eq.(1)}{=} \underbrace{e^{2\mathrm{i}\phi}}_{=1}\Psi(\vec{x}_1, \vec{x}_2).
$$

$$
\begin{aligned}
e^{\mathrm{i}\phi} = 1 \quad & \text{bosons} \\
e^{\mathrm{i}\phi} = -1 \quad & \text{fermions}
\end{aligned}
$$

Only bosons and fermions?

Wave function of two indiscernible particles at positions $\vec{x}_1$, $\vec{x}_2$.

$$\begin{aligned} |\Psi(\vec{x}_1,\vec{x}_2)|^2 = |\Psi(\vec{x}_2,\vec{x}_1)|^2 \Leftrightarrow \\ \Psi(\vec{x}_1,\vec{x}_2) = e^{\mathrm{i}\phi}\Psi(\vec{x}_2,\vec{x}_1) \end{aligned} \qquad (1)$$

$$\Psi(\vec{x}_1,\vec{x}_2) \overset{Eq.(1)}{=} e^{\mathrm{i}\phi}\Psi(\vec{x}_2,\vec{x}_1) \overset{Eq.(1)}{=} \underbrace{e^{2\mathrm{i}\phi}}_{=1}\Psi(\vec{x}_1,\vec{x}_2).$$

$$\begin{aligned} e^{\mathrm{i}\phi} &= 1 \quad \text{bosons} \\ e^{\mathrm{i}\phi} &= -1 \quad \text{fermions} \end{aligned}$$

What did we do wrong?

Only bosons and fermions?

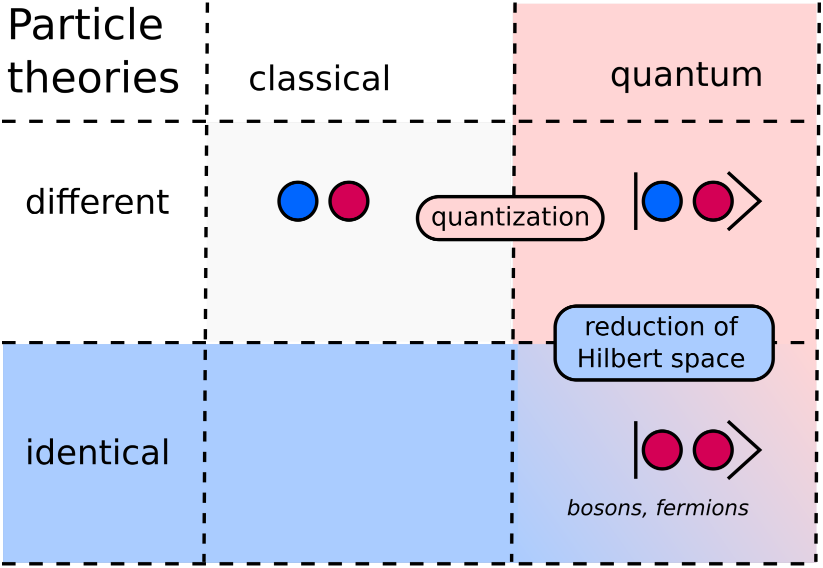
Problem

Intrinsic problem with proof:

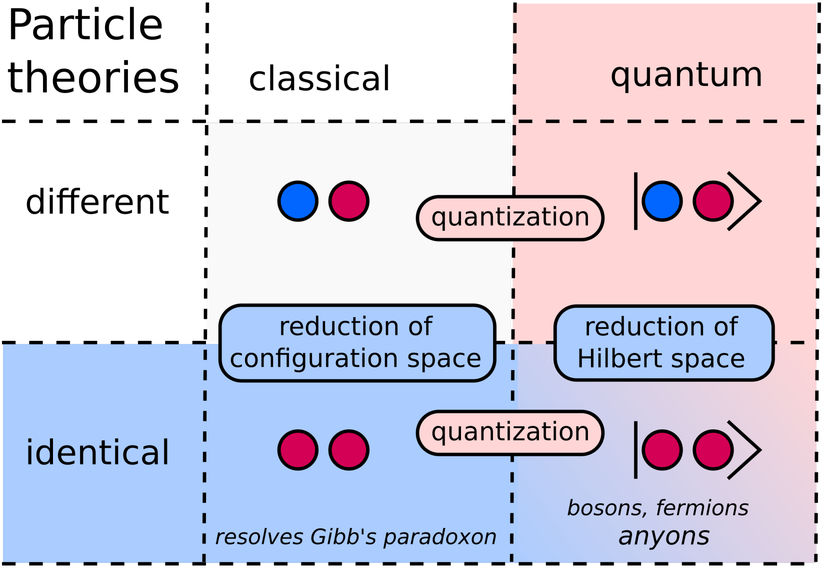
The labeling $\vec{x}_1$, $\vec{x}_2$ adds unobservable elements to the theory.

- $|\Psi(\vec{x}_1, \vec{x}_2)|^2 = |\Psi(\vec{x}_2, \vec{x}_1)|^2$ at most reflects notational redundancy

Underlying idea



Underlying idea



Classical Theory

Configuration space

Configuration space $\mathcal{C}$: Set containing all possible spatial configurations

For distinguishable particles

$$\mathcal{C}_{n,d}^{\text{dist.}} = \left\{ (\vec{x}_1, \ldots, \vec{x}_n) \, | \vec{x}_i \in \mathbb{R}^d \right\}$$

Classical Theory

Configuration space

Configuration space $\mathcal{C}$: Set containing all possible spatial configurations

For distinguishable particles

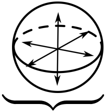
$$\mathcal{C}_{n,d}^{\text{dist.}} = \left\{ (\vec{x}_1, \ldots, \vec{x}_n) \,|\vec{x}_i \in \mathbb{R}^d \right\}$$

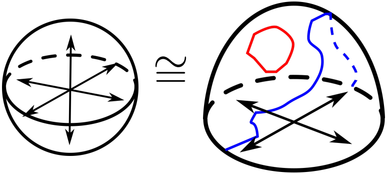
For identical particles

$$\mathcal{C}_{n,d}^{\text{id.}} = \left\{ \{\vec{x}_1, \ldots, \vec{x}_n\} \,|\vec{x}_i \in \mathbb{R}^d, \vec{x}_i \neq \vec{x}_j \right\}$$

Configuration space in $3D$

$$\mathcal{C}^{\text{ind.}}_{2,3D} = \left\{ \{\vec{x}_1, \vec{x}_2\} \,\middle|\, \vec{x}_i \in \mathbb{R}^3, \vec{x}_1 \neq \vec{x}_2 \right\} = \underbrace{\mathbb{R}^3}_{\text{center of mass}} \times \underbrace{(0,\infty)}_{\text{distance}} \times$$





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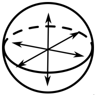
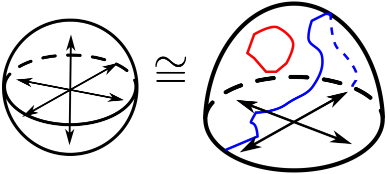


relative angles



$$\pi_1\left(\mathcal{C}^{\text{ind}}_{2,3D}\right) \cong \mathbb{Z}_2$$

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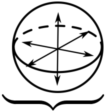

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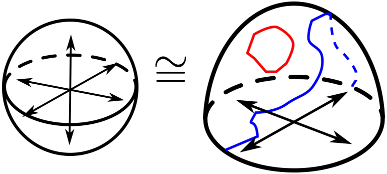

$$\pi_1\left(\mathcal{C}^{\text{ind}}_{2,3D}\right) \cong \mathbb{Z}_2 \quad \Rightarrow$$

two particle species: bosons & fermions

Configuration space in $2D$

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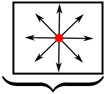

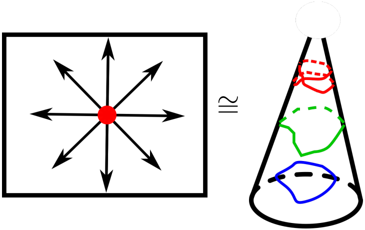

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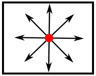

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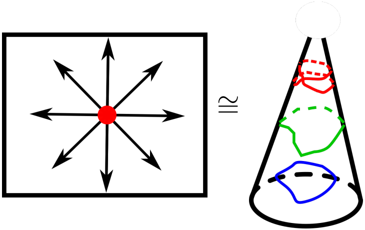

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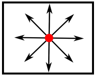

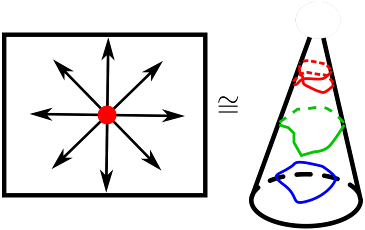

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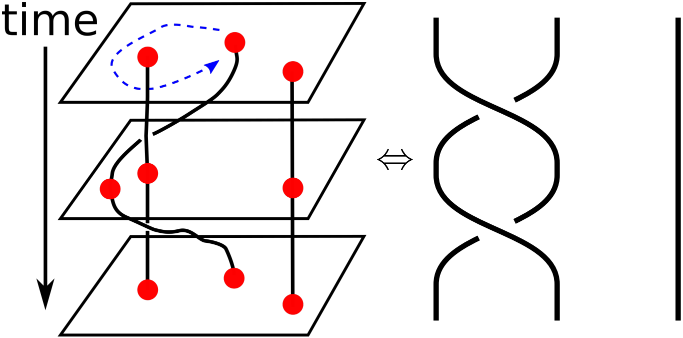
infinitely many representations: **anyons**

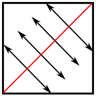
Braid group for anyons



World lines ⇔ braids

Configuration space in $1D$

$$\mathcal{C}^{\text{ind.}}_{2,1D} = \{\{x_1, x_2\}\,|x_i \in \mathbb{R}, x_1 \neq x_2\} =$$



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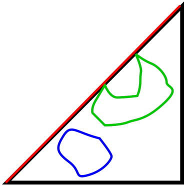

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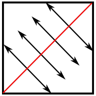

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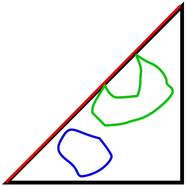

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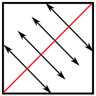
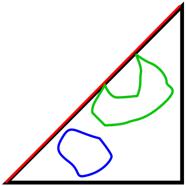
$$\mathcal{C}^{\text{ind.}}_{2,1D} = \{\{x_1, x_2\}\,|x_i \in \mathbb{R}, x_1 \neq x_2\} =$$  $=$ 

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Conservation of probability: $\int_{\mathcal{C}} d\vec{x}_1\, dx_2\ \Psi^*(\{x_1, x_2\}, t)\Psi(\{x_1, x_2\}, t) = 1.$

$$\partial_{x_1 - x_2}\Psi(\{x_1, x_2\}, t)|_{x_1 - x_2 \to 0^+} = \eta\Psi(\{x_1, x_2\}, t)|_{x_1 - x_2 \to 0}$$

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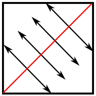

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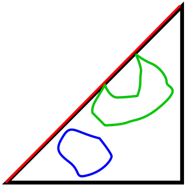

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η: 1D statistical parameter.

- $\eta \to \infty$: fermions
- $\eta = 0$: bosons
- η general: **anyons** (same name, different model)

Posske, Thorwart, Trauzettel: PRB **96**, 195422 (2017)

Anyonic systems and their applications

1 History of anyons

2 Mathematical prerequisites

3 Anyons aux Leinaas and Myrheim

4 Anyonic systems and their applications
- Quantum computing with anyons
- Concrete anyon "anyon-like" models

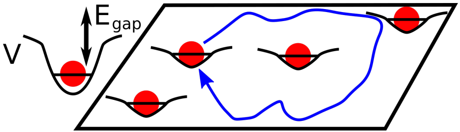
5 Remarks on statistical transmutation

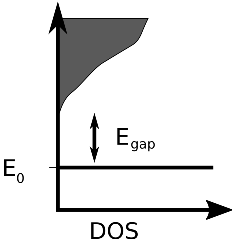
Degenerate subspaces by anyons

- n localized (noninteracting) anyons

$$\mathcal{H} = \sum_{i}^{n} -\nabla^2_{\vec{x}_i}/(2m) + V\left(\{\vec{x}_1, \ldots, \vec{x}_n\}\right)$$

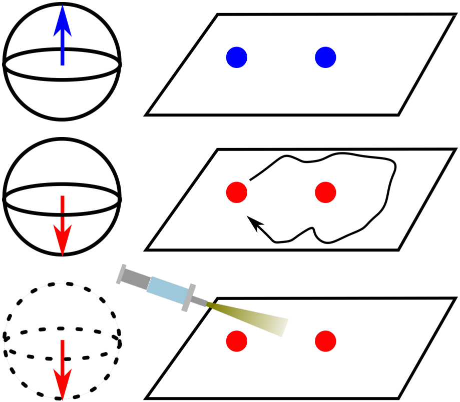
- adiabatic braiding: transformation U with n-dependent dimension
- Different state after braid possible





Topological quantum computing by anyons

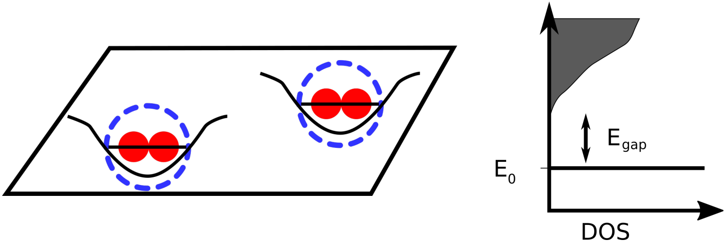
1. input state $\mathbb{C}^n$
2. unitary evolution U
3. readout (measurement)



- Dephasing, decoherence depend on overlap of particles.
- Topologically protected gates

Anyon fusion

- Bring anyons close together
- only measure collective properties ⇒ composite anyon
- combined braiding properties, fusion rules
- repeated fusion possible

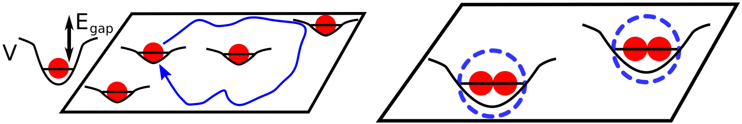


Low energy anyon model

Abstract anyon model

anyon labels $\mathcal{L}$	$1, \tau, \psi, \ldots$	$\bar{1}, \bar{\tau}, \bar{\psi}, \ldots$
fusion rules	$a \times b = \sum_c N^c_{a,b\in\mathcal{L}} c$	

(→ Belén Paredes) (→ Sabine Hossenfelder)



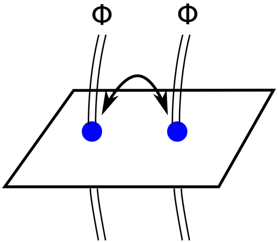
Concrete Anyon and "anyon-like" models

- general D (n)
  - Closely related: Potts, clock models and Gentile statistics
  - $(n-1)$D particles in nD (e.g., strings in 3D)
- 2D
  - **Quantum Hall effects**
  - Type-II superconductors flux-charge composites
  - **Majorana zero modes in 1D superconductors**
  - Majorana zero modes in type II SC vortices
  - Kitaev's toric code
- 1D
  - Haldane-Shastry chain
  - Calogero-Sutherland model
  - fractional excitations in Tomonaga-Luttinger liquids
  - Double-delta interacting Bose gas
  - **1D Anyon-Hubbard model**
  - Cross-over from 2D anyons

Wilczek anyons

Wilczek "anyons"

- flux-charge composite with any exchange phase: **anyons**
- Exchange phase by Aharonov-Bohm effect



Wilczek 1982:

Wilczek: PRL **49**, 957

Since interchange of two of these particles can give *any* phase, I will call them generically anyons.

Wilczek anyons

Wilczek "anyons"

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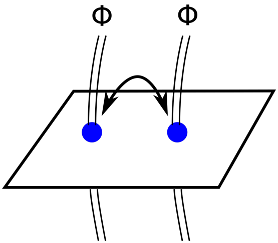

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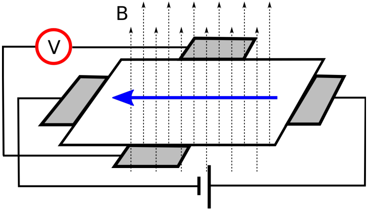
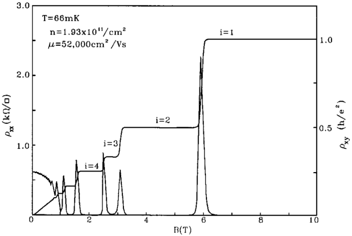
Since interchange of two of these particles can give *any* phase, I will call them generically anyons.

The universal covering space seems very awkward to parametrize and I have not made much progress with it. It is certainly an intriguing mathematical problem to see how the statistical mechanics of many free anyons interpolates between bosons and fermions.

Quantum Hall effect

Quantum Hall effect: $2D$ electrons with strong magnetic field





Tsui: Rev Mod Phys **71**, 891 (1999)

- Plateaus in the Hall resistance at $R_H = \frac{h}{e^2}\frac{1}{\nu} = \frac{1}{\nu}25812.8075\ldots\Omega$

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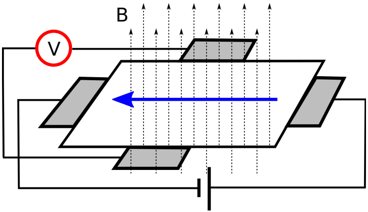

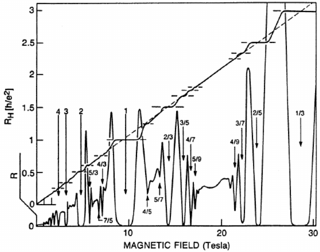

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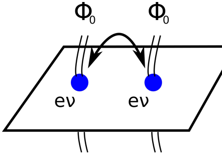

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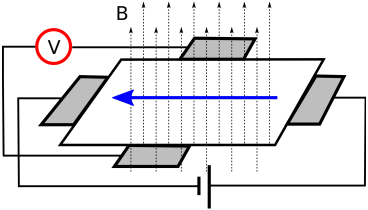

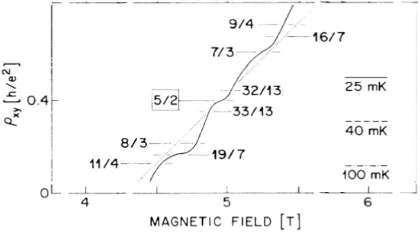

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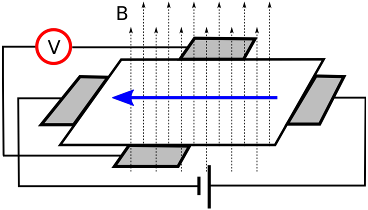

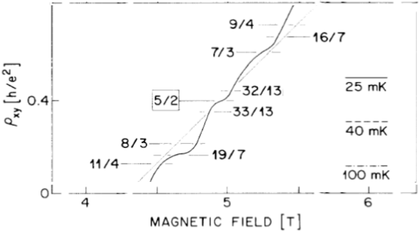

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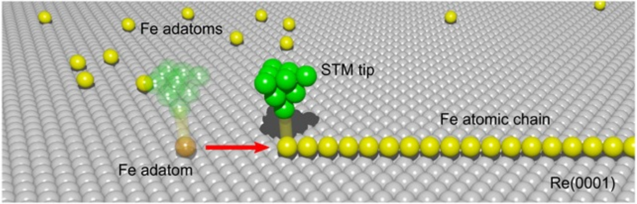
- on lattices ($\rightarrow$ Anne Nielsen)

Majorana zero modes in 1D superconductors



Kim, Palacio-Moralez, Posske, Thowart, .., Wiesendanger: Science Adv. **4**, 5 (2018)

- Fe on Re, helical magnetism, 1D topological superconductor
- Majorana operator γ with $\gamma = \gamma^\dagger$, $\{\gamma_i, \gamma_j\} = \delta_{i,j}$
- one Dirac fermion ⇒ two Majoranas
  $\gamma_1 = 1/\sqrt{2}\left(c + c^\dagger\right)$, $\gamma_2 = \mathrm{i}/\sqrt{2}\left(c - c^\dagger\right)$
- Exchange ⇒ $U \propto e^{\pi/4\gamma_1\gamma_2}$, representation of braid group
  No universal quantum computing.
- (→ Mikhail Baranov)(→ Joachim Brand)

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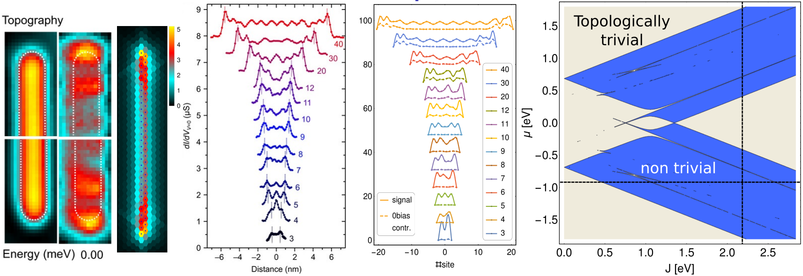

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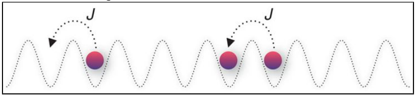
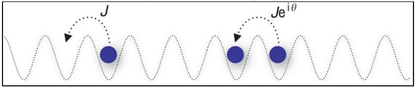
1D Anyon-Hubbard model

Anyon-Hubbard model



conditionally hopping bosons



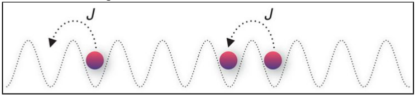
Keilmann, Lanzmich, McCulloch, Roncaglia: Nat Comm. **2**, 361 (2011)

$$\mathcal{H}^a = -t \sum_{j=1}^{L} \left(a_j^\dagger a_{j+1} + h.c. \right) + U/2 \sum_{j=1}^{L} n_j(n_j - 1)$$

$$a_j a_k - e^{i\theta \, \mathrm{sgn}(j-k)} a_k a_j = 0, \quad a_j a_k^\dagger - e^{-i\theta \, \mathrm{sgn}(j-k)} a_k^\dagger a_j = \delta_{k,j}$$

1D Anyon-Hubbard model

Anyon-Hubbard model



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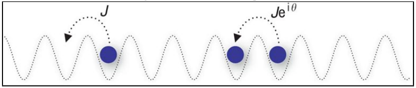


Keilmann, Lanzmich, McCulloch, Roncaglia: Nat Comm. **2**, 361 (2011)

$$\mathcal{H}^a = -t \sum_{j=1}^{L} \left(a_j^\dagger a_{j+1} + h.c. \right) + U/2 \sum_{j=1}^{L} n_j (n_j - 1)$$

$$a_j a_k - e^{\mathrm{i}\theta\, \mathrm{sgn}(j-k)} a_k a_j = 0, \quad a_j a_k^\dagger - e^{-\mathrm{i}\theta\, \mathrm{sgn}(j-k)} a_k^\dagger a_j = \delta_{k,j}$$

Equivalent to bosonic model $a_j = b_j e^{\mathrm{i}\theta \sum_{i=1}^{j-1} n_i}$

$$\mathcal{H}^b = -t \sum_{j=1}^{L} \left(b_j^\dagger b_{j+1} e^{\mathrm{i}\theta n_j} + h.c. \right) + U/2 \sum_{j=1}^{L} n_j \left(n_j - 1 \right)$$

($\rightarrow$ Thore Posske 2nd talk)

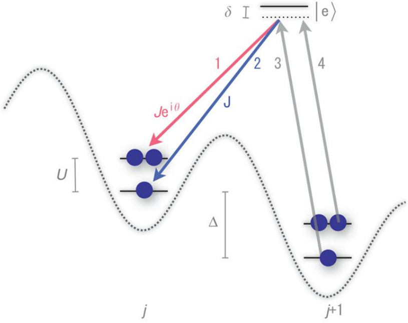
Extended model: ($\rightarrow$ Kevin Jägering) ($\rightarrow$ Martin Bonkhoff)

Tang, Eggert, Pelster: NJP **17**, (2015) and Greschner, Santos: PRL **115**, 053002 (2015)

1D Anyon-Hubbard model

Realization in ultracold atomic gases

- Realizable in ultracold atomic gases
  (→ Sebastian Greschner) (→ Philipp Preiss)
  - Raman-assisted hopping
  - lattice-shaking-induced resonant tunneling



Keilmann, Lanzmich, McCulloch, Roncaglia: Nat Comm. **2**, 361 (2011)

also: Ma, Tai, Preiss,.., Greiner: PRL **107**, 095301 (2011)

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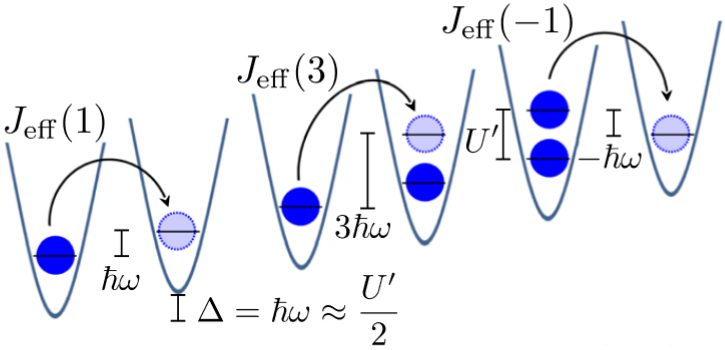


Sträter, Srivastava, Eckardt: PRL **117**, 205303 (2016)

also: Ma, Tai, Preiss,.., Greiner: PRL **107**, 095301 (2011)

2D anyons

Realization in ultracold atom systems

- Realizable in ultracold atoms systems ($\rightarrow$ Christof Weitenberg)
  - Rotating Bose-Einstein condensates

    Wilkin, Gunn: PRL **84**, 6 (2000) and N.Gemelke et al: arXiv:1007.2677 (2010)
  - Lattice fractional quantum Hall effect ($\rightarrow$ Anne Nielsen)
    - ★ Oscillating quadrupole field + modulated hopping

      A.S. Sorensen et al.: PRL **94**, 086803 (2005)
    - ★ laser-assisted tunneling + tilted optical potential

      Aidelsburger et al.: Phys. Rev. Lett. **111**, 185301 (2013)

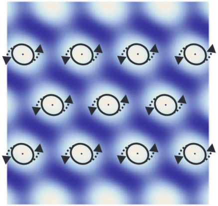
Condensation of "Composite Bosons" in a Rotating BEC

N. K. Wilkin[1,2] and J. M. F. Gunn[1,2]

[1]*School of Physics and Astronomy, University of Birmingham, Edgbaston, Birmingham B15 2TT, United Kingdom*
[2]*European Synchrotron Radiation Facility, BP. 220, 38043 Grenoble Cedex 9, France*
(Received 21 June 1999)

We provide evidence for several novel phases in the dilute limit of rotating Bose-Einstein condensates. By exact calculation of wave functions and energies for small numbers of particles, we show that the states near integer angular momentum per particle are best considered condensates of composite entities, involving vortices and atoms. We are led to this result by explicit comparison with a description purely in terms of vortices. Several parallels with the fractional quantum Hall effect emerge, including the presence of the Pfaffian state.

PACS numbers: 03.75.Fi, 05.30.Jp, 67.40.–w



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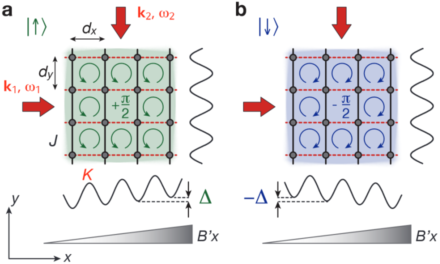
Fractional quantum Hall states of atoms in optical Lattices

Anders S. Sørensen[1,2], Eugene Demler[2], and Mikhail D. Lukin[1,2]
[1] *ITAMP, Harvard-Smithsonian Center for Astrophysics and*
[2] *Physics department, Harvard university, Cambridge Massachusetts 02138*

We describe a method to create fractional quantum Hall states of atoms confined in optical lattices. We show that the dynamics of the atoms in the lattice is analogous to the motion of a charged particle in a magnetic field if an oscillating quadrupole potential is applied together with a periodic modulation of the tunneling between lattice sites. We demonstrate that in a suitable parameter regime the ground state in the lattice is of the fractional quantum Hall type and we show how these states can be reached by melting a Mott insulator state in a super lattice potential. Finally we discuss techniques to observe these strongly correlated states.

PACS numbers: 03.75.Lm,73.43.-f



A.S. Sorensen et al.: PRL **94**, 086803 (2005) and Aidelsburger et al.: Phys. Rev. Lett. 111, 185301 (2013)

Remarks on statistical transmutation

1 History of anyons

2 Mathematical prerequisites

3 Anyons aux Leinaas and Myrheim

4 Anyonic systems and their applications

5 Remarks on statistical transmutation

Statistical transmutation in 1D

- bosonization $\leftrightarrow$ (re)fermionization

$$\Psi(x) = \frac{1}{\sqrt{a}} F e^{-\mathrm{i}\phi(x)}$$

- Jordan-Wigner transformation: fermion $\leftrightarrow$ spins

$$c_j = S^- e^{-\mathrm{i}\pi \sum_{k<j} S^+ S^-}$$

- generalized J-W transformation: bosons/fermions $\leftrightarrow$ anyons

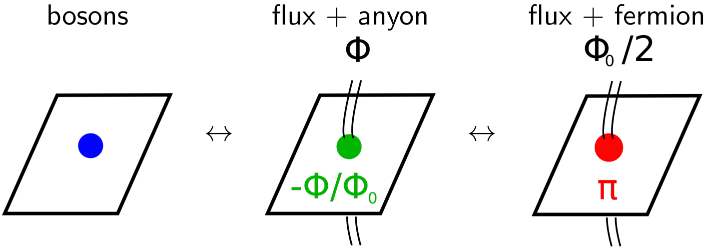
Tang, Eggert, Pelster: NJP **17**, (2015) and Posske, Thorwart, Trauzettel: PRB **96**, 195422 (2017)

$$a(j) = b(j) \lim_{\epsilon \to 0^+} e^{\mathrm{i} \int_{-\infty}^{\epsilon} dk \;\; b^\dagger(k) b(k) \phi_\eta(k-j)}$$

$$a_j = b_j e^{\mathrm{i}\theta \sum_{k<j} b_k^\dagger b_k}$$

Statistical transmutation in 2D

- statistical transmutation



- Composite fermion picture
- $\geq 2D$ bosonization (Haldane 1992, Fröhlich 1995)

J. Frohlich et al.: J. Phys. A: Math. Gen. **28**, 1169 (1995)

- 2D Jordan-Wigner transformation (Fradkin 1989)

Fradkin: PRL **63**, 322 (1989)

Summary

- Anyons: 1D or 2D identical particles that are neither bosons nor fermions
- Prospect for fault-proof quantum computing on degenerate ground states
- Different models with different phenomenology, could appear in cold-atom systems, quantum Hall effects or Topological superconductors or other realizations

Questions, comments, discussion

Recommendations:

- J.S. Dowker J. Phys. A **5**, 7 (1972)
- Leinaas & Myrheim , J. Nuovo Cim B **37**, 1(1977)
- Frank Wilczek. Phys. Rev. Lett. 49, 957 (1982)
- Preskill's lecture notes on topological quantum computation

UHH Universität Hamburg
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ERC ASTONISH

ERC Advanced Grant ASTONISH

ERC AIM

erc European Research Council
Established by the European Commission

DFG Deutsche Forschungsgemeinschaft

International School and Workshop

Anyon Physics of Ultracold Atomic Gases

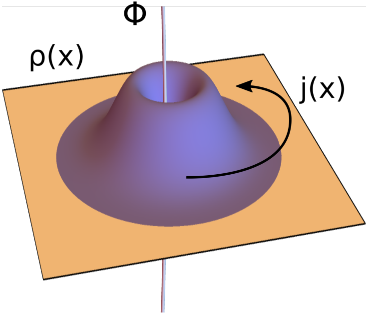
Technische Universität Kaiserslautern (Germany), December 10 - 14, 2018

Anyone, any anyon question?

Additional slides

Rotating an anyon around itself

- Anyon "wave function" is extended
- parts of the wave function can be rotated around other parts to give a statistical phase
- Total phase the same as if rotating around other anyon



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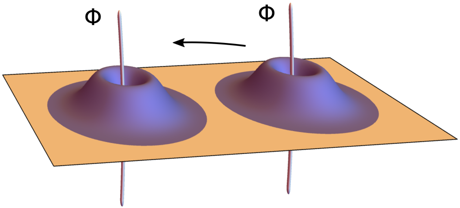
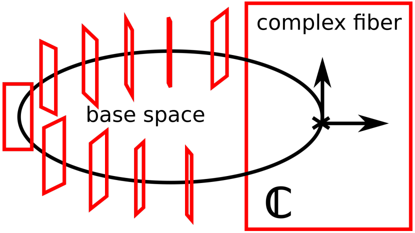


Canonical quantization of topologically nontrivial spaces



Change definition of wave function: $\Psi : \left(E \xrightarrow{\pi} \mathcal{C}\right) \to \mathbb{C}$ maps vector bundle to complex numbers

Canonical quantization:

$p \to \mathrm{i}\nabla$: Momentum becomes covariant derivative

$x \to x$: Spatial coordinate

$$\mathcal{H}_{\text{free}} = \frac{p^2}{2m} \to \frac{(\mathrm{i}\nabla_x + A(x))^2}{2m} \text{ for concrete representation}$$

With real $A(x)$. Mathematically equivalent to magnetic (vector) potential in minimal coupling.

$\Rightarrow$ Nontrivial topology $\Leftrightarrow$ Aharonov-Bohm phase

Low energy anyon model

Abstract model

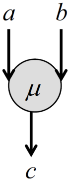
anyon labels $\mathcal{L}$	$1, \tau, \psi, \ldots \quad \bar{1}, \bar{\tau}, \bar{\psi}, \ldots$
fusion rules	$a \times b = \sum_c N^c_{a,b \in \mathcal{L}} c$

Examples

- trivial
  $\tau \times \tau = 1$
- non-abelian but non-universal (Majorana, Ising)
  $\tau \times \tau = 1 + \psi$, $\psi \times \tau = \tau$, $\Psi \times \Psi = 1$
- non-abelian and universal (Fibonacci)
  $\tau \times \tau = 1 + \tau$

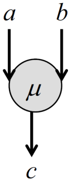
($\rightarrow$ Belén Paredes) ($\rightarrow$ Sabine Hossenfelder)

Manipulation

fusion 

$$= |\, ab; c, \mu \rangle \;,\; \mu \in \left\{ 1, \ldots, N^{c}_{a,b} \right\}$$

Manipulation

fusion 

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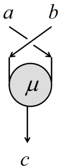
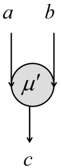
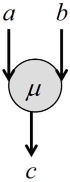
braiding 

$$= \sum_{\mu'} \left(R^{c}_{ba}\right)^{\mu'}_{\mu}$$



Manipulation

fusion



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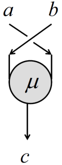
braiding



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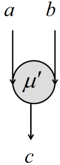
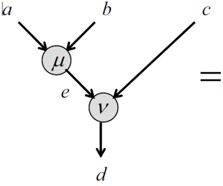
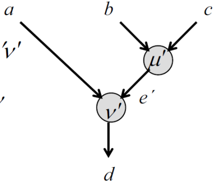


F-move

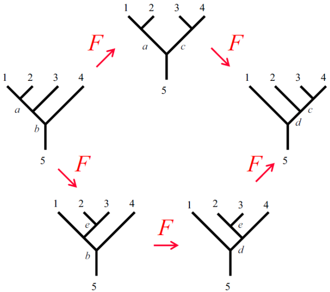


$$= \sum_{e'\mu'\nu'} \left(F^d_{abc}\right)^{e'\mu'\nu'}_{e\mu\nu}$$

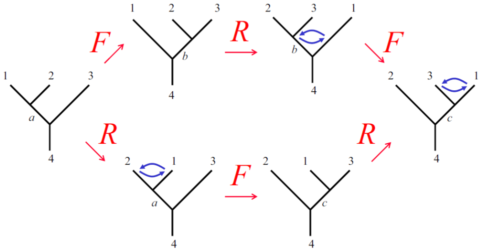


Consistency relations

pentagon equation



hexagon equation



Problems with Majorana zero modes