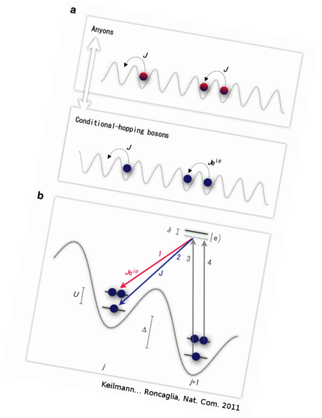
# (Abelian) Anyon-Hubbard Models in 1D (Optical) Lattices

I. (Floquet) Engineering with cold atomic quantum gases: Interaction modulation & Raman assisted hopping

II. Lattice shaking & Properties (ground state phase diagram and dynamics)





Sebastian Greschner - Anyon Workshop - December 2018

UNIVERSITÉ DE GENÈVE
FACULTÉ DES SCIENCES
Section de physique

DQMP Department of Quantum Matter Physics

MaNEP SWITZERLAND NETWORK

FNSNF
FONDS NATIONAL SUISSE
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FONDO NAZIONALE SVIZZERO
SWISS NATIONAL SCIENCE FOUNDATION

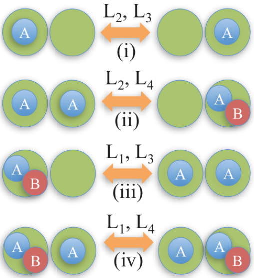
# Outline - Day I:

I. (Floquet) Engineering with cold atomic quantum gases

- Digest of cold atom physics
- Fermions, Bosons, Anyons - Jordan-Wigner-Transformation in 1D and 2D
- Floquet-Engineering
- Modulated Interactions and Experiments in Chicago
- Assisted Hopping Schemes

II. Properties

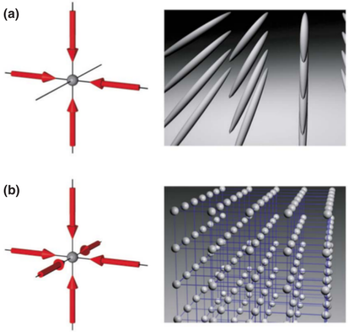
- Digest of 1D physics
- Anyon “Interferometer” on a ring
- Simple ladder model of braiding anyons
- 1D (Pseudo) Anyon Hubbard model



# Ultracold Gases in Optical Lattices

Ultracold quantum gases in optical lattices provide an excellent toolbox for...

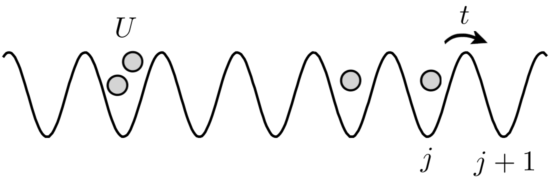
- ... strongly correlated many-body systems in and out of equilibrium
- ... **quantum simulation** of condensed matter paradigms, high energy physics
- ... quantum simulation of interacting **synthetic gauge field** theories




M. Greiner, PhD thesis

- clean, scalable lattice system
- control, adjustable (in real time)
- observable (momentum distribution, measurements with single site resolution, ...)

# Bose Hubbard model



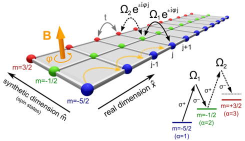

- Hamiltonian for interacting bosonic particles in a trapping potential
- Tight binding approximation: Expand bosonic field operator in basis of wannier functions $\hat{\Psi}(x) = \sum_i \hat{b}_i w(x - x_i)$

$$\hat{H}_{BH} = -\sum_{\langle ij \rangle} J_{ij} \hat{b}_i^\dagger \hat{b}_j + \sum_i (\epsilon_i - \mu) \hat{n}_i + \sum_i \frac{U}{2} \hat{n}_i (\hat{n}_i - 1)$$

- **Bose Hubbard Hamiltonian** with effective parameters for hopping $J_{ij}$ and interaction $U$

# Density-Dependent Gauge Fields

## static flux models

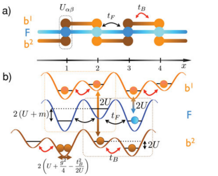



Mancini, ... Fallani Science 2015
Stuhl, ... Spielman, Science 2015,
Aidelsburger, ... Bloch, PRL 2013,
...

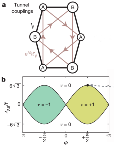
Anyons,
density dependent
fields, ...

Dynamic feedback of the particles on the gauge field - “Moving particles create magnetic field”

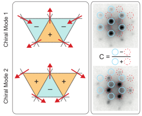
## dynamical LGT



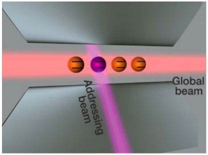

Banerjee, ... Zoller, PRL 2012,
Kasper, ... Berges New J. Phys 2017
Wiese, Ann. Phys 2013,
...




Jotzu... Esslinger, Nature 2014




Struck... Sengstock, Science 2011,
...




Martinez, ... Blatt, Nature 2016

# Bosons, Fermions, Anyons

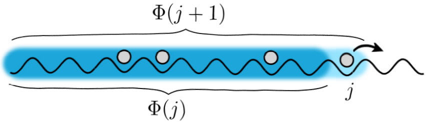
# Jordan-Wigner-Transformation in 1D

- Consider strong interaction side of the Bose-Hubbard phase diagram $U \to \infty$, at some fixed filling say $0 < n < 1$. Assume hard-core bosons $b_j$, i.e. $b_j^2 = 0$

*Can we describe hard-core bosons $b_j$ by fermions $c_j$?*

$$b_j^\dagger = a_j^\dagger \left(\mathrm{e}^{\mathrm{i}\alpha \sum_{l<j} \hat{n}_l}\right) \qquad a_j^\dagger = b_j^\dagger \left(\mathrm{e}^{-\mathrm{i}\alpha \sum_{l<j} \hat{n}_l}\right)$$

$$b_j = \left(\mathrm{e}^{-\mathrm{i}\alpha \sum_{l<j} \hat{n}_l}\right) a_j \qquad a_j = \left(\mathrm{e}^{\mathrm{i}\alpha \sum_{l<j} \hat{n}_l}\right) b_j$$




This keeps density operators invariant $n_j = b_j^\dagger b_j = a_j^\dagger \left(\mathrm{e}^{\mathrm{i}\alpha \sum_{l<j} \hat{n}_l}\right)\left(\mathrm{e}^{-\mathrm{i}\alpha \sum_{l<j} \hat{n}_l}\right) a_j = a_j^\dagger a_j$

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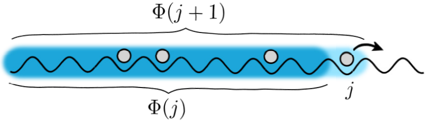


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What does this mean to commutations?

$$a_j a_k = b_j \left(\mathrm{e}^{\mathrm{i}\alpha \sum_{l<j} \hat{n}_l}\right)\left(\mathrm{e}^{\mathrm{i}\alpha \sum_{l<k} \hat{n}_l}\right) b_k = a_k a_j \begin{cases} e^{-\mathrm{i}\alpha}, & j > k \\ e^{\mathrm{i}\alpha}, & j < k \end{cases}$$

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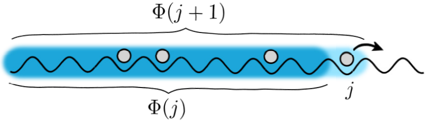


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What about the nearest neighbor hopping term?

$$b_j^\dagger b_{j+1} = a_j^\dagger \left(\mathrm{e}^{\mathrm{i}\alpha \sum_{l<j} \hat{n}_l}\right)\left(\mathrm{e}^{-\mathrm{i}\alpha \sum_{l<j+1} \hat{n}_l}\right) a_{j+1} = a_j^\dagger \left(\mathrm{e}^{\mathrm{i}\alpha \hat{n}_j}\right) a_{j+1} = a_j^\dagger a_{j+1} = c_j^\dagger c_{j+1}$$

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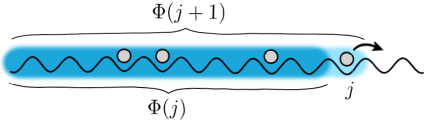


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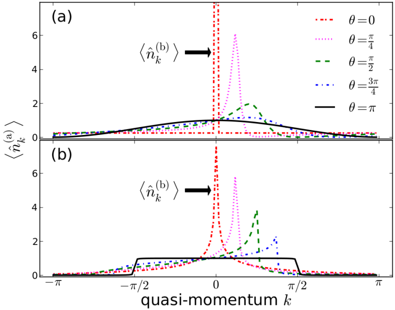
- We have solved analytically strongly interacting many-body problem!
- We have introduced 1D anyons! (and showed that they have a trivial spectrum...)

# Anyons

Certainly not trivial - e.g. momentum distribution

$$n(k) = \sum_{j,j'} \mathrm{e}^{\mathrm{i}k(j-j')} \underbrace{\langle a_j^\dagger a_{j'} \rangle}_{\langle b_j^\dagger \mathrm{e}^{-\mathrm{i}\Phi(j)} \mathrm{e}^{\mathrm{i}\Phi(j')} b_{j'} \rangle}$$



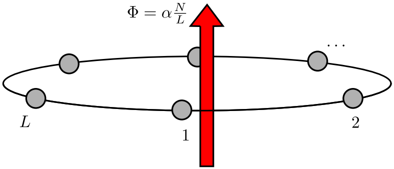

G. Tang, S. Eggert, A. Pelster, New J. Phys. 17, 123016 (2015)

Idea: Engineer anyon model in quantum gas experiment by local modification of the hopping.

Allow for exchange of anyons:

- 1D ring
- Particles may pass through each other: "Pseudo" anyons
- 2D system, ladder, ...

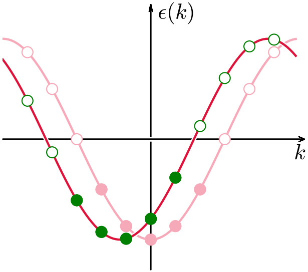
# Hard core anyons on a ring



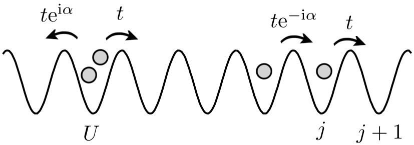

- 1D system with PBC, i.e. add a "long bond"

$$
\begin{aligned}
a_L^\dagger a_1 &= c_L^\dagger \mathrm{e}^{\mathrm{i}\alpha \sum_{l=1}^{L} n_l} c_1 \\
&= \mathrm{e}^{\mathrm{i}\alpha(N-1)} c_L^\dagger c_1
\end{aligned}
$$

- Fermions with flux $\alpha N/L$ depending on total particle number
- For finite system energy changes, time dynamics
- in the thermodynamic limit same as OBC system



# 1D Pseudo anyons - Correlated Tunneling




- Assume bosons on-site
- anyonic/deformed exchange statistics
  $a_j a_k^\dagger - \mathcal{F}_{j,k} a_k^\dagger a_j = \delta_{j,k}$

$$\mathcal{F}_{j,k} := \begin{cases} e^{-i\alpha}, & j > k, \\ 1, & j = k, \\ e^{i\alpha}, & j < k, \end{cases}$$

- **Correlated/Density dependent hopping** model for bosons

$$\hat{H} = -t \sum_j \left( b_j^\dagger e^{-i\alpha n_j} b_{j+1} + \text{H.c.} \right)$$

- similar: **two component anyons**

$$\hat{H} = -t \sum_{j,\sigma} \left( a_{j,\sigma}^\dagger a_{j+1,\sigma} + \text{H.c.} \right)$$

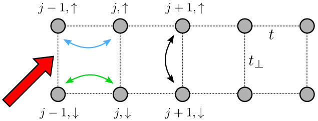
again $a_{j,\sigma} a_{k,\sigma'}^\dagger - \mathcal{F}_{j,k} a_{k,\sigma'}^\dagger a_{j,\sigma} = \delta_{j,k}\delta_{\sigma,\sigma'}$

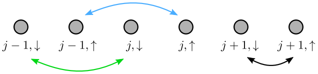
# Anyons on a ladder

| ladder | spins | anyons |
|---|---|---|
| × ×<br>o→× | $(\downarrow,0)\rightarrow(0,\downarrow)$ | 1 |
| o ×<br>o→× | $(2,0)\rightarrow(\uparrow,\downarrow)$ | $\mathrm{e}^{\mathrm{i}\alpha}$ |
| × o<br>o→× | $(2,0)\rightarrow(\uparrow,\downarrow)$ | 1 |
| o o<br>o→× | $(2,\uparrow)\rightarrow(\uparrow,2)$ | $\mathrm{e}^{\mathrm{i}\alpha}$ |
| o→×<br>o × | $(2,0)\rightarrow(\downarrow,\uparrow)$ | 1 |
| o→×<br>× o | $(2,0)\rightarrow(\downarrow,\uparrow)$ | $\mathrm{e}^{-\mathrm{i}\alpha}$ |
| … | … | |

+ Hermitian conjugate (← processes)

+ rung exchange



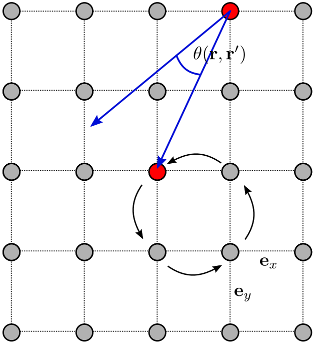

- Choose some order of lattice sites

$$\hat{H} = -t \sum_{j,\sigma=0,1} a^\dagger_{j,\sigma} a_{j+1,\sigma} + \mathrm{H.c.} + \\ - t_\perp \sum_j a^\dagger_{j,\uparrow} a_{j,\downarrow} + \mathrm{H.c.}$$

- particles "overcrossing" $\mathrm{e}^{-\mathrm{i}\alpha}$ and "undercrossing" $\mathrm{e}^{+\mathrm{i}\alpha}$

$$\hat{H} = -t \sum_j b^\dagger_{j,\uparrow} \mathrm{e}^{\mathrm{i}\alpha n_j} b_{j+1,\uparrow} + \mathrm{H.c.} + \\ - t \sum_j b^\dagger_{j,\downarrow} \mathrm{e}^{-\mathrm{i}\alpha n_{j+1}} b_{j+1,\downarrow} + \mathrm{H.c.} + \cdots$$

# Jordan-Wigner-Transformation in 2D (Fradkin, 1989)




There are several ways to generalize the Jordan Wigner transformation to 2D

$$a_{\mathbf{r}} = \mathrm{e}^{\mathrm{i}\alpha \sum_{\mathbf{k}} \theta_{\mathbf{k},\mathbf{r}} n_k} c_{\mathbf{r}}$$

$\theta_{\mathbf{k},\mathbf{r}}$ is the angle between $\mathbf{k} - \mathbf{r}$ and some direction.

- The resulting particles are indeed anyons

$$a_{\mathbf{r}} a_{\mathbf{r}'} = \mathrm{e}^{\mathrm{i}\alpha(\theta_{\mathbf{r},\mathbf{r}'} - \theta_{\mathbf{r}',\mathbf{r}})} a_{\mathbf{r}'} a_{\mathbf{r}}$$

- the hopping becomes

$$\hat{H} = \sum_{r,r'} c_r^\dagger \mathrm{e}^{\mathrm{i}A_{r,r'}} c_{r'} + \mathrm{H.c.} + \cdots$$

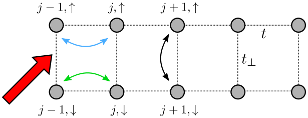
with $A_{r,r'} = \sum_{k \neq r,r'} (\theta_{k,r} - \theta_{k,r'}) \hat{n}_k$

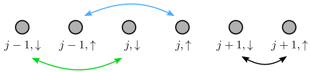
- equivalent to flux “attached” to particle:

$$\begin{aligned} \text{flux} &= A_{r,r+e_x} + A_{r+e_x,r+e_x+e_y} - A_{r+e_y,r+e_x+e_y} - A_{r,r+e_y} \\ &= \frac{\alpha}{2}\left(n_r + n_{r+e_x} - n_{r+e_x+e_y} - n_{r+e_y}\right) \end{aligned}$$

# Different "gauge"

$$\hat{H} = -\,t\sum_j b^\dagger_{j,\uparrow}\mathrm{e}^{\mathrm{i}\alpha n_{j,\downarrow}}b_{j+1,\uparrow} + \mathrm{H.c.} + \\ -\,t\sum_j b^\dagger_{j,\downarrow}\mathrm{e}^{-\mathrm{i}\alpha n_{j+1,\uparrow}}b_{j+1,\downarrow} + \mathrm{H.c.} + \\ -\,t_\perp\sum_j b^\dagger_{j,\downarrow}b_{j,\uparrow} + \mathrm{H.c.}$$




- Symmetric gauge $b_{j,\sigma} \to \mathrm{e}^{\mathrm{i}\sigma\frac{\alpha}{2}n_j}b_{j,\sigma}$.

$$\hat{H} = -t\sum_{j,\sigma=0,1} b^\dagger_{j,\sigma}\mathrm{e}^{-\mathrm{i}\sigma\frac{\alpha}{2}(n_j+n_{j+1})}b_{j+1,\sigma} + \mathrm{H.c.} - t_\perp\sum_j b^\dagger_{j,0}b_{j,1} + \mathrm{H.c.}$$

- Simple check: Put phases on rungs $b_{j,\sigma} \to \mathrm{e}^{\mathrm{i}\sigma\alpha\sum_{l<j}n_l}\mathrm{e}^{\sigma\mathrm{i}\frac{\alpha}{2}n_{j,\bar{\sigma}}}b_{j,\sigma}$.

$$\hat{H} = -t\sum_{j,\sigma=0,1} b^\dagger_{j,\sigma}b_{j+1,\sigma} + \mathrm{H.c.} - t_\perp\sum_j \mathrm{e}^{\mathrm{i}\sigma\frac{\alpha}{2}\sum_{l<j}(n_{j,\uparrow}-n_{j,\downarrow})}b^\dagger_{j,0}b_{j,1} + \mathrm{H.c.}$$

For $t_\perp \to 0$ we have two disconnected hardcore-chains and the exchange phase has any influence ✓

# Density-dependent phase

Dynamic feedback of the particles on the gauge field: Operator dependent phase. Engineering of a **density-dependent Peierls phase**

$$\hat{b}_j^\dagger e^{i\phi(\hat{n}_j,\hat{n}_k)} \hat{b}_k + H.c.$$

- Modulated interactions
  - 2-species Modulated interactions — *SG, G.Sun, D.Poletti, and L.Santos. PRL 113, 215303 (2014)*
  - s-p orbitals — *Clark... Cheng PRL 121, 030402 (2018)*
- Assisted tunneling
  - Raman assisted hopping — T. Keilmann, S. Lanzmich, L. McCulloch, and M. Roncaglia. *Nature Comm.* **2**, *361 (2011)*; SG, and L. Santos. *PRL 115, 053002 (2015)*
  - **lattice shaking** — *C. Sträter, S. C. L. Srivastava, and A. Eckardt. PRL 117(20), 205303 (2016)*; *L. Cardarelli, SG, and L. Santos. PRA, 94(2), 023615 (2016)*
  - ...
- Various further cold atom proposals

*B. Paredes, P. Fedichev, J. I. Cirac, and P. Zoller, PRL 87, 010402 (2001)*
*L.-M. Duan, E. Demler, and M. D. Lukin, PRL 91, 090402 (2003)*
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*M. Aguado, G. K. Brennen, F. Verstraete, and J. I. Cirac, PRL 101, 260501 (2008)*
*L. Jiang, ... P. Zoller, Nat Phys 4, 482 (2008)*
**...**

# Intensive work on Abelian lattice anyons...

- Bethe-Ansatz, integrable systems

*A. Osterloh, L. Amico, U. Eckern J. Phys. A (2000)*
*...*

- asymmetric momentum distributions

*P. Calabrese and M. Mintchev, Phys. Rev. B 75, 233104 (2007)*
*O. I. Ptu, V. E. Korepin, and D. V. Averin, J. Phys. A 40, 14963 (2007)*
*Y. Hao, Y. Zhang, and S. Chen, Phys. Rev. A 78, 023631 (2008)*
*P. Calabrese and R. Santachiara, J. Stat. Mech. Theory Exp 2009, P03002 (2009)*
*G. Tang, S. Eggert, and A. Pelster, New J. Phys 17, 123016 (2015)*
*...*

- particle dynamics

*A. del Campo, Phys. Rev. A 78, 045602 (2008)*
*Y. Hao and S. Chen, Phys. Rev. A 86, 043631 (2012)*
*L. Wang, L. Wang, and Y. Zhang, Phys. Rev. A 90, 063618 (2014)*
*L. Piroli and P. Calabrese, Phys. Rev. A 96, 023611 (2017)*
*N. T. Zinner, Phys. Rev. A 92, 063634 (2015)*
*...*

- entanglement properties

*R. Santachiara, F. Stauffer, and D. C. Cabra, J. Stat. Mech. Theory Exp 2007, L05003 (2007)*
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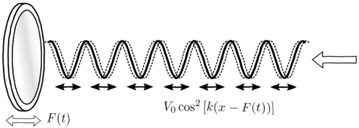
*G. Marmorini, M. Pepe, and P. Calabrese, J. Stat. Mech. Theory Exp 2016, 073106 (2016)*
*...*

- quantum phase transitions

*T. Keilmann, S. Lanzmich, I. McCulloch, and M. Roncaglia, Nature Comm. 2, 361 EP (2011)*
*J. Arcila-Forero, R. Franco, and J. Silva-Valencia, Phys. Rev. A 94, 013611 (2016)*
*F. Lange, S. Ejima, H. Fehske, Phys. Rev. Lett. 118, 120401 ...*

- ... (incomplete)

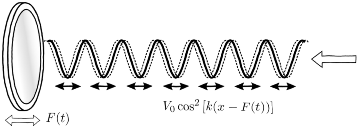
# Floquet Systems - Lattice shaking etc.



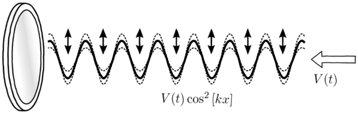
# Realizing effective models in periodically driven systems

- Hamiltonian with time dependent (fast) periodic driving (e.g. lattice shaking)

$$H(t) = H(t+T)$$




- ubiquitous tool in many areas of physics for the engineering and probing of various systems (NMR, atom-light interactions, Raman-dressing...)




- basic concept of Floquet engineering: Time evolution

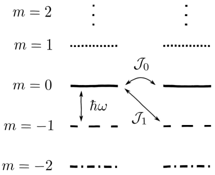
$$\hat{\mathcal{U}}(T,0) = \mathcal{T}\exp\left[-\mathrm{i}\int_0^T dt\hat{H}(t)\right] \equiv \mathrm{e}^{-\mathrm{i}T\hat{H}_{eff}}$$

is described stroboscopically by *effective time-independent* $\hat{H}_{eff}$

- works for a window of frequencies: High compare to “low energy scales”, low compared to higher bands: $J, U, \ldots \gg \omega \gg \Delta_{BG}$

A. Eckardt, Rev. Mod. Phys. 89, 011004 (2017)

# Floquet analysis




- Time periodic Hamiltonian with period $T = 2\pi/\omega$

$$\hat{H}(t) = \hat{H}(t+T)$$

- Analog to Bloch-wave functions Floquet-theorem ensures existence of solution

$$\Psi_n(t) = u_n(t)\mathrm{e}^{\mathrm{i}\epsilon_n t/\hbar}.$$

  with time-periodic functions $u_n(t) = u_n(t+T)$ being the eigenstates of $H(t) - \mathrm{i}\hbar\partial_t$

- real quasi-energies $\epsilon_n$ defined up to a multiple of a photon-energy $m\hbar\omega$
- Floquet basis in a composite Hilbert space and a scalar product

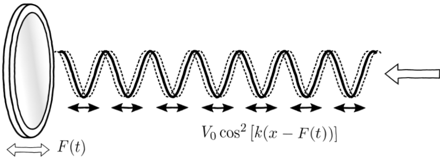
$$|\{n_j\}, m\rangle = |\{n_j\}\rangle\,\mathrm{e}^{\mathrm{i}m\omega t}, \quad \langle\langle\cdot|\cdot\rangle\rangle = \frac{1}{T}\int_0^T dt\langle\cdot|\cdot\rangle$$

- resulting time independent Hamiltonian $\langle\langle\{n'_j\}, m'|H(t) - i\hbar\partial_t|\{n_j\}, m\rangle\rangle$, composed of several blocks separated by $\hbar\omega$

e.g. Eckardt, Weiss, Holthaus, PRL 2005

# Lattice shaking

- shaking mirrors or periodically detuning lasers of optical lattice
- Periodic forcing translates into driven tilt in tight binding comoving frame, e.g $V(t) = V_0 \sin(\omega t)$




$$H(t) = -J \sum b_j^\dagger b_{j'} + \mathrm{H.c.} + V(t) \sum_j j \hat{n}_j + H_{int}$$

- Analyis becomes easier if we go to $\hat{U}(t) = \mathrm{e}^{-i\tilde{V}(t)\sum_j j n_j}$ with $\tilde{V}(t) = \int_0^t V(t')dt'$ and $\tilde{\Theta} = \langle \tilde{V}(t) \rangle_T$ fixes the gauge
- Hamiltonian in new frame $H \to U^\dagger H U - iU^\dagger \dot{U}$

$$H_{tun}(t) = -J \sum_{\langle j,j' \rangle} \mathrm{e}^{i\left(\tilde{V}(t)(j-j')+\tilde{\Theta}_j\right)} b_j^\dagger b_{j'} + H.c.$$

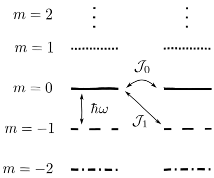
# Lattice shaking

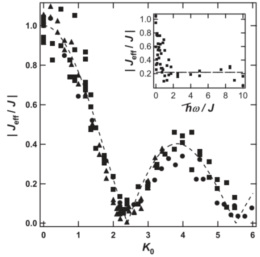
$$H_{tun}(t) = -J \sum_{\langle j,j' \rangle} \mathrm{e}^{i\left(\tilde{V}(t)(j-j')+\tilde{\Theta}_j\right)} a_j^\dagger a_{j'} + H.c.$$

- Floquet-matrix

$$\langle\langle \{n_j'\}, m' | H(t) - i\hbar\partial_t | \{n_j\}, m \rangle\rangle = \delta_{m,m'} \left[ \langle \{n_j'\} | H_{int} | \{n_j\} | \rangle + \hbar\omega m \right] +$$

$$+ \left( \sum_j j(n_j' - n_j) \right)^{m'-m} \mathcal{J}_{m'-m}\left(\frac{V_0}{\hbar\omega}\right) \langle \{n_j'\} | H_{tun} | \{n_j\} | \rangle$$




Lignier, ... Arimondo, *PRL 2007*

Approximation *Just keep diagonal terms $m' = m$*

- Assuming that $t, U \ll \omega$

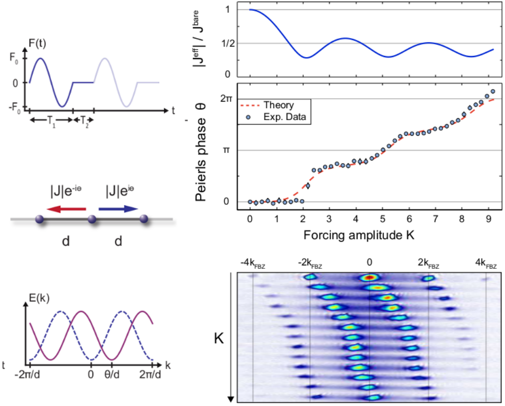
$$H \to -J_{eff} \sum b_j^\dagger b_{j'} + \mathrm{H.c.} + H_{int}$$

with an effective hopping

$$J_{eff} \to -J\mathcal{J}_0\left(\frac{V_0}{\omega}\right)$$

- Interesting situation for $\frac{V_0}{\omega} = 2.404\ldots$ roots of the Besselfunction, *coherent destruction of tunneling*

# Complex phases




Struck, ... Sengstock, *PRL 2012*,
Struck, ... Sengstock, *Nature Physics 2013*

- effective hopping becomes negative for $\frac{V_0}{\omega} > 2.404\ldots$

$$J_{eff} \to J\mathcal{J}_0\left(\frac{V_0}{\omega}\right)$$

Realization of frustrated magnetism in triangular lattices

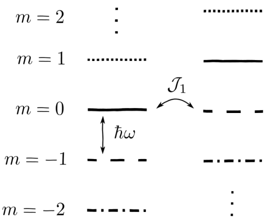
- hopping may become complex (if certain time-reflection symmetries are broken)

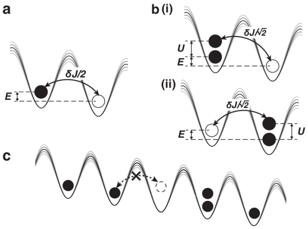
$$J_{eff} \to J\left|F\left(\frac{V_0}{\omega}\right)\right| \mathrm{e}^{i\Phi\left(\frac{V_0}{\omega}\right)}$$

Measurement of the quasi-momentum distribution shows shift $\sim \Phi$

$$n(k) = \sum_{ij} \mathrm{e}^{k(i-j)} \langle b_i^\dagger b_j \rangle$$

# Resonances




Ma,... Greiner, PRL 2011

- Consider a tilted lattice with an energy difference of $\Delta$ between sites
- Hopping is strongly suppressed due to energy conservation
- Modulation $\hbar\omega \approx \Delta$ provides a photon to tunnel the atom

Examples:

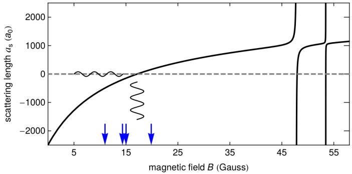
- Generation of synthetic magnetism in tilted optical lattices

*Aidelsburger,... Bloch, PRL 2013*
*Miyake, ... Ketterle, PRL 2013*

- Modulation spectroscopy of excitations on a Mott-insulator

*Ma... Greiner PRL 2011*

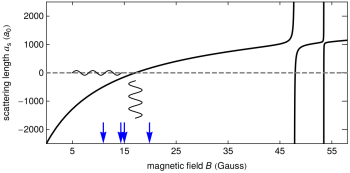
# Engineering anyon models by modulated interactions




Meinert, ... Nägerl, PRL 2016

# Modulated interactions

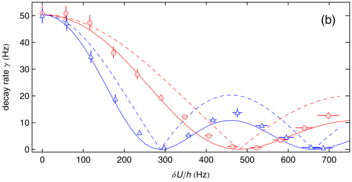
- Modulate magnetic field $B(t)$ close to a Feshbach-resonance in order to create modulated interactions




$$H = -J \sum b_j^\dagger b_{j'} + \mathrm{H.c.} + \frac{U(t)}{2} \sum n_j(n_j - 1)$$

The effective Hamiltonian now has a density dependent hopping, e.g for $U(t) = U_0 + U_1 \cos(\omega t)$

$$H_{eff} = -J \sum_j b_j^\dagger \mathcal{J}_0\left(\frac{U_1}{\omega}(n_j - n_{j'})\right) \mathrm{e}^{i\Phi(n_j - n_{j'})} b_{j'} + \mathrm{H.c.}$$




Meinert, ... Nägerl, PRL 2016

- Decay/Creation of doublons $|11\rangle \to |20\rangle$ and $|11\rangle \to |02\rangle$ with amplitude $\mathcal{J}_0\left(\frac{U_1}{\omega}\right) = \mathcal{J}_0\left(-\frac{U_1}{\omega}\right)$
- measurement of decay of single occupancy after quench from MI regime

# Complex density dependent hoppings

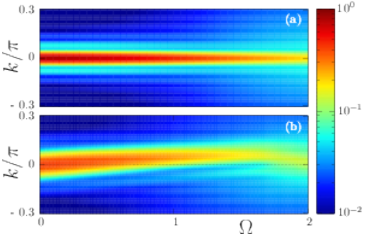
- What about a shaking scheme such which create as complex hopping?!

$$\hat{H}_{eff} = -J \sum b_j^\dagger e^{i\phi(n_j - n_{j+1})} b_{j+1} + \mathrm{H.c.}$$

- Effect of phase can be gauged out, with $\hat{H} = -J \sum \tilde{b}_j^\dagger \tilde{b}_{j+1} + \mathrm{H.c.}$

$$\tilde{b}_j^\dagger \to b_j^\dagger e^{i\phi n_j}$$

where $\tilde{b}_j$ is still a boson, hence spectrum unchanged

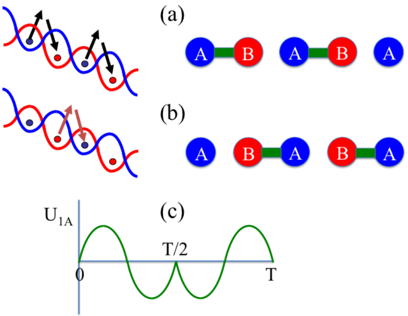


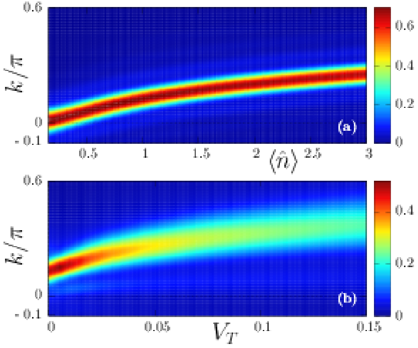

- **Momentum distribution** shows presence of a phase: blurring due to density fluctuations
- Density gradient leads to shift!

Need to break the space inversion symmetry (while keeping the system homogeneous)!

# Density dependent Peierls phases: AB-model

$$H_{eff} = \frac{-J}{2} \sum_x a^\dagger_{2x} e^{i\Phi n^a_{2x}} b_{2x+1} + b^\dagger_{2x+1} e^{i\Phi n^a_{2x+2}} a_{2x+2} + h.c.$$




Density depended drift in momentum space of ground state...

*Quite complicated scheme (2 species, Raman coupling, interaction modulation)...*

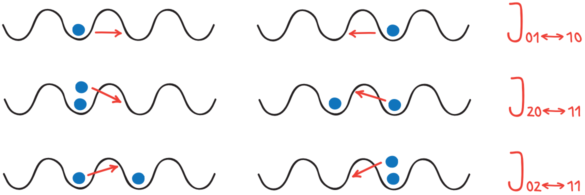
# Doubly-shaken Hamiltonians

Both periodically modulated position of the lattice and short-range interactions:

$$\hat{H}(t) = -J\sum_{\langle i,j\rangle} \hat{b}_i^\dagger \hat{b}_j + \underbrace{\frac{U(t)}{2}\sum_i \hat{n}_i(\hat{n}_i - 1)}_{\text{interaction shaking}} + \underbrace{F(t)\sum_i i\,\hat{n}_i}_{\text{lattice shaking}}$$

Consider driving $U(t) = U_1\cos(\omega t) + U_0$ and $F(t) = F_1\cos(\omega t)$.

$$\hat{H}_{eff} = -J\sum_{\langle i,j\rangle} \hat{b}_i^\dagger \mathcal{J}_0\left[\frac{F_1}{\omega}(i-j) + \frac{U_1}{\omega}(\hat{n}_i - \hat{n}_j)\right]\hat{b}_j + \frac{U_0}{2}\sum_i \hat{n}_i(\hat{n}_i - 1)$$




*Direction and density dependent effective tunneling (can be complex)!*

# Cheng Chin Group

PHYSICAL REVIEW LETTERS **121**, 030402 (2018)

## Observation of Density-Dependent Gauge Fields in a Bose-Einstein Condensate Based on Micromotion Control in a Shaken Two-Dimensional Lattice

Logan W. Clark,[1,2,*] Brandon M. Anderson,[1] Lei Feng,[1,2] Anita Gaj,[1,2] K. Levin,[1] and Cheng Chin[1,2]
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(Received 13 February 2018; revised manuscript received 27 May 2018; published 18 July 2018)

We demonstrate a density-dependent gauge field, induced by atomic interactions, for quantum gases. The gauge field results from the synchronous coupling between the interactions and micromotion of the atoms in a modulated two-dimensional optical lattice. As a first step, we show that a coherent shaking of the lattice in two directions can couple the momentum and interactions of atoms and break the fourfold symmetry of the lattice. We then create a full interaction-induced gauge field by modulating the interaction strength in synchrony with the lattice shaking. When a condensate is loaded into this shaken lattice, the gauge field acts to preferentially prepare the system in different quasimomentum ground states depending on the modulation phase. We envision that these interaction-induced fields, created by fine control of micromotion, will provide a stepping stone to model new quantum phenomena within and beyond condensed matter physics.

DOI: 10.1103/PhysRevLett.121.030402

Synthesizing gauge fields for cold atoms opens the door to investigate novel quantum phenomena associated with

atom systems [30], enabling exciting developments including topological bands [31–33]. In our recent work, lattice

# Lattice shaking with s-p orbitals

- Near resonant modulation
  $\omega = \epsilon_p - \epsilon_s + \delta$
- Two-Band model

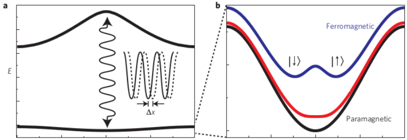
$$H = \begin{pmatrix} \epsilon_s(\mathbf{q}) & 0 \\ 0 & \epsilon_p(\mathbf{q}) \end{pmatrix} - \frac{\omega}{2}\sigma_z + \frac{\Omega}{2}\sigma_x$$

- eigenenergies for coupling

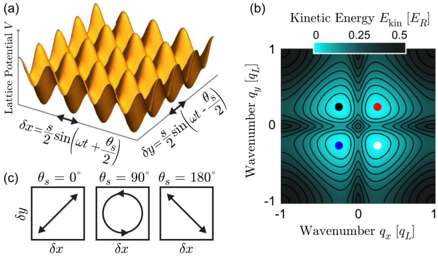
$$A_k/2 \pm \sqrt{(\epsilon_p - \epsilon_s - \omega)^2/4 + \Omega^2}$$

- degenerate for critical shaking amplitude $\Delta$

Lab frame wave-function
$\sim \Psi_q^{(s)}(x) + \alpha e^{i(\omega t + \phi_x)} \Psi_q^{(p)}(x)$




Parker, ... Chin, *Nat. Phys. 2013*




Clark, ... Chin, *PRL 2018*

# Density dependent gauge fields with s-p-bands

- 1D tight binding picture, Wannier functions

$$w_j(x,t) = \psi^{(s)}(x-dj) + \varepsilon \mathrm{e}^{\mathrm{i}(\omega t - \phi_s)} \psi^{(p)}(x-dj)$$

- effective time independent Hamiltonian

$$H_{eff} = -J \sum b_j^\dagger b_{j'} + \mathrm{H.c.} + \frac{1}{2} \sum U_{jlmn} b_j^\dagger b_l^\dagger b_m b_n$$

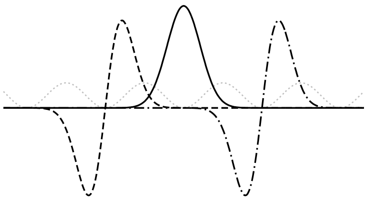
with

$$U_{jlmn} = \frac{1}{T} \int_0^T g(t) \int dx\, w_j^*(x,t)\, w_l^*(x,t)\, w_m(x,t)\, w_n(x,t)$$

For a shallow lattice we may have to include

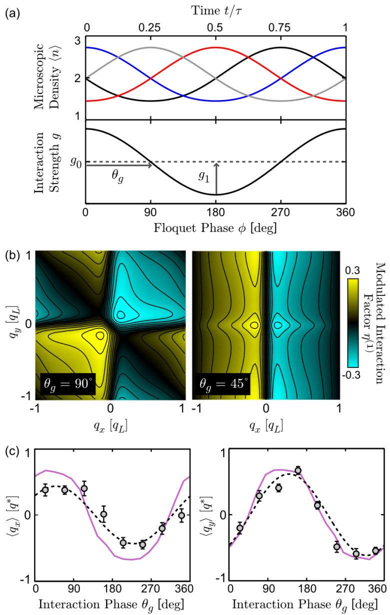
$$U_{jjj,j+1} \sim \frac{\varepsilon}{T} \int_0^T dt \mathrm{e}^{\mathrm{i}(\omega t - \phi_s)} g(t) \times$$

$$\times \int dx \psi_{x+d}^{(p)*} \psi_x^{(s)*} \psi_x^{(s)} \psi_x^{(s)}$$



Hence, $U_{jjj,j+1} = -U_{jjj,j-1}$ and $U_{jjj,j+1} \sim \mathrm{e}^{-\mathrm{i}(\phi_s - \phi_g)}$

# Density dependent gauge fields with s-p-bands




- So with $U_1 \equiv U_{jjj,j+1}$ one finds an effective hopping

$$H_{eff} = -J \sum b_j^\dagger \tilde{J}_{j,j'} b_{j'} + \mathrm{H.c.}$$

with

$$\tilde{J}_{j,j'} = 1 - U_1 \hat{n}_j - U_{-1}^* \hat{n}_{j'}$$
$$\sim \mathrm{e}^{-\mathrm{i}\phi_A(\hat{n}_j + \hat{n}_{j'})}$$

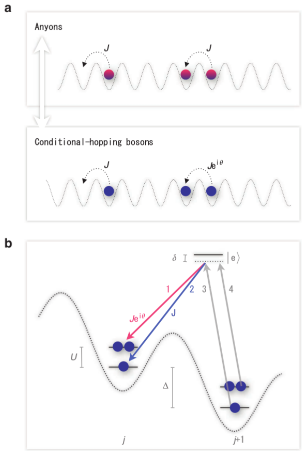
if the anyonic phase $\phi_A$ is small

- In 2D similar argument

$\mathcal{E} \sim \rho\, \mathbf{e}_\Theta \cdot \mathbf{q}$ with unit vector in direction of $\Theta = \theta_g - \theta_s$

- Average momentum depends on $\theta_g$

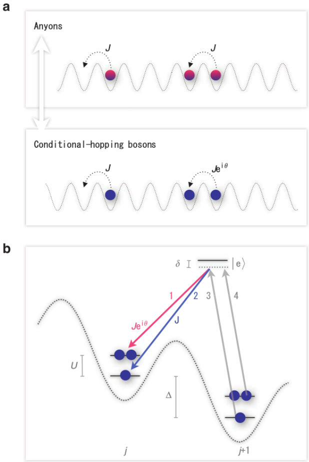
# Assisted Hopping Schemes




Keilmann, ... Roncaglia, Nat. Com. 2011

# Resonances

$$b_{j+1}^{\dagger} e^{i\alpha n_j} b_j + \text{H.c.}$$




Keilmann,... Roncaglia, Nat. Com. 2011

Idea of *T. Keilmann et al.*: Restore in a tilted lattice hopping by a set Raman-lasers $L_i$ imprint correct phases

| | process | $\Delta E_i$ | ME |
|---|---|---|---|
| i | $(1,0) \rightarrow (0,1)$ | $-\Delta$ | $1$ |
| ii | $(1,1) \rightarrow (0,2)$ | $-\Delta + U$ | $\sqrt{2}$ |
| iii | $(2,0) \rightarrow (1,1)$ | $-\Delta - U$ | $\sqrt{2}e^{i\alpha}$ |
| iv | $(2,1) \rightarrow (1,2)$ | $-\Delta$ | $2e^{i\alpha}$ |
| ... | | | |

For higher fillings further processes could be included.

- frequencies $\omega_i$ of the 4 Raman lasers $L_1$, $L_2$, $L_3$, $L_4$ far detuned ($\delta \sim \text{GHz}$) to avoid losses
- Resonance condition determines coupling:

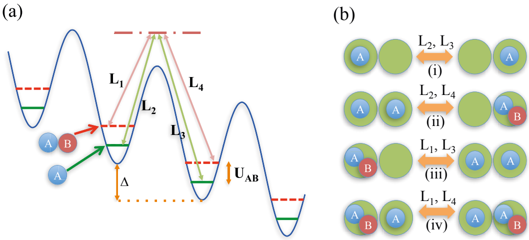
$$\omega_2 - \omega_3 = \Delta E_{i}, \quad \omega_2 - \omega_4 = \Delta E_{ii},$$
$$\omega_1 - \omega_3 = \Delta E_{iii}, \quad \omega_1 - \omega_4 = \Delta E_{iv}$$

# Raman assisted hopping

Problem: $(1,0) \rightarrow (0,1)$ and $(2,1) \rightarrow (1,2)$ are degenerate

Solution:

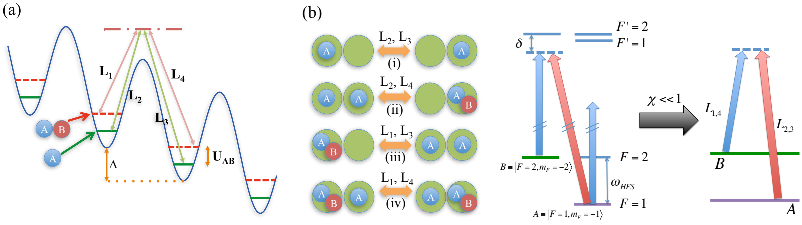
- Restrict to low density limit, where $(2,1) \rightarrow (1,2)$ is less relevant...
- Choose species with different coupling to simulate bosonic model




- 2 internal states $|A\rangle$ and $|B\rangle$
- e.g. for $^{87}$Rb choose $|A\rangle \equiv |F=1, m_F=-1\rangle$ and $|B\rangle \equiv |F=2, m_F=-2\rangle$
- Only want to couple (0), (A) and (AB)

# Raman assisted hopping

SG, and L. Santos. *PRL 115, 053002 (2015)*




- e.g. for [87]Rb choose $|A\rangle \equiv |F=1, m_F=-1\rangle$ and $|B\rangle \equiv |F=2, m_F=-2\rangle$
- $L_{1,4}$ to have linear polarization and $L_{2,3}$ circular $\sigma_-$ polarization
- $|B\rangle$ is just affected by lasers $L_{1,4}$ due to selection rules
- again both $L_{2,3}$ and $L_{1,4}$ couple to $|A\rangle$, the coupling with $L_{1,4}$ can be made much smaller than that of $L_{2,3}$ due to different coupling strengths
- Keep spurious processes off-resonant ($U_{AB}$, $U_{AA}$ should be sufficiently different)
- Realization possible for various bosonic and fermionic species

*Similar model possible by lattice shaking!*

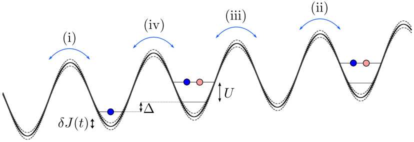
# Outline - Day II:

I. (Floquet) Engineering with cold atomic quantum gases

- Digest of cold atom physics
- Fermions, Bosons, Anyons - Jordan-Wigner-Transformation in 1D and 2D
- Floquet-Engineering
- Modulated Interactions and Experiments in Chicago
- Raman Assisted Hopping

**Today:**

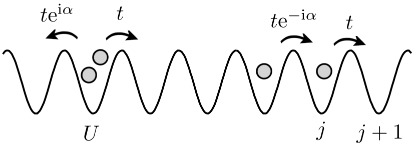
- Anyon hubbard model by lattice shaking




II. Properties

- Digest of 1D physics
- Anyon “Interferometer” on a ring
- Simple ladder model of braiding anyons
- 1D (Pseudo) Anyon Hubbard model

# 1D (pseudo) anyons - Correlated Tunneling




- Assume bosons on-site
- anyonic/deformed exchange statistics $a_j a_k^\dagger - \mathcal{F}_{j,k} a_k^\dagger a_j = \delta_{j,k}$

$$\mathcal{F}_{j,k} := \begin{cases} e^{-\mathrm{i}\alpha}, & j > k, \\ 1, & j = k, \\ e^{\mathrm{i}\alpha}, & j < k, \end{cases}$$

- **Correlated/Density dependent hopping** model for bosons

$$\hat{H} = -t \sum_j \left( b_j^\dagger \mathrm{e}^{-\mathrm{i}\alpha n_j} b_{j+1} + \mathrm{H.c.} \right)$$

<u>Experimental ideas:</u>

- Modulated interactions

  *SG, G.Sun, D.Poletti, and L.Santos. PRL 113, 215303 (2014)*
  Clark... Cheng *PRL 121, 030402 (2018)*

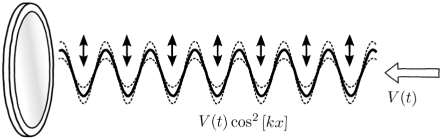
- Assisted tunneling

  T. Keilmann, S. Lanzmich, L. McCulloch, and M. Roncaglia. *Nature Comm.* **2**, *361 (2011)*
  SG, and L. Santos. *PRL 115, 053002 (2015)*
  C. Sträter, S. C. L. Srivastava, and A. Eckardt. *PRL 117(20), 205303 (2016)*
  L. Cardarelli, SG, and L. Santos. *PRA, 94(2), 023615 (2016)*

- ...

# Lattice depth modulation

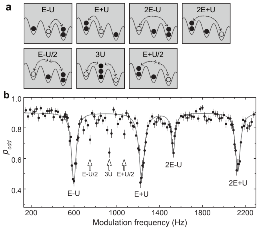
Periodic modulation of tunneling element
(neglect effect on interactions)




$$\hat{H} = (J + \delta J(t))\,\hat{H}_{tun} + \hat{H}_{int}$$

Sinusoidal modulation $\delta J(t) = \delta J \sin(\omega t + \phi)$

$$\langle\!\langle \{n'_j\}, m' | \hat{H}(t) - i\hbar\partial_t | \{n_j\}, m \rangle\!\rangle = \delta_{m,m'}\left[\langle \{n'_j\} | J\hat{H}_{tun} + \hat{H}_{int} | \{n_j\} | \rangle + \hbar\omega m\right] + \\ + \delta_{m',m+1}\mathrm{i}\frac{\delta J}{2}\mathrm{e}^{\mathrm{i}\phi}\langle \{n'_j\} | \hat{H}_{tun} | \{n_j\} | \rangle + \\ - \delta_{m',m-1}\mathrm{i}\frac{\delta J}{2}\mathrm{e}^{-\mathrm{i}\phi}\langle \{n'_j\} | \hat{H}_{tun} | \{n_j\} | \rangle \,.$$




Ma,... Greiner, PRL 2011

# Higher order corrections

- One may systematically include higher order corrections (Magnus expansion, perturbatively coupling higher Floquet-sectors...)

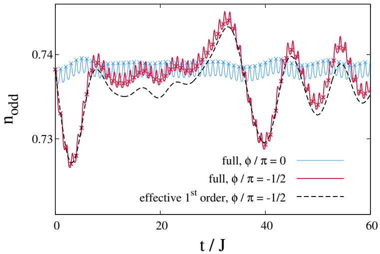
$$H_{eff} = \frac{1}{T}\int_0^T dt H(t) \; + \; \frac{-\mathrm{i}}{2T}\int_0^T dt_2 \int_0^{t_2} dt_1 \left[H(t_2), H(t_1)\right] + \mathcal{O}\left(\frac{1}{\omega^2}\right)$$

- convergence?
- expanding as a Fourier series: $\hat{H}(t) = \hat{H}_0 + \sum V^{(k)} \mathrm{e}^{ik\omega t}$

$$H_{eff} = \hat{H}_0 \; + \; \frac{1}{\omega}\sum_k \frac{1}{k}\left(\left[V^k, V^{-k}\right] - \left[V^k, \hat{H}_0\right] + \left[V^{-k}, \hat{H}_0\right]\right) + \cdots$$

- illustrate the influence of these corrections for the case of lattice depths modulation

$$\hat{H}^{(1)}_{eff} = \frac{\mathrm{i}U\delta J}{\omega} \sin\phi \sum_i b_i^\dagger b_{i+1} - n_i b_i^\dagger b_{i+1} + H.c.$$

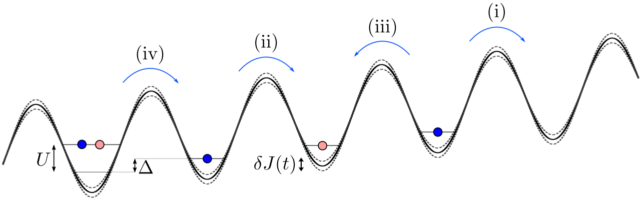


# Anyons by lattice modulation

# Three-color modulation on tilted lattice

- L. Cardarelli, SG, and L. Santos. *PRA, 94(2), 023615 (2016)*
- C. Sträter, S. C. L. Srivastava, and A. Eckardt, PRL 117, 205303 (2016)

$$\hat{H}(t) = -\left(J_0 + \delta J(t)\right) \sum_{j,\sigma} \left[c^\dagger_{j+1,\sigma} c_{j,\sigma} + \mathrm{H.c.}\right] + U\hat{H}_{\mathrm{int}} + \Delta \hat{H}_{\mathrm{tilt}},$$




| | process | $\Delta E_s$ | ME |
|---|---|---|---|
| i | $(1,0) \to (0,1)$ | $\Delta$ | $\delta J_i \mathrm{e}^{-\mathrm{i}\phi_i}$ |
| ii | $(1,1) \to (0,2)$ | $\Delta + U$ | $\delta J_{ii} \mathrm{e}^{-\mathrm{i}\phi_{ii}}$ |
| iii | $(2,0) \to (1,1)$ | $\Delta - U$ | $\delta J_{iii} \mathrm{e}^{-\mathrm{i}\phi_{iii}}$ |
| iv | $(2,1) \to (1,2)$ | $\Delta$ | $\delta J_i \mathrm{e}^{-\mathrm{i}\phi_i}$ |

$$\delta J(t) = \sum_s \delta J_s \cos(\omega_s t + \phi_s)$$

and $\omega_s = \Delta E_s$

Idea: Use bichromatic lattice to resolve iv and i!

# Three-color modulation on tilted lattice

$$\delta J(t) = \delta J_1 \left[\cos(\omega_1 t) + \beta \cos(\omega_2 t + \phi) + \beta \cos(\omega_3 t + \phi)\right]$$

- keeping only resonant terms

$$\hat{H}_{\text{eff}} = -\frac{\delta J_1}{2} \sum_{j,\sigma} \underbrace{c^{\dagger}_{j+1,\sigma} F\left[\left|n_{\mathrm{J}+1,\bar{\sigma}} - n_{\bar{\sigma},j}\right|\right] c_{j,\sigma}}_{c^{\dagger}_{j+1,\sigma} \mathrm{e}^{\mathrm{i}\phi|n_{\bar{\sigma},j+1} - n_{\bar{\sigma},j}|} c_{j,\sigma}} + \tilde{U}\hat{H}_{\text{int}} + \hat{H}^{\text{2nd}}_{\text{NN}},$$

where $F[0] = 1$, and $F[1] = \beta e^{i\phi}$

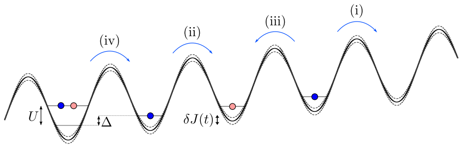
- **two component anyons** (neglecting process (iv))

$$\hat{H}_{\text{AHM}} = -\frac{\delta J_1}{2} \sum_{j,\sigma} \left( f^{\dagger}_{j,\sigma} f_{j+1,\sigma} + \text{H.c.} \right) + \tilde{U}\hat{H}_{\text{int}}.$$

deformed exchange statistics $f_{j,\sigma} f^{\dagger}_{k,\sigma'} + \mathcal{F}_{j,k} f^{\dagger}_{k,\sigma'} f_{j,\sigma} = \delta_{j,k}\delta_{\sigma,\sigma'}$

$$\mathcal{F}_{j,k} := \begin{cases} e^{-\mathrm{i}2\phi}, & j > k, \\ 1, & j = k, \\ e^{\mathrm{i}2\phi}, & j < k, \end{cases}$$

# Effective interactions



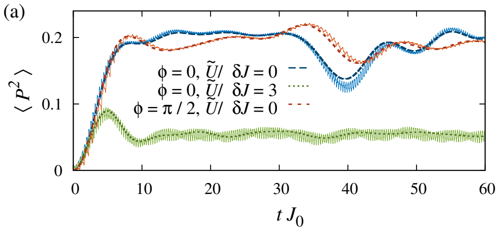

- detune slightly from resonance to create effective on-site interaction-energy $\tilde{U} \ll U$ for doublon creation

$$\omega_i = \Delta\,, \quad \omega_{ii} = \Delta + U - \tilde{U}\,, \quad \omega_{iii} = -\Delta + U - \tilde{U}$$

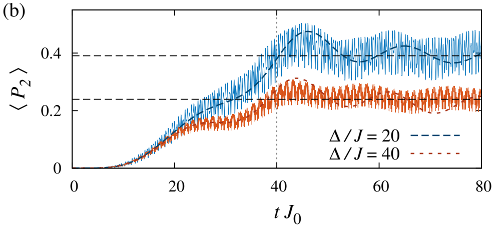
- virtual hoppings create effective nearest neighbor interactions (same derivation via Magnus expansion etc.)

$$\begin{aligned} \hat{H}_{\mathrm{NN}} &= \sum_{\langle i,j\rangle}\Bigg[\frac{2J_0^2}{\Delta+U}P_i^0P_j^2 - \frac{2J_0^2}{\Delta-U}P_i^2P_j^0 \\ &+ \frac{J_0^2}{\Delta}\Big((1-n_i)P_j^1 - P_i^1(1-n_j)\Big) \\ &+ \frac{2UJ_0^2}{\Delta^2-U^2}\left(P_i^{1\uparrow}P_j^{1\downarrow}+P_i^{1\downarrow}P_j^{1\uparrow}-S_i^+S_j^- - S_i^-S_j^+\right)\Bigg] \end{aligned}$$

# Effective model time dependence



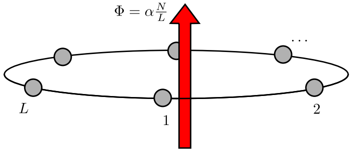

Evolution after sudden quench ($k_B T = J_0$)




Quasi-adiabatic preparation for different tilting $\Delta$

Time-evolution of the average double occupancy of full Hamiltonian and effective model neglecting $\mathcal{O}\left(\delta J\right)$-terms

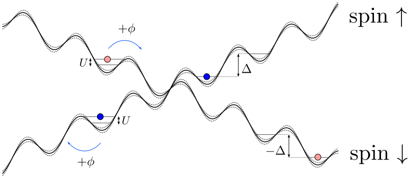
# Anyons on a Ring



# Three-color modulation on a spin dependent tilted lattice

| | spins | $\Delta E$ | ME |
|---|---|---|---|
| × ×<br>○→× | $(\downarrow, 0) \to (0, \downarrow)$ | $-\Delta$ | 1 |
| ○ ×<br>○→× | $(2, 0) \to (\uparrow, \downarrow)$ | $-\Delta - U$ | $\mathrm{e}^{\mathrm{i}\phi}$ |
| × ○<br>○→× | $(\downarrow, \uparrow) \to (0, 2)$ | $-\Delta + U$ | $\mathrm{e}^{\mathrm{i}\phi}$ |
| ○ ○<br>○→× | $(2, \uparrow) \to (\uparrow, 2)$ | $-\Delta$ | 1 |
| ○→×<br>○ × | $(2, 0) \to (\downarrow, \uparrow)$ | $\Delta - U$ | $\mathrm{e}^{-\mathrm{i}\phi}$ |
| ○→×<br>× ○ | $(\uparrow, \downarrow) \to (0, 2)$ | $\Delta + U$ | $\mathrm{e}^{-\mathrm{i}\phi}$ |
| … | … | | |

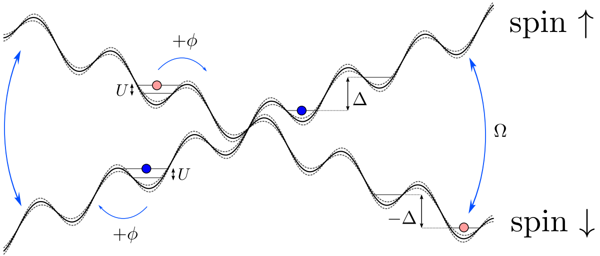
+ Hermitian conjugate (← processes)




$$\hat{H}(t) = -J(t) \sum_{j,\sigma} \left[ c^\dagger_{j+1,\sigma} c_{j,\sigma} + \mathrm{H.c.} \right] + U \hat{H}_{\mathrm{int}} + \Delta \hat{H}_{\mathrm{tilt}},$$

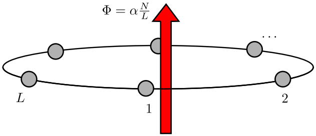
$$J(t) = J_0 + \delta J_1 \left[ \cos(\omega_1 t) + \cos(\omega_2 t - \phi) + \cos(\omega_3 t + \phi) \right]$$

# Hardcore anyons on a ring




Additional microwave fields $\Omega$ couple the boundaries of the system

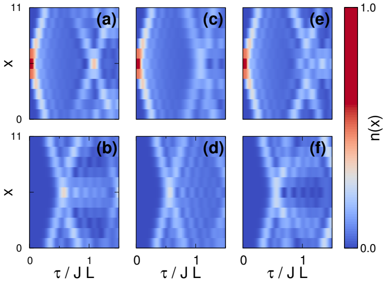
$$H = -\frac{\delta J}{2} \sum_{j,\sigma} c^{\dagger}_{j,\sigma} \mathrm{e}^{\mathrm{i}\sigma\phi |n_{j+1,\bar{\sigma}} - n_{j,\bar{\sigma}}|} c_{i+1,\sigma} + \mathrm{H.c.} - \Omega \left( c^{\dagger}_{0,1} c_{0,0} + c^{\dagger}_{L,1} c_{L,0} + \mathrm{H.c.} \right)$$



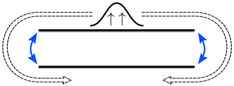

- This can be rewritten as hardcore-anyon model on a ring!

$$H = \sum_{i=0\cdots 2L} \alpha^{\dagger}_{i} \alpha_{i+1} + \alpha^{\dagger}_{L} \alpha_{0} + \mathrm{H.c.}$$

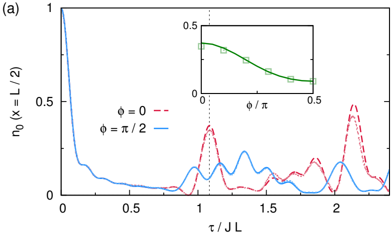
# "Interferometer“



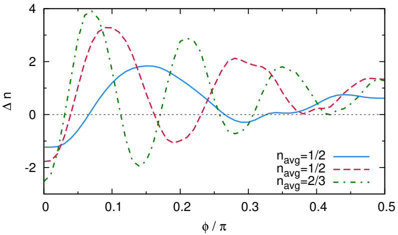

- Expansion of an initially prepared cloud of particles



- Density imbalance after evolution as witness of anyonic exchange relations



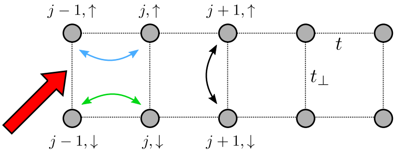

Fixed particle number

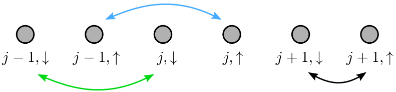



Density average

# Braiding Anyons on a Ladder

# Jordan-Wigner-Transformation on a ladder




- Choose some order of lattice sites

$$\hat{H} = t \sum_{j,\sigma=0,1} a^\dagger_{j,\sigma} a_{j+1,\sigma} + \mathrm{H.c.} + \\ + t_\perp \sum_j a^\dagger_{j,0} a_{j,1} + \mathrm{H.c.}$$

- particle exchange overcrossing: $\mathrm{e}^{-\mathrm{i}\alpha}$ - Undercrossing $\mathrm{e}^{+\mathrm{i}\alpha}$

$$\hat{H} = t \sum_{j,\sigma=0,1} b^\dagger_{j,\sigma} \mathrm{e}^{\mathrm{i}\sigma\alpha(n_{j,\sigma}+n_{j+1,\sigma})} b_{j+1,\sigma} + \mathrm{H.c.} + t_\perp \sum_j b^\dagger_{j,0} b_{j,1} + \mathrm{H.c.}$$

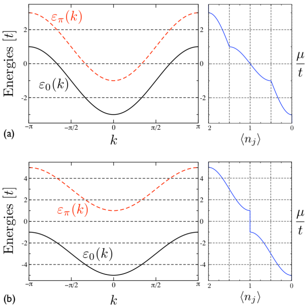
*Smooth connection between two well known limits!*

# Fermions on a Ladder ($\alpha \to \pi$)

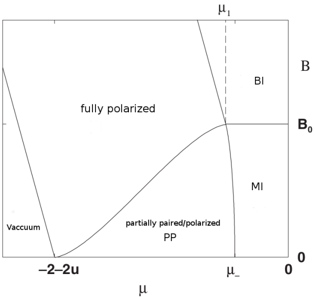
$$H = -t \sum_{j,\sigma} c^{\dagger}_{j,\sigma} c_{j+1,\sigma} + H.c. \; - \; t_{\perp} \sum_{j} c^{\dagger}_{j,0} c_{j,1} + H.c.$$

- Rotate basis to diagonalize $t_{\perp}$ part

$$\tilde{c}_{j,\uparrow} = \frac{1}{\sqrt{2}} \left( \tilde{c}_{j,\uparrow} + \tilde{c}_{j,\downarrow} \right) \;, \quad \tilde{c}_{j,\downarrow} = \frac{1}{\sqrt{2}} \left( \tilde{c}_{j,\uparrow} - \tilde{c}_{j,\downarrow} \right)$$




Crépin, Laflorencie, Roux, Simon, PRB 2011




Essler, Frahm, Göhmann, Klümper, Korepin. Cambridge 2005

$$H \to -t \underbrace{\sum_{j,\sigma} \tilde{c}^{\dagger}_{j,\sigma} \tilde{c}_{j+1,\sigma} + H.c.}_{\sum_k 2\cos(k) \tilde{c}^{\dagger}_{k,\sigma} \tilde{c}_{k,\sigma}} - t_{\perp} \sum_{j} \underbrace{\left( \tilde{n}_{j,\uparrow} - \tilde{n}_{j,\downarrow} \right)}_{S^z_j}$$

- Two (decoupled) chains with magnetic field $t_{\perp}$
- liquid phases with different number of Fermi-points $c = 1$ and $c = 2$, Band-insulator state

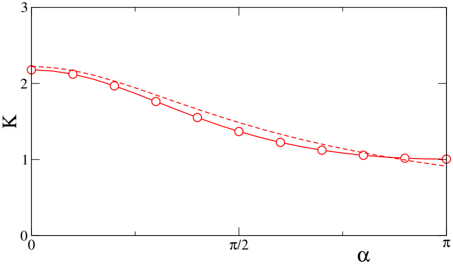
# Bosonizing the anyons - weak coupling

- Start with **one leg** for the moment (same treatment for pseudo-anyons, on-site boson)

$$H = -t \sum_j b_j^\dagger \mathrm{e}^{\mathrm{i}\alpha n_j} b_{j+1} + \mathrm{H.c.} + \frac{U}{2} \sum_j n_j (n_j - 1)$$

- weak coupling $U, \alpha \ll 1$ (where $\gamma_{1,2,3} \sim \rho\alpha$)

$$\sim \frac{v}{2\pi} \int dx \left[ \frac{\pi^2}{K} (\partial_x \phi)^2 + K (\partial_x \theta)^2 \right] + \underbrace{\gamma_1 (\partial_x \theta)} + \gamma_2 (\partial_x \phi) + \gamma_3 (\partial_x \phi)(\partial_x \theta) + \cdots$$




- absorb $\gamma_1, \gamma_2$ in redefinition $\theta$ and $\phi$

$$K^2 = \frac{\pi^2}{\alpha^2 + \frac{U}{2\rho t}}$$

- $\gamma_3$ can be eliminated when calculating correlations e.g. $\langle b_i^\dagger b_j \rangle \sim |i - j|^{1/2K}$ and $\langle n_i n_j \rangle \sim \rho^2 - \frac{K}{2\pi^2 |i-j|^2}$
- extract $K$ numerically and compare to wk

# (Handwaving) Bosonization of the ladder

- two copies for the ladder

$$\hat{H} = -t \sum_{j,\sigma=0,1} b^\dagger_{j,\sigma} \mathrm{e}^{\mathrm{i}\sigma\alpha(n_{j,\sigma}+n_{j+1,\sigma})} b_{j+1,\sigma} + \mathrm{H.c.} - t_\perp \sum_j b^\dagger_{j,0} b_{j,1} + \mathrm{H.c.} + U \sum_j n_{j,\uparrow} n_{j,\downarrow}$$

- Introduce anti-symmetric and symmetric combinations (spin and charge) $\phi_{A/S} = (\phi_\uparrow \pm \phi_\downarrow)/\sqrt{2}$ and $\theta_{A/S} = (\theta_\uparrow \pm \theta_\downarrow)/\sqrt{2}$

$$\hat{H} \rightarrow H^0_A + H^0_S - \frac{4t_\perp}{2\pi} \int dx \cos\left[\sqrt{2}\theta_A(x)\right] + \frac{2U}{(2\pi)^2} \int dx \cos\left[\sqrt{8}\phi_A(x)\right]$$

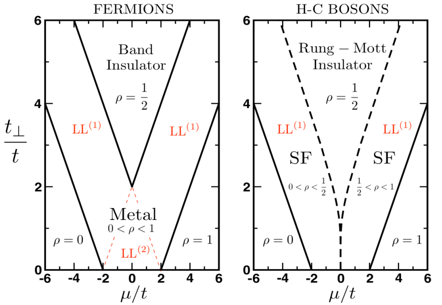
- (density dependent) magnetic field acts like a chemical potential for $\theta_A$

$$H^0_A \rightarrow \frac{v_A}{2\pi} \int dx \left[ \frac{\pi^2}{K_A} (\partial_x \phi_A)^2 + K_A (\partial_x \theta_\sigma - \alpha^2 \rho)^2 \right]$$

- rung-coupling

$$-\frac{4t_\perp}{2\pi} \int dx \cos\left[\sqrt{2}\theta_A(x)\right]$$

# Bosons on a Ladder




Crépin, Laflorencie, Roux, Simon, PRB 2011

$$\hat{H} \rightarrow H_A^0 + H_S^0 + \underbrace{+ \frac{4t_\perp}{2\pi} \int dx \cos\left[\frac{\sqrt{2}\alpha}{\pi}\phi_A(x)\right] \cos\left[\sqrt{2}\theta_A(x)\right]}_{\text{easy for bosons, difficult for fermions!?}}$$

- Sine-Gordon Hamiltonian ($\alpha = 0$) for A-fields

$$H_A^0 - \frac{4t_\perp}{2\pi} \int dx \cos\left[\sqrt{2}\theta_A(x)\right]$$

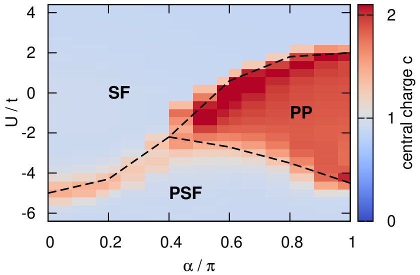
- Sine-Gordon-term opens a gap (for $K_A > 1$) - hence, there is no $c = 2$ phase for the Bose-ladder
- For incommensurate fillings S(charge)-field remains free
- neglecting terms relevant at commensurate densities, etc.
- Mott-insulator gap opens for any $t_\perp > 0$ at half filling

# Anyons on a Ladder - incommensurate filling

- density dependent magnetic field induces commensurate-incommensurate transition for $\theta_A$: For critical $\alpha \cdot \rho$ gap will open an system enters 2 component LL phase

$$H_A^0 \to \frac{v_A}{2\pi} \int dx \left[ \frac{\pi^2}{K_A} (\partial_x \phi_A)^2 + K_A (\partial_x \theta_A - \alpha^2 \rho)^2 \right]$$

- same mechanism as for chemical potential or Meissner-Vortex-phase transitions

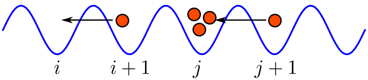



- Competition between sine-Gordon terms

$$-\frac{4t_\perp}{2\pi} \int dx \cos\left[\sqrt{2}\theta_A(x)\right] + \frac{2U}{(2\pi)^2} \int dx \cos\left[\sqrt{8}\phi_A(x)\right]$$

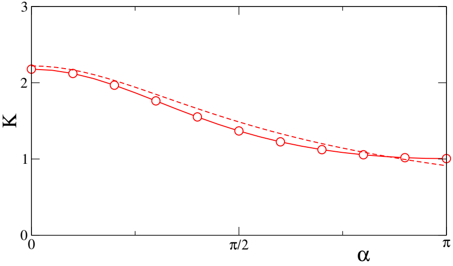
- Double sine-Gordon allows for Ising-type transition between SF and PSF phase

# 1D Anyons



# Pseudo-anyon Hubbard model

$$H = -t \sum_j b_j^\dagger \mathrm{e}^{\mathrm{i}\alpha n_j} b_{j+1} + \mathrm{H.c.} + \frac{U}{2} \sum_j n_j(n_j - 1)$$




- weak coupling $K^2 = \frac{\pi^2}{\alpha^2 + \frac{U}{2\rho t}}$
- anyonic statistics $\alpha$ has similar effect as repulsive interaction
- $K \to 1$ (free hardcore(!) fermions) for $\alpha = \pi$ ?!

*What happens for "Pseudo-fermions"?*

# Naive mean field approach to 1D AHM

Want to understand bosonic Pseudo-anyon Hubbard model

$$H = -t \sum_j \underbrace{b_j^\dagger e^{i\theta n_j} b_{j+1}}_{\equiv c_j^\dagger b_{j+1}} + \frac{U}{2} \sum_j n_j \left( n_j - 1 \right)$$

- Mean field approach may give some intuition into observable phenomena (in 1D probably terribly wrong)
- Idea from Keilmann et al. $b_j^\dagger e^{i\theta n_j} b_{j+1} = c_j^\dagger b_{j+1}$ is decoupled as two fields $\Psi_{1,j}$, $\Psi_{2,j}$

$$c_j^\dagger b_{j+1} \approx \Psi_{2,j}^* b_{j+1} + c_j^\dagger \Psi_{1,j+1} - \Psi_{2,j}^* \Psi_{1,j+1} \ ,$$

- solution has to be found self consistently

$$\Psi_{1,j}^* = \langle b_j \rangle \ , \quad \Psi_{2,j} = \langle c_j \rangle = \langle b_j^\dagger e^{i\theta n_j} \rangle$$

- Solutions may be coupled and depend on each other
- The solution minimizes the energy functional $E\left( \Psi_1, \Psi_2 \right)$
- Here: focus on homogeneous solution $\Psi_{1,j} = \Psi_1$ and $\Psi_{2,j} = \Psi_2$
- Better solution, should include incommensurabilities (Tang, Eggert, Pelster, New J. Phys 17, 123016 (2015))

# Mean Field Pseudo-Fermions: self-consistant solutions

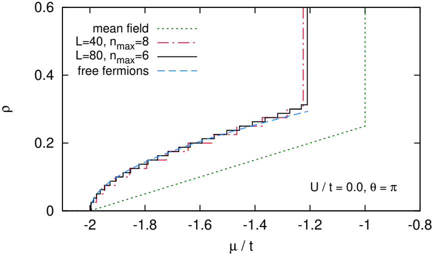
$$H = -zt\big(\Psi_2 b^\dagger + \Psi_2^* b + \Psi_1 c^\dagger + \Psi_1^* c - \Psi_1^*\Psi_2 - \Psi_2^*\Psi_1\big) + \frac{U}{2}n(n-1) - \mu n$$

- For $b^\dagger(-1)^n b$ ($\theta \to \pi$) one finds a self-consistent solution with $\Psi_1 = \Psi_2 \equiv \Psi$
- The MF-Hamiltonian becomes block diagonal, decoupling into blocks of $\{n, n+1\}$, e.g. $\{0, 1\}$, $\{2, 3\}$, etc.

- Hamiltonian in sector of 0 and 1 particles

$$H_{0,1} = 2t\begin{pmatrix} |\Psi|^2 & -\Psi \\ -\Psi^* & |\Psi|^2 - \mu/2t \end{pmatrix}$$

- solution $\Psi_{0,1} \to \sqrt{1-(\mu/zt)^2}/2$



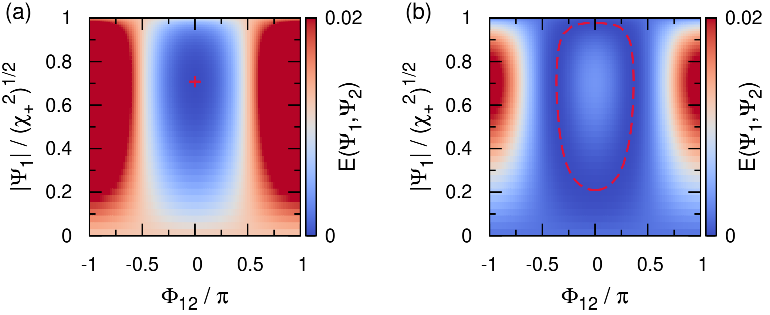

*Effective Pauli principle*

# Mean Field: second type of solutions

- second class of solutions $\Psi_1 \neq \Psi_2$

$$\chi_+^2 \equiv |\Psi_1|^2 + |\Psi_2|^2$$
$$\chi_-^2 \equiv \max||\Psi_1|^2 - |\Psi_2|^2|$$

**Partially paired (PP) phase** "hardcore (fermionic)" liquid + superfluid quasi condensate of pairs $\langle b^2 \rangle \sim \chi_-^2$



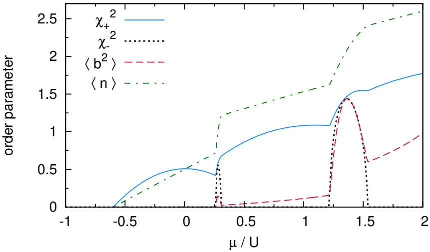
# Mean Field: phase diagram

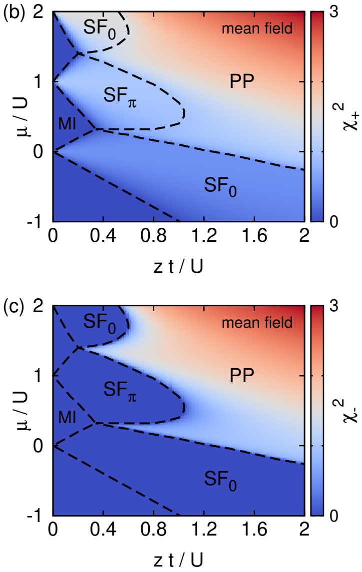
- For SF phase

$$\chi_+^2(n, n+1) = \frac{(zt)^2(1+n)^2 - (\mu - nU)^2}{2(zt)^2(1+n)}$$

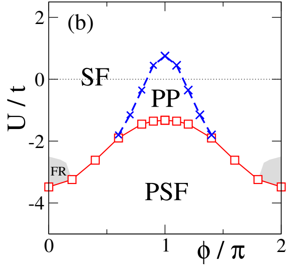
and $\chi_-^2 = 0 \sim \langle b^2 \rangle$

- larger unit cell allows to define phase between different sites: $SF_0$ and $SF_\pi$ phases

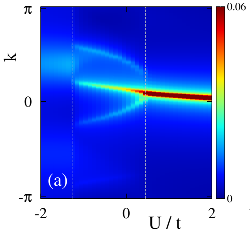


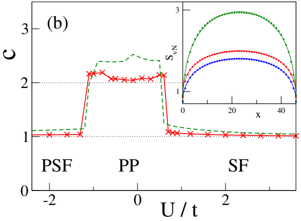



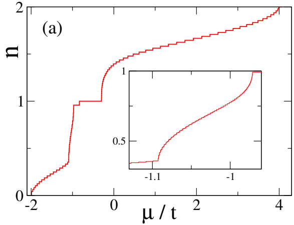
# Anyon Hubbard Model in 1D - DMRG results




- Realization of 3-body hardcore constraint by construction $\left(b^{\dagger}\right)^{3} = 0$
- momentum distribution: shifts due to fluctuations
- two component PP phase realized for $\theta \sim \pi$
- multi-peak momentum distribution



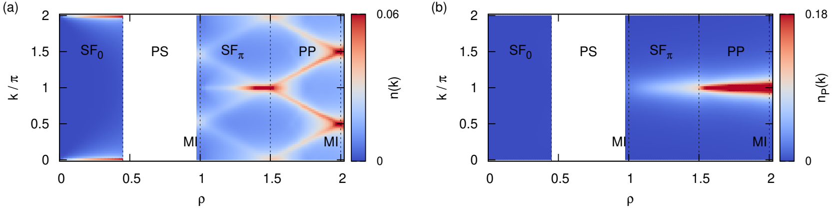




SG, Santos, *PRL 2015*

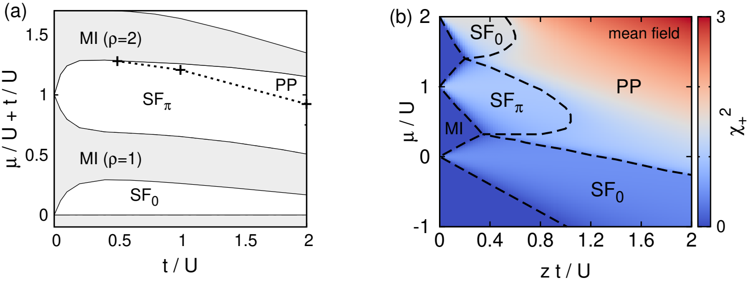
# Partially paired phase for one-component pseudo-fermions

Zhang, SG, Fan, Scott, Zhang, PRA 2017

- (Pair) momentum distribution




- phase diagram and MF



# Dilute limit

- simple analytical description in dilute limit from solution of the 2-particle problem
- A general two-particle state may be described by

$$\left|\Psi_K\right\rangle = \sum_x c_{x,x}\left(b_x^\dagger\right)^2\left|0\right\rangle + \sum_{x,y>x} c_{x,y} b_x^\dagger b_y^\dagger \left|0\right\rangle .$$

- due to the conservation of total momentum $Q = k_1 + k_2$ in the scattering process one may write $c_{x,x+r} = C_r \mathrm{e}^{iQ\left(x+\frac{r}{2}\right)}$
- insert into Schrödinger equation $H\left|\Psi\right\rangle = \Omega\left|\Psi\right\rangle$
- simple system of coupled equations

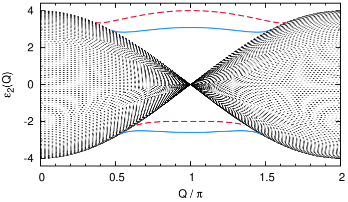
$$r = 0, 1, \quad \begin{cases} (\epsilon_2 - U) C_0 = -\sqrt{2}t\left(\mathrm{e}^{-\mathrm{i}\frac{Q}{2}} + \mathrm{e}^{\mathrm{i}\left(\frac{Q}{2}+\theta\right)}\right) C_1 \\ \epsilon_2 C_1 = -\sqrt{2}t\left(\mathrm{e}^{\mathrm{i}\frac{Q}{2}} + \mathrm{e}^{-\mathrm{i}\left(\frac{Q}{2}+\theta\right)}\right) C_0 + 2t\cos\left(\frac{Q}{2}\right) C_2 \end{cases}$$

$$r \geq 2, \quad \left\{ \epsilon_2 C_r = -2t\cos\left(\frac{Q}{2}\right)\left(C_{r-1} + C_{r+1}\right) \right.$$

- Calculate scattering and bound states
- Two particle solution may offer analytical insight into frustrated systems

Kolezhuk, Heidrich-Meisner, SG, Vekua, PRA 2011

# Scattering solution




- scattering states of two particles energy with total and relative momentum $Q = k_1 + k_2$, $q = \frac{k_1 - k_2}{2}$

$$\epsilon_2 = \epsilon(k_1) + \epsilon(k_2) = -4t\cos(q)\cos\left(\frac{Q}{2}\right)$$

- ansatz $C_r = e^{-\mathrm{i}qr} + e^{2\mathrm{i}\delta} e^{\mathrm{i}qr}$ solves Eqs. for $r \geq 2$
- $C_0$ and $\delta$ determined by $r = 0$ and 1

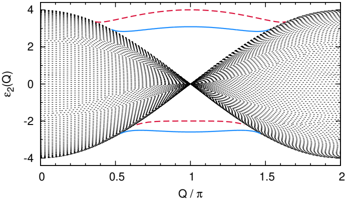
- from the scattering phase shift $\delta$, we can extract the scattering length,

$$a = \frac{t(1 + \cos\theta)}{-2(2t + U) + 4t\cos\theta}$$

- effective interaction strength $g = -2/(am)$
- $\lim_{\theta \to 0, 2\pi, U \to 0} a = \infty$
- $\lim_{\theta \to \pi, U \to 0} a = 0$, the system approaches the Tonks limit $K \to 1$ of a hardcore fermions

*Effective Pauli principle for $\theta \to \pi$*

## Bound states



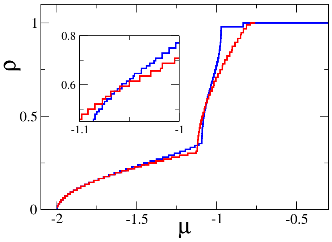

- Ansatz $C_r = \alpha^r$ with $|\alpha| < 1$ (exponentially localized to center of mass)
- for $\theta = 0$ repulsively bound-pairs only for high energies
- for $\theta = \pi$ two different bound-state solutions close to $Q \sim \pi$

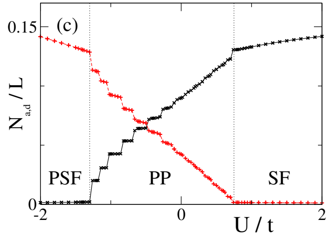
$$\epsilon_{\pm}^{B} = \frac{2U\cos(k) \pm (\cos(k) - 1)\sqrt{U^2 + 8t^2(1 - 3\cos(k))}}{3\cos(k) - 1}$$

- For any $U > 0$ one low energy solution $\epsilon_{+}^{B} < 0$
- For $U < 2t$ it exhibits a local minimum at $Q = \pi$

*In spite of this effective hardcore character of the two particle scattering state, nonetheless low-lying bound states of two particles may exist!*

# PP phase model




- Naive description of the PP-phase via

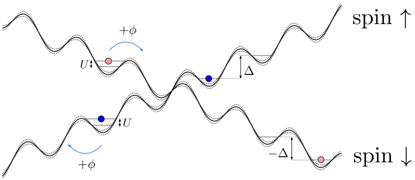
$$H_{eff} = -2t \sum_k \cos(k) a_k^\dagger a_k + \sum_k \epsilon_-^B(k) b_k^\dagger b_k$$

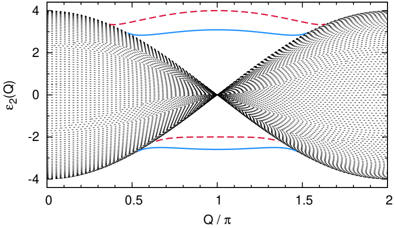
minimized under the constraint $\rho_a + 2\rho_p = \rho$

- at low densities the ground state only contains species $a$; for higher fillings both species are present
- finite size structure of 2 and 1 particle jumps due to constraint
- approximately measure the densities $\rho_a$ and $\rho_d$

$$N_a = \sum_i \left\langle b_i^\dagger b_{i+1} \right\rangle, \quad N_d = \sum_i \left\langle (b_i^\dagger)^2 (b_{i+1})^2 \right\rangle \;,$$

# Summary




- Anyons about to be explored with cold atoms!
- Various techniques: Modulated interactions, lattice shaking, Raman-assisted hopping, ...

characteristic features of 1D anyon lattice model

- shift of quasi-momentum
- effective repulsion
- PP-phase

# Much, much more not mentioned...

- Bethe-Ansatz, integrable systems

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- asymmetric momentum distributions

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# Thank you very much!

**Thanks to collaborators!!**

Thierry Giamarchi (U Geneva), Temo Vekua (Indiana U), Luis Santos (ITP Hannover), Lorenzo Cardarelli (ITP Hannover), Dario Poletti (SUTD Singapore), Wanzhou Zhang (U Shanxi), Yunbo Zhang (U Shanxi)

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