UNIVERSITY OF INNSBRUCK

IQOQI
AUSTRIAN ACADEMY OF SCIENCES

Lecture 2: Majorana Fermions as an example of non-Abelian Anyons

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Anyon Physics of Ultracold Atomic Gases

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Lecture 2:

Majorana fermions as non-Abelian anyons

Majorana fermions in Kitaev wire

Braiding protocol

Demonstration of non-Abelian statistics

Using Majorana fermions for QC

- Deutsch-Jozsa algorithm

Conclusion

Ettore Majorana,
1906-1938?

Majorana “fermions” as non-Abelian anyons

Introducing Majorana “fermions”

For a (complex or Dirac) fermionic operators $\hat{a}$ and $\hat{a}^+$

with algebra $$\{\hat{a},\hat{a}^+\}=1, \{\hat{a},\hat{a}\}=\{\hat{a}^+,\hat{a}^+\}=0$$

two hermitian(!) Majorana operators (Majorana fermions)

$$\gamma_1=\hat{a}+\hat{a}^+=\gamma_1^+$$

$$\gamma_2=(\hat{a}-\hat{a}^+)/i=\gamma_2^+$$

with algebra $$\{\gamma_k,\gamma_l\}=2\delta_{kl}$$ or $$\gamma_1^2=\gamma_2^2=1,$$ $$\gamma_1\gamma_2=-\gamma_2\gamma_1$$

Inverse: $\hat{a}=(\gamma_1+i\gamma_2)/2$ and $\hat{a}^+=(\gamma_1-i\gamma_2)/2$

One fermionic mode $\Longleftrightarrow$ Two Majoranas

Fermionic states and Majorana fusion

States: $\{|0\rangle, |1\rangle\}: \; a|0\rangle = 0, |1\rangle = a^{+}|0\rangle$

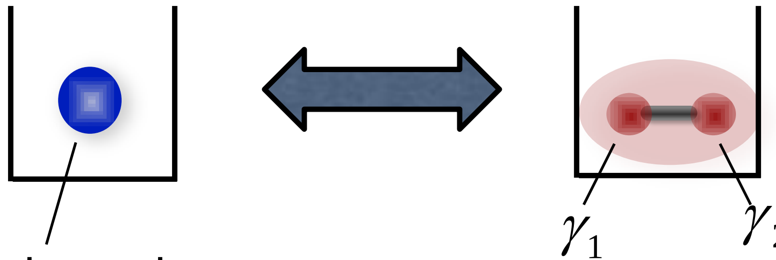
$$\hat{n} = \hat{a}^{+}\hat{a} = \frac{i}{2}\gamma_1\gamma_2 + \frac{1}{2}$$

$$\hat{n}|0\rangle = 0|0\rangle \qquad \qquad -i\gamma_1\gamma_2|0\rangle = |0\rangle, \quad |0\rangle \equiv |+\rangle$$

$$\hat{n}|1\rangle = 1|1\rangle \qquad \Longrightarrow \qquad -i\gamma_1\gamma_2|1\rangle = -|1\rangle, \quad |1\rangle \equiv |-\rangle$$

fermionic parity

$$P_F = (-1)^{\hat{a}^{+}\hat{a}} = -i\gamma_1\gamma_2$$



fermionic mode
(0 or 1 fermion)

states of two Majoranas
(different fusion channels)

Hamiltonian and Hilbert space (states)

Complex (Dirac) fermion a

Majorana fermions γ_1, γ_2

$$H = \varepsilon(a^+ a - \frac{1}{2})$$

$$H = -\frac{i}{2}\varepsilon\,\gamma_1\gamma_2$$

States: $\{|0\rangle, |1\rangle\}:\ a|0\rangle = 0, |1\rangle = a^+|0\rangle$

$$P_F = (-1)^{a^+ a} = i\gamma_1\gamma_2$$

$$H|0\rangle = -\frac{\varepsilon}{2}|0\rangle$$

$$H|1\rangle = +\frac{\varepsilon}{2}|1\rangle$$

$$i\gamma_1\gamma_2|0\rangle = |0\rangle, \quad |0\rangle \equiv |+\rangle$$

$$i\gamma_1\gamma_2|1\rangle = -|1\rangle, \quad |1\rangle \equiv |-\rangle$$

fermionic parity

Two Majorana fermions can correspond to

either fermionic vacuum state (fuse to vacuum) $|0\rangle$ (even parity)

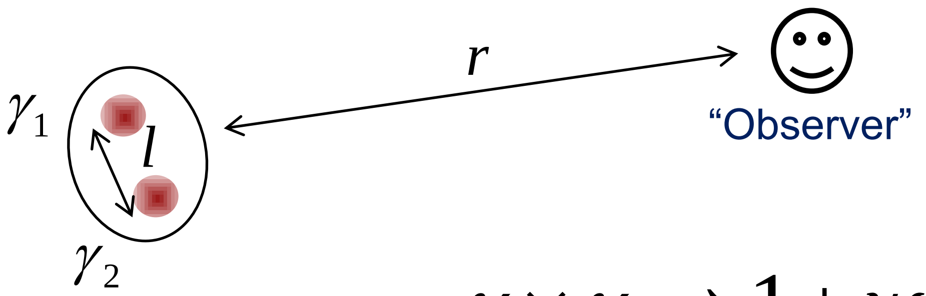
or single-fermion state (fuse to fermion) $|1\rangle$ (odd parity)

Two-Majorana states: Fusion of Majoranas (reminder)

State with NO fermion $|0\rangle$ and state with ONE fermion $|1\rangle$
are BOTH described by two Majorana fermions (anyons)

Fusion of two Majoranas γ_1, γ_2 :

how do they behave as a combined object seen from distances much large than the separation between them $r >> l$



The result is either
fermionic vacuum $|0\rangle\ (=1)$ or
single-fermion $|1\rangle\ (=\psi)$

$$\gamma \times \gamma \rightarrow 1 + \psi$$

- Majorana fusion rules

More degrees of freedom

For N complex (Dirac) fermions a_j ($j=1,\dots,N$):

with algebra $\{a_k, a_l^+\} = \delta_{kl}, \{a_k, a_l\} = \{a_k^+, a_l^+\} = 0$

we define $2N$ hermitian Majorana operators γ_m ($m=1,\dots,2N$)

$$\gamma_{2j-1} = a_j + a_j^+ \qquad \gamma_{2j} = (a_j - a_j^+)/i$$

with algebra $\gamma_m\gamma_n + \gamma_n\gamma_m = 2\delta_{mn}$ - Clifford algebra

Inverse:

For $2N$ hermitian Majorana operators γ_m ($m=1,\dots,2N$)

we define N complex (Dirac) fermions a_j ($j=1,\dots,N$)

$$a_j = (\gamma_{2j-1} + i\gamma_{2j})/2 \qquad a_j^+ = (\gamma_{2j-1} - i\gamma_{2j})/2$$

Hamiltonian:

$$\gamma_{2j-1} = a_j + a_j^+ \qquad \gamma_{2j} = (a_j - a_j^+)/i$$

$$a_j = (\gamma_{2j-1} + i\gamma_{2j})/2 \qquad a_j^+ = (\gamma_{2j-1} - i\gamma_{2j})/2$$

$$H = \sum_{j=1}^{N} \left(\varepsilon_j - \frac{1}{2} \right) a_j^+ a_j = \frac{i}{2} \sum_{j=1}^{N} \varepsilon_j \gamma_{2j-1} \gamma_{2j}$$

Fermionic parity operator:

$$P_F = (-1)^{\sum_{j=1}^{N} a_j^+ a_j} = \prod_{j=1}^{N} (-i\gamma_{2j-1}\gamma_{2j})$$

State description

2^N possible states can be described in two equivalent ways:

1. By occupations $n_j = 0, 1$ of the N fermionic modes a_j $(j = 1, \ldots, N)$

or $\hat{n}_j = a_j^+ a_j = (1 + i\gamma_{2j-1}\gamma_{2j})/2$

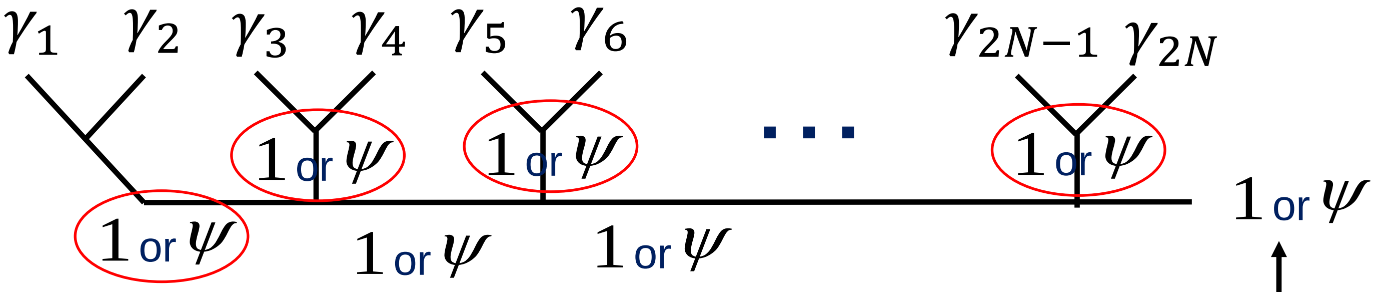
2. By fusion channels $(1, \psi)_j$ for N pairs $\gamma_{2j-1}, \gamma_{2j}$ of Majoranas

$(j = 1, \ldots, N)$

$n_j = 0$ or $-i\gamma_{2j-1}\gamma_{2j}\left|0_j\right\rangle = \left|0_j\right\rangle$ corresponds to fusion channel 1_j

$n_j = 1$ or $-i\gamma_{2j-1}\gamma_{2j}\left|1_j\right\rangle = -\left|1_j\right\rangle$ corresponds to fusion channel ψ_j

State description (reminder)



$n_j = 0$ or $-i\gamma_{2j-1}\gamma_{2j}|0_j\rangle = |0_j\rangle$ fusion channel 1_j

$n_j = 1$ or $-i\gamma_{2j-1}\gamma_{2j}|1_j\rangle = -|1_j\rangle$ fusion channel ψ_j

Fermionic parity

ψ for odd n_ψ

1 for even n_ψ

When it becomes nontrivial?

The formal mapping one fermion $\rightarrow$ two Majorana fermions becomes of interest if we can make spatially separated Majorana fermions (non-local fermion)

Spatially separated Majorana fermions can be braided to test and make use of their non-abelian statistics

Majorana fermions in Kitaev wire

Majorana edge states in Kitaev wire

A.Y. Kitaev, Phys. Usp. (2001)

Kitaev wire: spinless fermions with “p-wave” pairing on a1D chain of size L

$$H=\sum_{j=1}^{L-1}\left(\underbrace{-J\hat{a}_j^{+}\hat{a}_{j+1}}_{\text{hopping}}+\underbrace{\Delta\hat{a}_j\hat{a}_{j+1}}_{\text{pairing}}+\mathrm{h.c.}\underbrace{-\mu\hat{a}_j^{+}\hat{a}_j}_{\text{chemical potential}}\right)$$

Symmetries: The pairing amplitude Δ breaks the $U(1)$ gauge symmetry

$$a_j \rightarrow e^{i\varphi}a_j$$

down to the Z_2 symmetry

$$a_j \rightarrow -a_j$$

Parity is a conserved quantum number, not the number of particles

can be measured in cold-atom systems!

Solving Kitaev wire

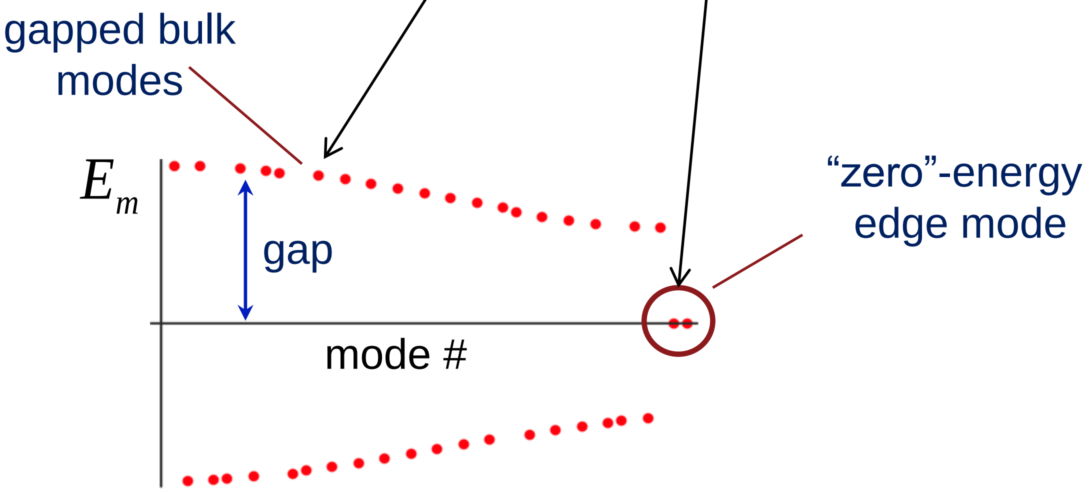
$$\Delta \neq J > 0, |\mu| < 2J$$

$$H = \sum_{j=1}^{L-1} \left(-J \hat{a}_j^+ \hat{a}_{j+1} + \Delta \hat{a}_j \hat{a}_{j+1} + \text{h.c.} - \mu \hat{a}_j^+ \hat{a}_j \right)$$

$\Downarrow$ Bogoliubov transformation

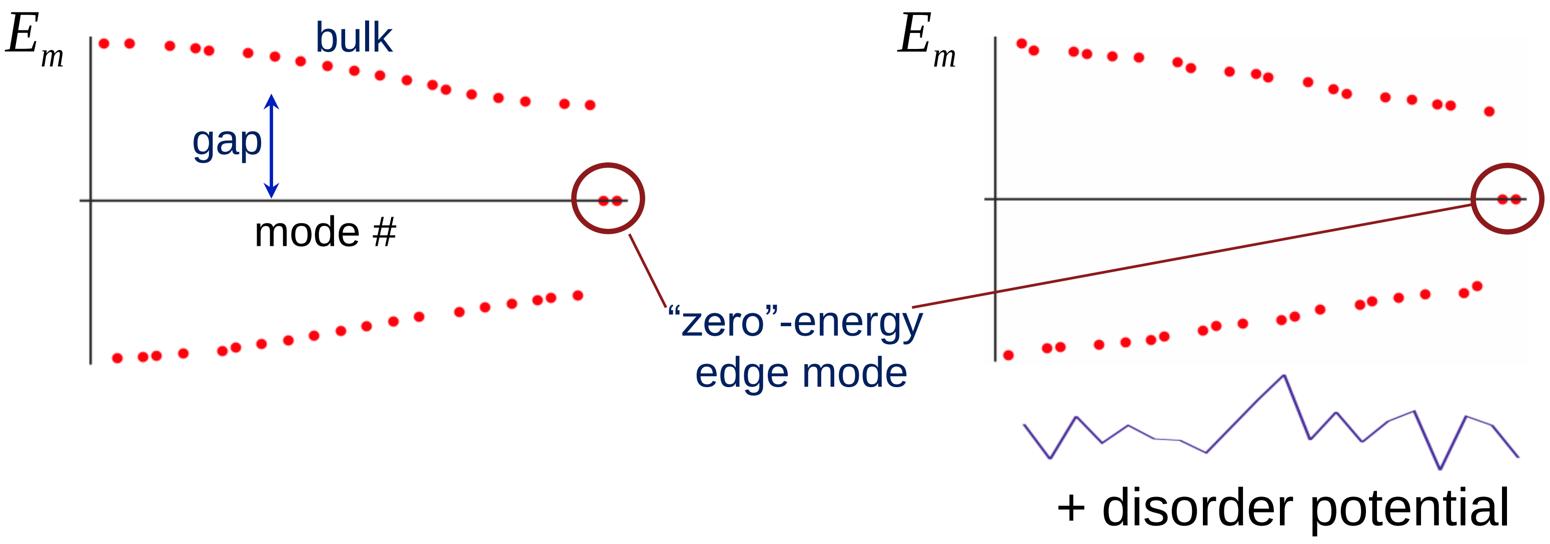
$$\hat{\alpha}_m = \sum_j \left(u_{mj}^* \hat{a}_j + v_{mj}^* \hat{a}_j^+ \right)$$

$$H = \sum_{m=1}^{L} E_m \hat{\alpha}_m^+ \hat{\alpha}_m = \sum_{\nu=1}^{L-1} E_\nu \hat{\alpha}_\nu^+ \hat{\alpha}_\nu + E_M \hat{\alpha}_M^+ \hat{\alpha}_M$$



Robustness

"Zero-energy" eigenvalue is robust against static disorder



This robustness against imperfection is a consequence of the topological order in the bulk – topological protection

Topological order in the bulk

Hamiltonian in (quasi)momentum space

(Δ -real)

$$H = \sum_{k\in(-\pi,\pi)} (a_k^+, a_{-k}) \underbrace{\begin{pmatrix} \xi_k & \Delta_k \\ \Delta_k^* & -\xi_k \end{pmatrix}}_{\text{matrix } \mathscr{H}_k} \begin{pmatrix} a_k \\ a_{-k}^+ \end{pmatrix}$$

with
$$\xi_k = -J\cos k - \mu/2$$
$$\Delta_k = -i\Delta \sin k,$$

Excitation spectrum

$$E_k = 2\sqrt{\xi_k^2 + |\Delta_k|^2}$$

(has to be gapped !)

Topological order in the bulk

Ground state (BCS)

$$\left| BCS \right\rangle = \prod_{k \in \mathrm{BZ}} (u_{\vec{k}} + v_{\vec{k}} a_{-k}^{+} a_{k}^{+}) \left| 0 \right\rangle$$

with $$u_{\vec{k}} = \sqrt{(E_k + \xi_k)/2E_k}, \qquad v_{\vec{k}} = \Delta_k / \sqrt{2E_k(E_k + \xi_k)}$$

Unit vector $\vec{n}_{\vec{k}}$

$$n_{x,k} = u_k v_k^* + u_k^* v_k = -\mathrm{Re}(\Delta_k)/E_k = 0$$

$$n_{y,k} = i(u_k v_k^* - u_k^* v_k) = \mathrm{Im}(\Delta_k)/E_k$$

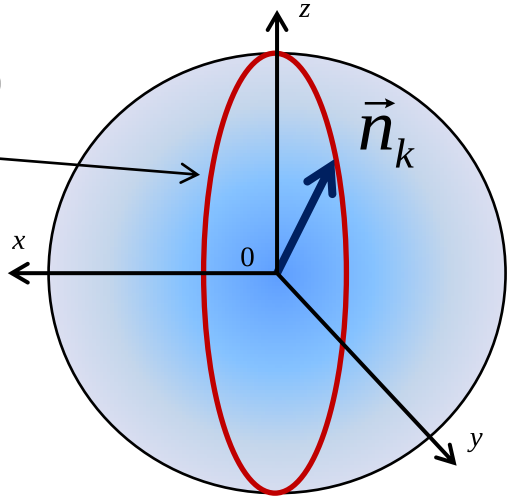
$$n_{z,k} = u_k u_k^* - v_k v_k^* = \xi_k / E_k$$

is well-defined for $E_k = 2\sqrt{\xi_k^2 + |\Delta_k|^2} > 0$ (gapped state)

Topological order in the bulk

Important: unit vector $\vec{n}_k$ is in the yz-plane for all $k \in \mathrm{BZ} = (-\pi, \pi) \sim S^1$

end of $\vec{n}_k$ lying on a circle $\quad S^1 : \left|\vec{n}_k\right| = 1,\, n_{x,k} = 0$



Unit vector $\vec{n}_k$ determines mapping $S^1 \to S^1$

classified by $\quad \pi_1(S^1) = \mathrm{Z}$

Winding number

$$\nu = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk\, \vec{e}_x \cdot (\vec{n}_k \times \partial_k \vec{n}_k) \in \mathrm{Z}$$

counts the number of times $\vec{n}_k$ winds around the origin

characterizes topological order (in this case!)

For Kitaev wire with $|\mu| \leq 2J$

$\nu = 1$ indicates nontrivial topological order

Closer look at the "zero-energy" mode

$$\boxed{\hat{\alpha}_M = (\gamma_L + i\gamma_R)/2}$$

γ_L, γ_R - Majorana operators

In Majorana basis $\gamma_{2j-1} = \hat{a}_j + \hat{a}_j^+ \quad \gamma_{2j} = (\hat{a}_j - \hat{a}_j^+)/i$

$$\boxed{\begin{aligned} \gamma_L &\sim \sum_j (x_+^j - x_-^j)\gamma_{2j-1} \\ \gamma_R &\sim \sum_j (x_+^{L-j} - x_-^{L-j})\gamma_{2j} \end{aligned}}$$

$$x_\pm = \frac{-\mu \pm \sqrt{\mu^2 + 4\Delta^2 - 4J^2}}{2(\Delta + J)}$$

$|x_+|, |x_-| < 1$ for $\Delta \neq J > 0, |\mu| < 2J$

γ_L "lives" near the left edge

γ_R "lives" near the right edge

$$\left.\begin{aligned} x_+^j - x_-^j &\sim \exp(-\kappa j) \\ -\kappa &= \ln\min(|x_\pm|) \end{aligned}\right\}$$

Majorana edge modes

γ_L  γ_R

$\hat{\alpha}_M$ - non-local fermion living on both edges

The energy of the “zero-energy” mode

The energy of the non-local fermion $\hat{\alpha}_M$

$$\boxed{E_M \sim \Delta \frac{x_+^{L+1} - x_-^{L+1}}{x_+ - x_-} \sim \exp(-\kappa L)}$$

is exponentially small with the size of the wire L

The Hamiltonian of the non-local fermion $\hat{\alpha}_M$

$$H_M = E_M \hat{\alpha}_M^+ \hat{\alpha}_M = \frac{i}{2} \underset{\uparrow}{E_M} \gamma_L \gamma_R + \frac{1}{2} E_M$$

$E_M \sim \exp(-\kappa L)$ - coupling between Majorana modes

Quasi degenerate ground state: $|0\rangle$ ($\hat{\alpha}_m |0\rangle = 0$) and

with different fermionic parity $|M\rangle = \hat{\alpha}_M^+ |0\rangle$

have exponentially close energies

In the “ideal” case $\Delta = J > 0,\ \mu = 0$

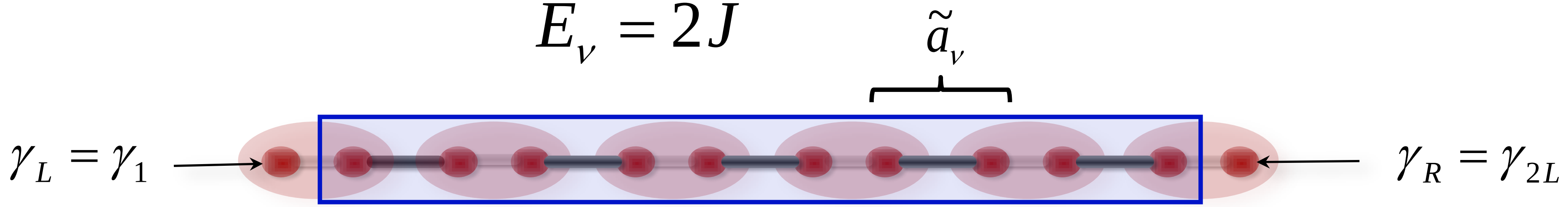
Zero-energy mode $\hat{\alpha}_M = (\gamma_1 + i\gamma_{2L})/2 = (\hat{a}_1 + \hat{a}_1^+ + \hat{a}_L - \hat{a}_L^+)/2$

$E_M = 0$

Majoranas $\gamma_L = \gamma_1$ and $\gamma_R = \gamma_{2L}$ are completely decoupled

Gapped modes $\hat{\alpha}_\nu = (\gamma_{2\nu} + i\gamma_{2\nu+1})/2 = i(\hat{a}_{\nu+1} + \hat{a}_{\nu+1}^+ - \hat{a}_\nu + \hat{a}_\nu^+)/2$

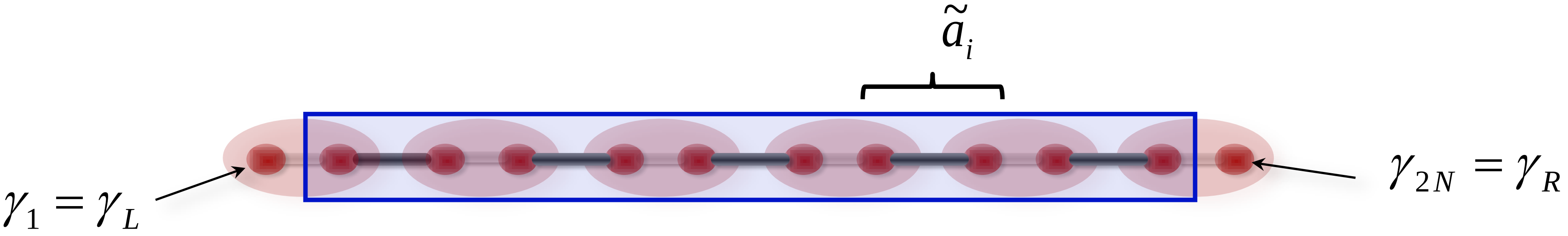
$E_\nu = 2J$



Degenerate ground state: states $|0\rangle$ ($\hat{\alpha}_m|0\rangle = 0$) and $|M\rangle = \hat{\alpha}_M^+|0\rangle$

have the same energy but different parity

Long-range fermionic correlations



$$\langle \pm | P_F | \pm \rangle = \langle \pm | (-1)^{\sum_j a_j^+ a_j} | \pm \rangle = \langle \pm | (-i)\gamma_1 \gamma_2 (-i)\gamma_3 \gamma_4 \cdots \quad \cdots (-i)\gamma_{2N-1} \gamma_{2N} | \pm \rangle$$

$$= \langle \pm | (-i)\gamma_1 \overbrace{\boxed{\gamma_2 (-i)\gamma_3}}^{=1} \overbrace{\boxed{\gamma_4 \cdots}}^{=1\ \cdots} \quad \overbrace{\boxed{\cdots (-i)\gamma_{2N-1}}}^{=1} \gamma_{2N} | \pm \rangle$$

$$\rightarrow \langle \pm | -i\gamma_1 \gamma_{2N} | \pm \rangle = -\langle \pm | (a_1 + a_1^+)(a_N - a_N^+) | \pm \rangle = \pm 1$$

fermionic correlations between sites 1 and N

Explicit ground state wave functions

$$|+\rangle = \frac{1}{2^N}\left[1 + \sum_{p=1}^{N} \sum_{i_1<\ldots<i_{2p}}^{2N+1} a_{i_{2p}}^{+} \cdots a_{i_1}^{+}\right] |\mathrm{vac}\rangle$$

$$|-\rangle = \frac{1}{2^N} \sum_{p=0}^{N} \sum_{i_1<\ldots<i_{2p+1}}^{2N+1} a_{i_{2p+1}}^{+} \cdots a_{i_1}^{+} |\mathrm{vac}\rangle$$

These states have identical local properties

but different fermionic number parity $\langle \pm | P_F | \pm \rangle = \pm 1$

“Making” Kitaev wire with cold atoms

System: fermionic atoms in an optical lattice

hopping term $-J\sum_i (a_i^+ a_{i+1} + \mathrm{h.c.})$

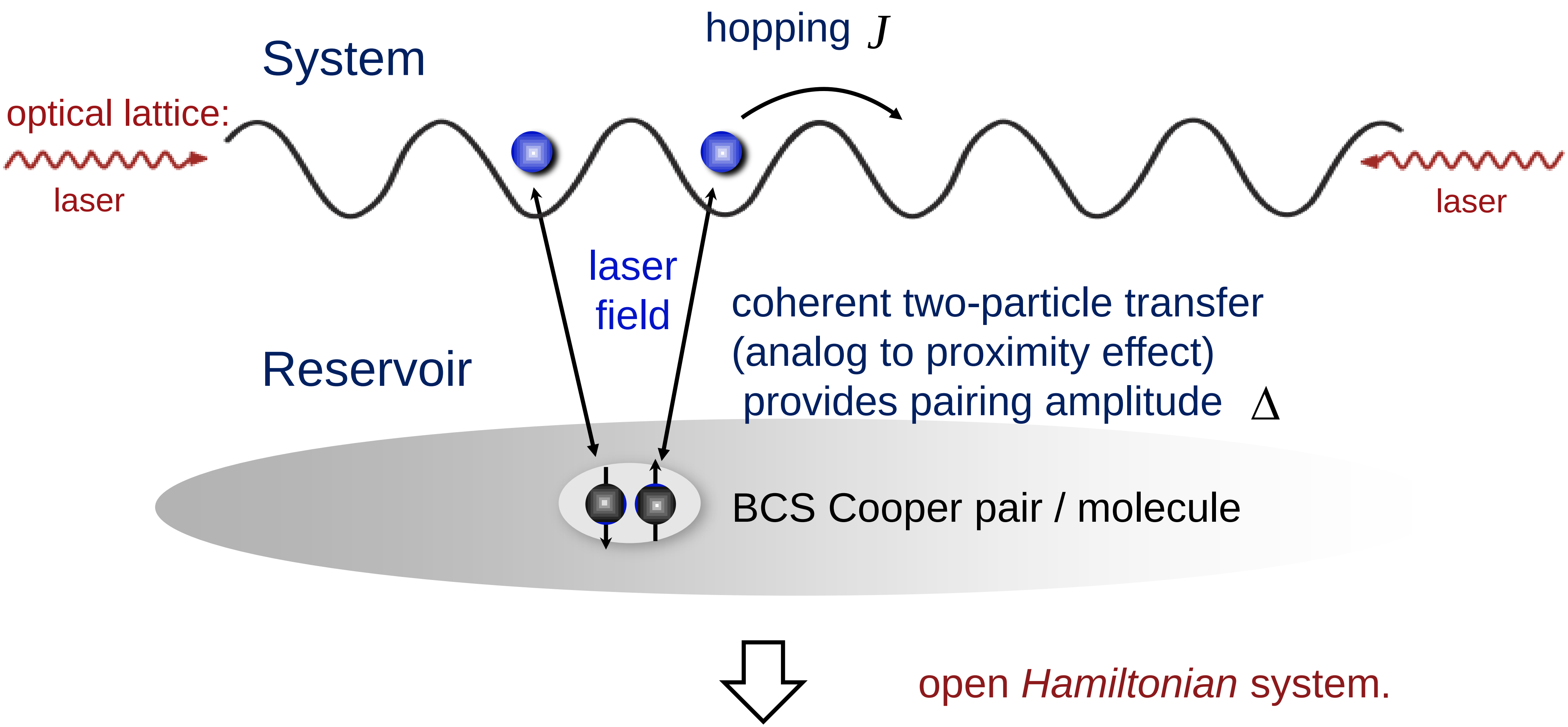
continuous version $-(\hbar^2/2m)\int d\vec{r}\,\hat{\psi}^+\Delta\hat{\psi}$

Reservoir: molecular BEC (or BCS) cloud

pairing term $\sum_i (\Delta a_i^+ a_{i+1}^+ + \mathrm{h.c.})$

continuous version $\Delta_0\int d\vec{r}\,(\hat{\psi}^+\nabla\hat{\psi}^+ + h.c.)$

Basic idea



open *Hamiltonian* system.

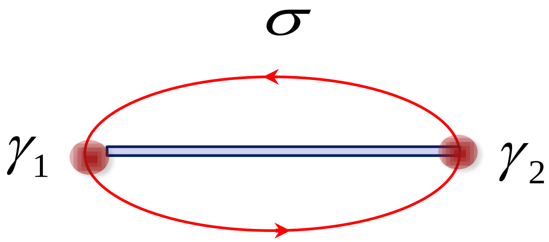
$$H = \sum_{i=1}^{N-1} \left(-J a_i^+ a_{i+1} + \Delta a_i a_{i+1} + \text{h.c.} - \mu a_i^+ a_i \right)$$

L. Jiang, et al, Phys. Rev. Lett. 106, 22042 (2011)

S. Nascimbène, J. Phys. B 46, 134005 (2013)

Braiding protocol

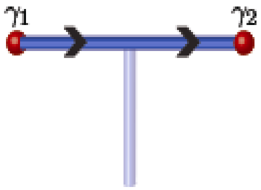
Braiding of Majorana fermions



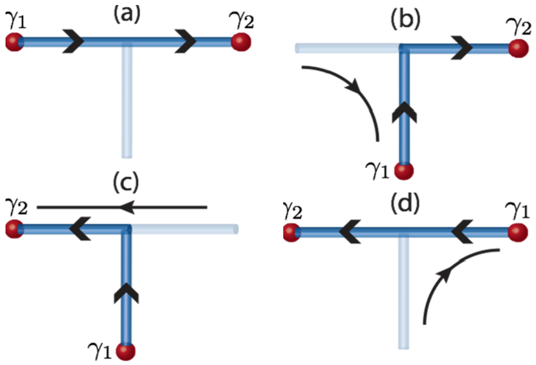
$$\gamma_1 \to -\gamma_2$$
$$\gamma_2 \to \gamma_1$$

How to realize?

T-junction:



J. Alicea et al, Nat. Phys. 7 412 (2011)

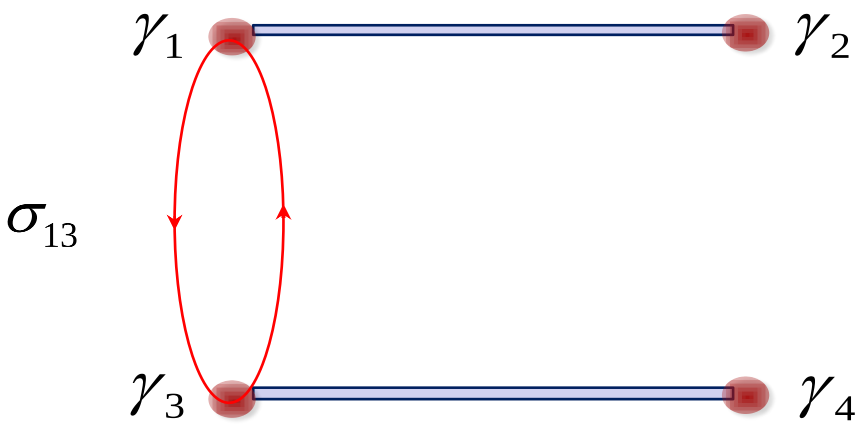


Moving Majoranas around by changing the local potential

Can also be done in atomic wires setup.

Could cold atoms provide something else?

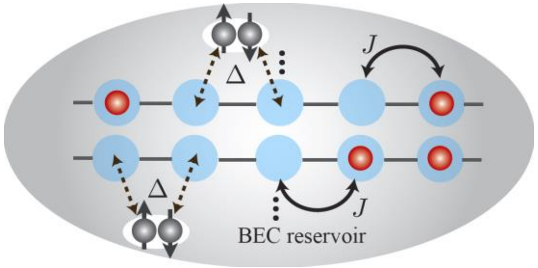
Braiding of Majorana fermions in atomic wires setup



$$\gamma_1 \to -\gamma_3 = U_{13}^{-1}\gamma_1 U_{13}$$
$$\gamma_3 \to \gamma_1 \quad = U_{13}^{-1}\gamma_3 U_{13}$$

$$U_{13} = e^{i\frac{\pi}{8}} \exp\left(-\frac{\pi}{4}\gamma_1\gamma_3\right) = e^{i\frac{\pi}{8}} \frac{1}{\sqrt{2}}(1-\gamma_1\gamma_3)$$

Braiding of Majorana fermions in atomic wires

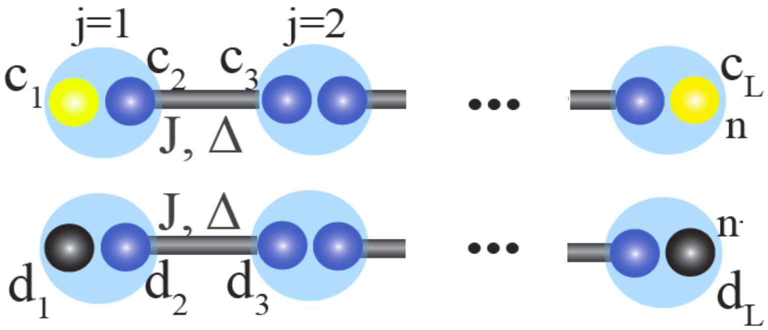


Two (nearest) Kitaev wires:

$$H=\sum_j\left(-Ja^+_{u,j}a_{u,j+1}+\Delta a_{u,j}a_{u,j+1}+\text{h.c.}-\mu a^+_{u,j}a_{u,j}\right)\quad\leftarrow\quad\text{upper wire}$$

$$+\sum_j\left(-Ja^+_{l,j}a_{l,j+1}+\Delta a_{l,j}a_{l,j+1}+\text{h.c.}-\mu a^+_{l,j}a_{l,j}\right)\quad\leftarrow\quad\text{lower wire}$$

γ_1 γ_3



γ_2 γ_4

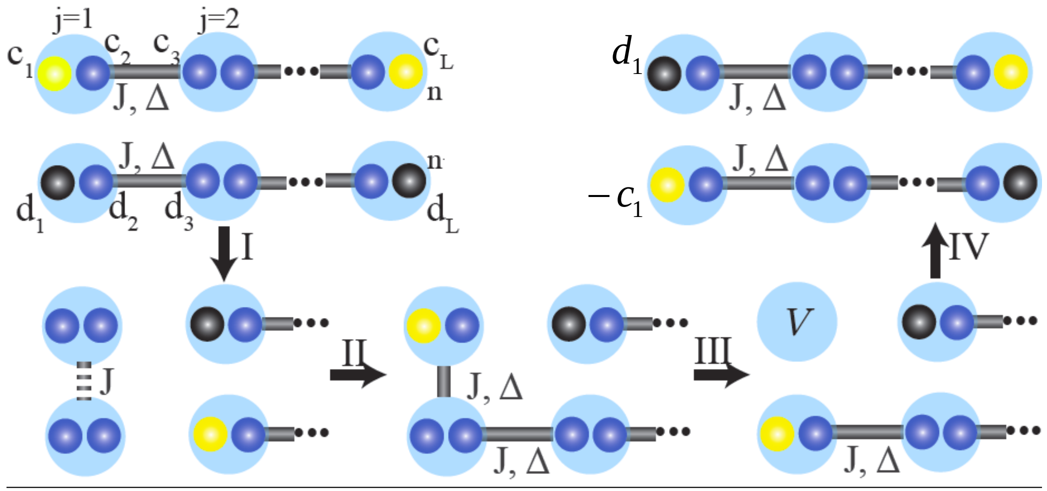
$$a_{u,j}=\frac{1}{2}(c_{2j-1}+ic_{2j})$$

$$a_{l,j}=\frac{1}{2}(d_{2j-1}+id_{2j})$$

Four Majorana fermions $\gamma_1,\gamma_2,\gamma_3,\gamma_4$, we braid $\gamma_1=c_1$ and $\gamma_3=d_1$

Braiding protocol:

C.V. Kraus, P. Zoller, and M.A. Baranov, PRL 111, 203001 (2013)



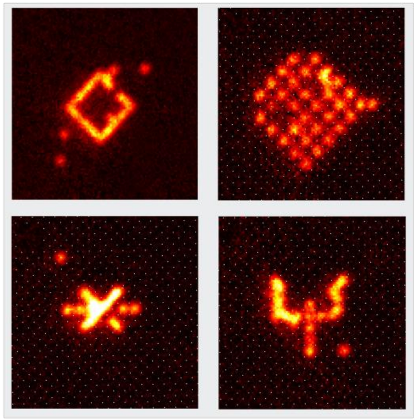
Advantages:
- small number of steps
- only four sites and links between them are involved (local)

Requirement:
- local cite/link addressing

J. Simon, et. al, Nature (London) 473:307-312, 2011

C. Weitenberg, et. al, Nature (London) 471:319-324, 2011

T. Fukuhara, et. al, Nat. Phys. 9:235, 2011



Needed local operations:

Single-link: switching on/off adiabatically

hopping $H_{jl}^{(J)} = -Ja_j^+ a_l - \text{h.c.}$ and pairing $H_{jl}^{(p)} = \Delta a_j^+ a_l^+ + \text{h.c.}$

between nearest sites j and l

Together give "Kitaev coupling" $H_{jl}^{(K)} = H_{jl}^{(J)} + H_{jl}^{(p)}$

Single-site: switching on/off adiabatically

on-site potential $H_j^{(loc)} = Va_j^+ a_j$

Braiding protocol: Step I

ϕ_t changes adiabatically from 0 to $\pi / 2$

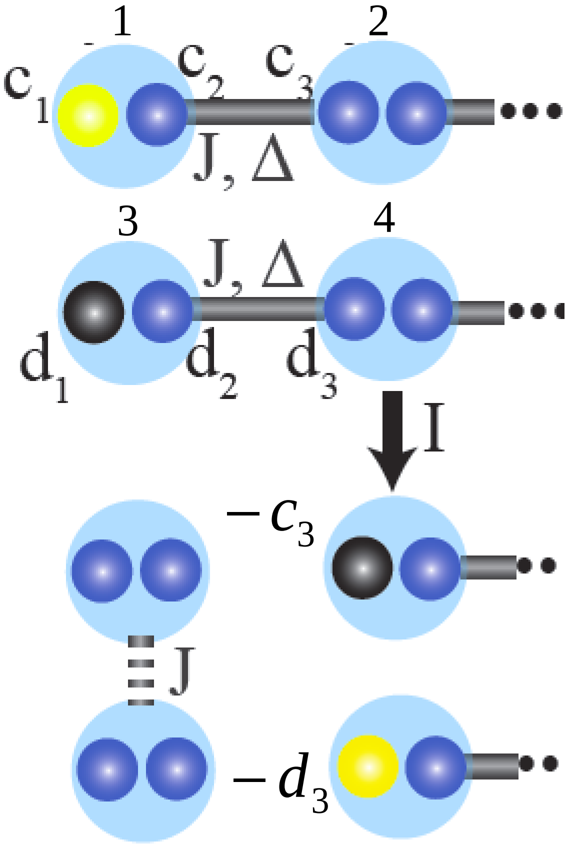
Turn off the couplings between sites 1-2 and 3-4;
turn on hopping between sites 1-3

$$H_I = \left(H_{12}^{(K)} + H_{34}^{(K)}\right)\cos\phi_t + H_{13}^{(J)}\sin\phi_t$$

$$\gamma_1(\phi_t) = (2c_1\cos\phi_t - d_3\sin\phi_t)/\sqrt{1+3\cos^2\phi_t}$$

$$\gamma_3(\phi_t) = (2d_1\cos\phi_t - c_3\sin\phi_t)/\sqrt{1+3\cos^2\phi_t}$$

$$\boxed{\begin{aligned}\gamma_1 &= c_1 \to -d_3 \\ \gamma_3 &= d_1 \to -c_3\end{aligned}}$$



Braiding protocol: Step II

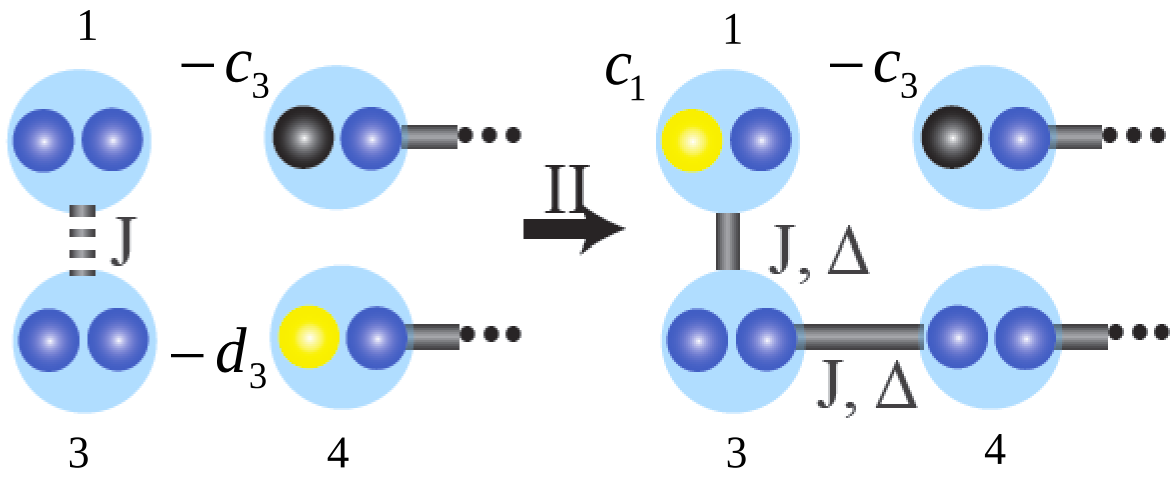
ϕ_t changes adiabatically from 0 to $\pi/2$

Turn on the couplings between sites 3-4;
turn on pairing between sites 1-3

$$H_{II} = H_{13}^{(J)} + \left(H_{13}^{(p)} + H_{34}^{(K)}\right)\sin\phi_t$$

$$\gamma_1(\phi_t) = [2c_1 \sin\phi_t - d_3(1-\sin\phi_t)] / \sqrt{4\sin^2\phi_t + (1-\sin\phi_t)^2}$$
$$\gamma_3(\phi_t) = -c_3$$

$$\boxed{\begin{aligned}\gamma_1 &\to c_1 \\ \gamma_3 &\to -c_3\end{aligned}}$$



Braiding protocol: Step III

ϕ_t changes adiabatically from 0 to $\pi/2$

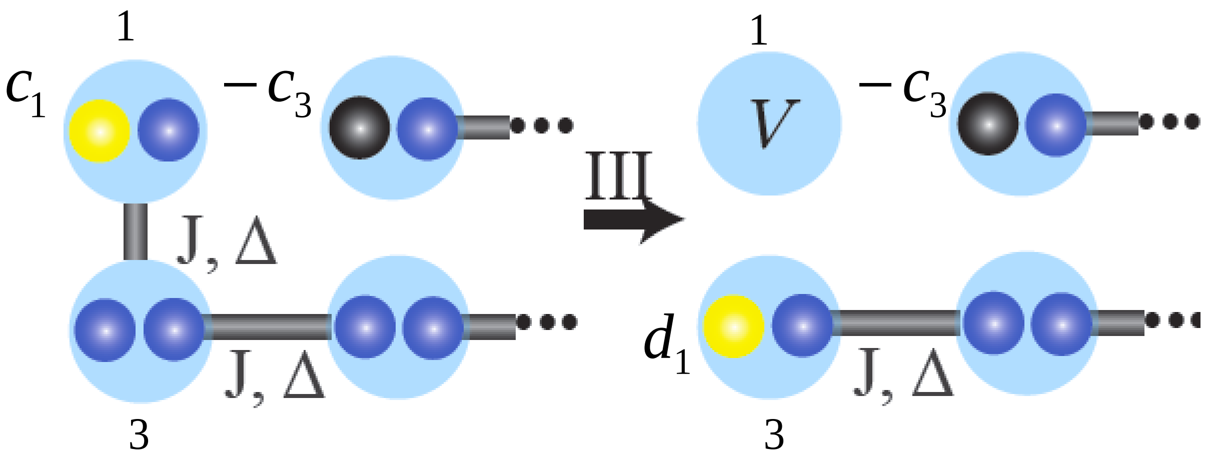
Ramp up local potential on site 1;
turn off couplings between sites1-3

$$H_{III} = H_1^{(loc)} \sin\phi_t + H_{13}^{(K)} \cos\phi_t + H_{34}^{(K)}$$

$$\gamma_1(\phi_t) = (Jc_1 \cos\phi_t + Vd_1 \sin\phi_t) / \sqrt{(J\cos\phi_t)^2 + (V\sin\phi_t)^2}$$

$$\gamma_3(\phi_t) = -c_3$$

$$\gamma_1 \rightarrow d_1$$
$$\gamma_3 \rightarrow -c_3$$



Braiding protocol: Step IV

ϕ_t changes adiabatically from 0 to $\pi/2$

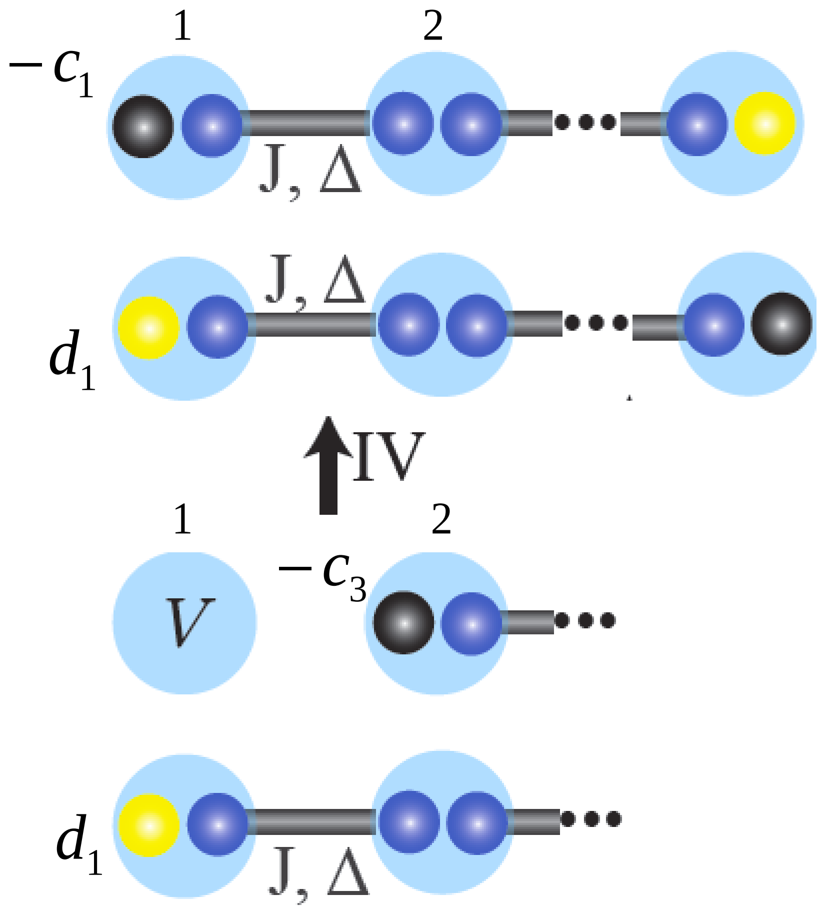
Ramp down local potential on site1;
turn on couplings between sites 1-2

$$H_{IV} = H_{12}^{(K)} \sin\phi_t + H_1^{(loc)} \cos\phi_t + H_{34}^{(K)}$$

$$\gamma_1(\phi_t) = d_1$$

$$\gamma_3(\phi_t) = -(Jc_1 \sin\phi_t + Vc_3 \cos\phi_t) / \sqrt{(J\sin\phi_t)^2 + (V\cos\phi_t)^2}$$

$$\boxed{\begin{aligned} \gamma_1 &\to d_1 \\ \gamma_3 &\to -c_1 \end{aligned}}$$



Result of the braiding protocol:

$$\gamma_1 \to -\gamma_3$$
$$\gamma_3 \to \gamma_1$$

generated by

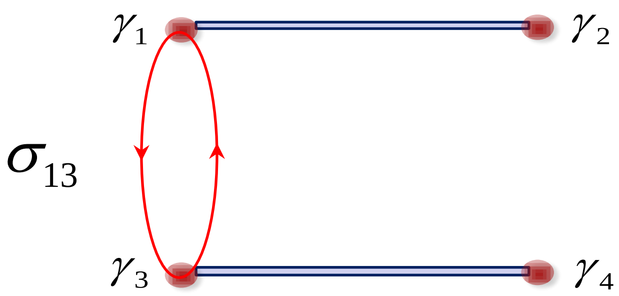
$$U_{13} = e^{i\frac{\pi}{8}} \exp\left(-\frac{\pi}{4}\gamma_1\gamma_3\right)$$
$$= e^{i\frac{\pi}{8}} \frac{1}{\sqrt{2}}\left(1-\gamma_1\gamma_3\right)$$

$$\gamma_1 \to -\gamma_3 = U_{13}^{-1}\gamma_1 U_{13}$$
$$\gamma_3 \to \gamma_1 \quad = U_{13}^{-1}\gamma_3 U_{13}$$

Physics behind

one fermion is taken from the system
(either from the lower or from the upper wire)
and inserted into the lower wire

Physical consequences:



$$U_{13} = e^{i\frac{\pi}{8}} \frac{1}{\sqrt{2}} \left(1 - \gamma_1 \gamma_3\right)$$

In the basis $\{|++\rangle, |--\rangle\}$ of eigenfunctions of $-i\gamma_1\gamma_2$ and $-i\gamma_3\gamma_4$

$$\left.\begin{matrix} -i\gamma_1\gamma_2 \\ -i\gamma_3\gamma_4 \end{matrix}\right\} | p_1, p_2 \rangle = \left.\begin{matrix} p_1 \\ p_2 \end{matrix}\right\} | p_1, p_2 \rangle$$

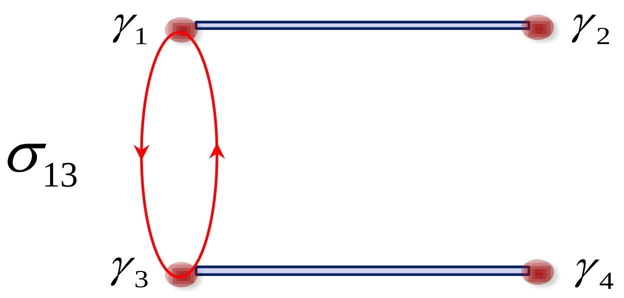
parity of the upper wire

parity of the lower wire

we have

$$U_{13} = \frac{1}{\sqrt{2}} e^{i\frac{\pi}{8}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \quad \text{and} \quad U_{13}^2 = e^{-i\frac{\pi}{4}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Physical consequences:



$$U_{13} = e^{i\frac{\pi}{8}} \frac{1}{\sqrt{2}} \left(1 - \gamma_1 \gamma_3\right)$$

$$U_{13}^2 = e^{-i\frac{\pi}{4}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Starting from $\left|++\right\rangle$

$$\sigma_{13} \qquad \left|++\right\rangle \xrightarrow{\sigma_{13}} U_{13}\left|++\right\rangle = e^{i\frac{\pi}{8}} \frac{1}{\sqrt{2}} \left[\left|++\right\rangle - i\left|--\right\rangle\right]$$

even-even parity

superposition of even-even and odd-odd

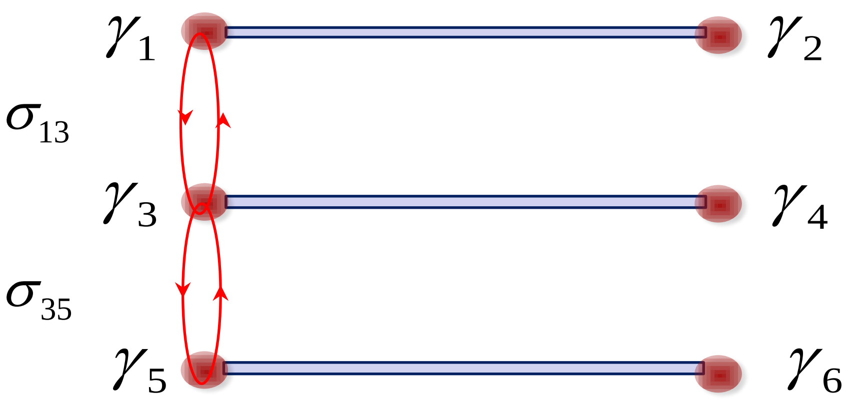
$$\sigma_{13}^2 \qquad \left|++\right\rangle \xrightarrow{\sigma_{13}^2} U_{13}^2\left|++\right\rangle = e^{-i\frac{\pi}{4}} \left|--\right\rangle$$

even-even parity

odd-odd parity

Demonstration of non-Abelian character

Three wires



$$U_{13} == e^{i\frac{\pi}{8}} \frac{1}{\sqrt{2}} (1 - \gamma_1 \gamma_3)$$

$$U_{35} == e^{i\frac{\pi}{8}} \frac{1}{\sqrt{2}} (1 - \gamma_3 \gamma_5)$$

Starting from $|+++\rangle$ - eigenstate of $-i\gamma_1\gamma_2, -i\gamma_3\gamma_4, -i\gamma_5\gamma_6$

$$|+++\rangle \xrightarrow{\sigma_{35}\sigma_{13}} U_{35}U_{13}|+++\rangle = e^{i\frac{\pi}{4}} \frac{1}{2}[|+++\rangle - i|+--\rangle - i|--+\rangle + |-+-\rangle]$$

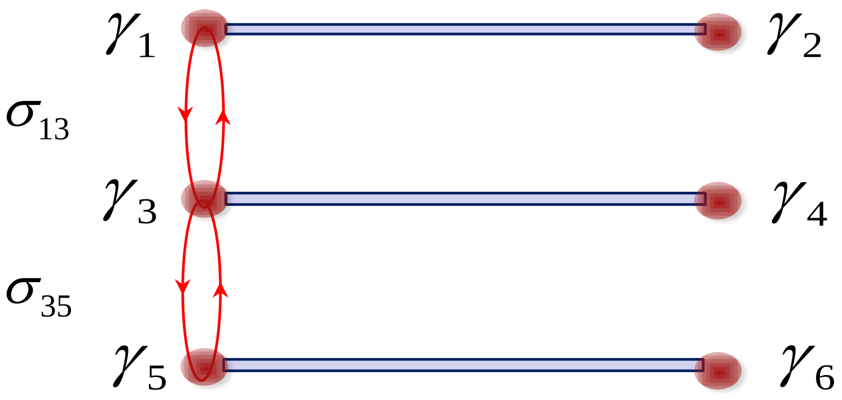
different

$$|+++\rangle \xrightarrow{\sigma_{13}\sigma_{35}} U_{13}U_{35}|+++\rangle = e^{i\frac{\pi}{4}} \frac{1}{2}[|+++\rangle - i|+--\rangle - i|--+\rangle - |-+-\rangle]$$

$$\boxed{\sigma_{13}\sigma_{35} \neq \sigma_{35}\sigma_{13}}$$

do not commute!

Another possibility: $\sigma_{13}\sigma_{35}$ and $\sigma_{35}\sigma_{13}$



$$U_{13} == e^{i\frac{\pi}{8}} \frac{1}{\sqrt{2}}(1-\gamma_1\gamma_3)$$

$$U_{35} == e^{i\frac{\pi}{8}} \frac{1}{\sqrt{2}}(1-\gamma_3\gamma_5)$$

Starting from $|+++\rangle$

$$|+++\rangle \xrightarrow{(\sigma_{35}\sigma_{13})(\sigma_{13}\sigma_{35})} U_{35}U_{13}^2U_{35}|+++\rangle = i|--+\rangle$$

different

$$|+++\rangle \xrightarrow{(\sigma_{13}\sigma_{35})(\sigma_{35}\sigma_{13})} U_{13}U_{35}^2U_{13}|+++\rangle = i|+--\rangle$$

$$\boxed{(\sigma_{13}\sigma_{35})(\sigma_{35}\sigma_{13}) \neq (\sigma_{35}\sigma_{13})(\sigma_{13}\sigma_{35})}$$ do not commute!

Using Majorana fermions for QC

Implementation of the Deutsch-Jozsa algorithm for two qubits

Although braiding does not provide a tool to build a universal set of gates,
it still can be used for QC.

Example: Deutsch-Jozsa algorithm

Deutsch-Jozsa algorithm (2 qubits)

Function $g: \{|0\rangle,|1\rangle\}\otimes\{|0\rangle,|1\rangle\}\mapsto\{0,1\}$ (oracle)

can be either constant or balanced

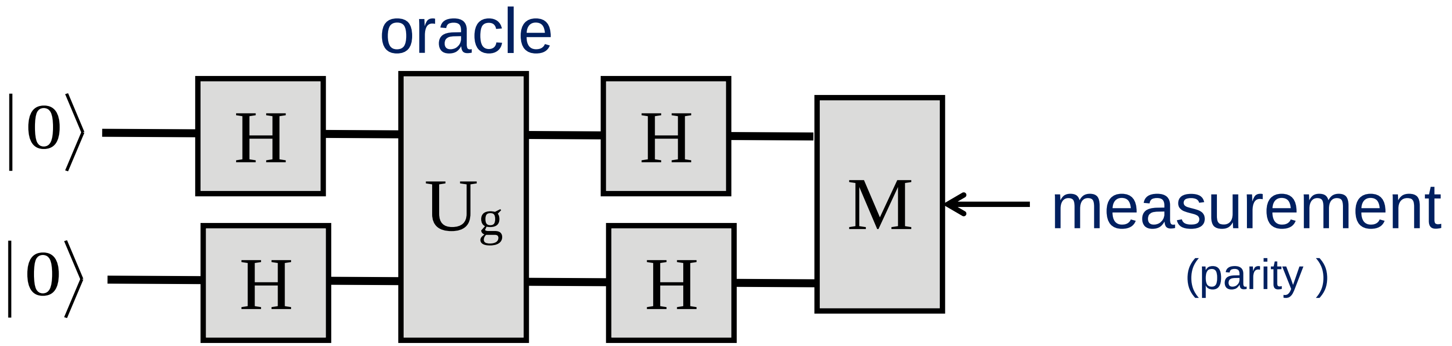
	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|11\rangle$	
g_0	0	0	0	0	constant
g_1	0	0	1	1	balanced
g_2	0	1	1	0	balanced
g_3	0	1	0	1	balanced

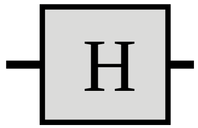
Question: is a given but unknown g constant or balanced?

Naïve way: three measurements (in the worst case)

Deutsch-Jozsa algorithm for two qubits: only one measurements!

When oracle is realized as a unitary $U_g|x\rangle = (-1)^{g(x)}|x\rangle$





Hadamard gate

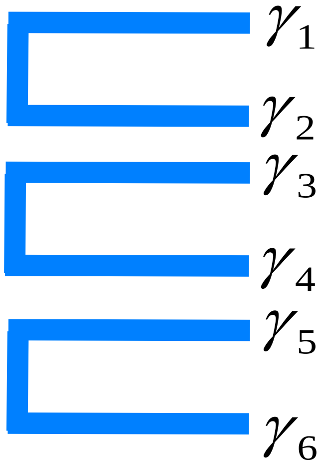
$g(x)$ constant: probability to measure $|00\rangle$ is 1

$g(x)$ balanced: probability to measure $|00\rangle$ is 0

Realization of the algorithm via braiding

C.V. Kraus, P. Zoller, and M.A. Baranov, PRL 111, 203001 (2013)

Setup: 3 Kitaev wires



f_1

f_2 controls parity

f_3

$$f_1 = (\gamma_1 + i\gamma_2)/2$$

$$f_2 = (\gamma_3 + i\gamma_4)/2$$

$$f_3 = (\gamma_5 + i\gamma_6)/2$$

encode 2 qubits

$$|00\rangle = f_2^+ |0\rangle_f$$

$$|01\rangle = f_3^+ |0\rangle_f$$

$$|10\rangle = f_1^+ |0\rangle_f$$

$$|11\rangle = f_1^+ f_2^+ f_3^+ |0\rangle_f$$

L. Georgiev, Phys. Rev. B 74 (2006)

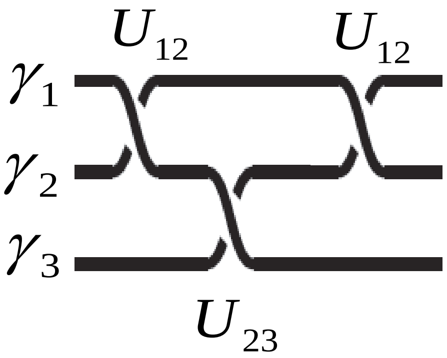
Hadamard gate

$$H \otimes H$$

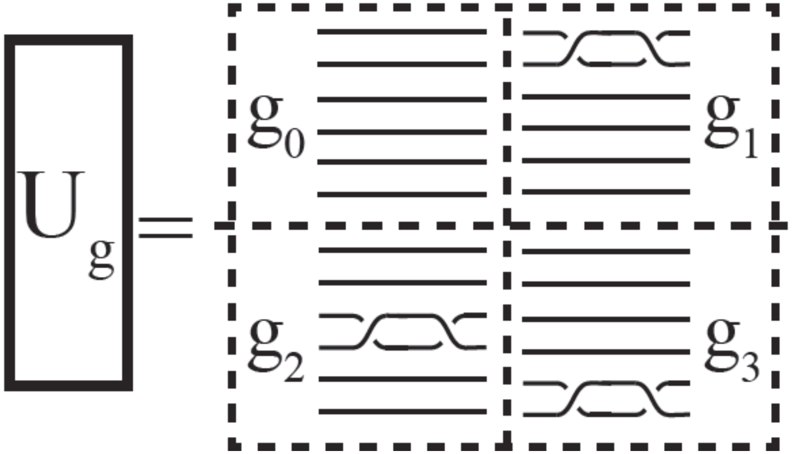
$$H \otimes 1 = U_{12} U_{23} U_{12}$$

$$1 \otimes H = U_{56} U_{45} U_{56}$$

$$H \otimes 1 =$$



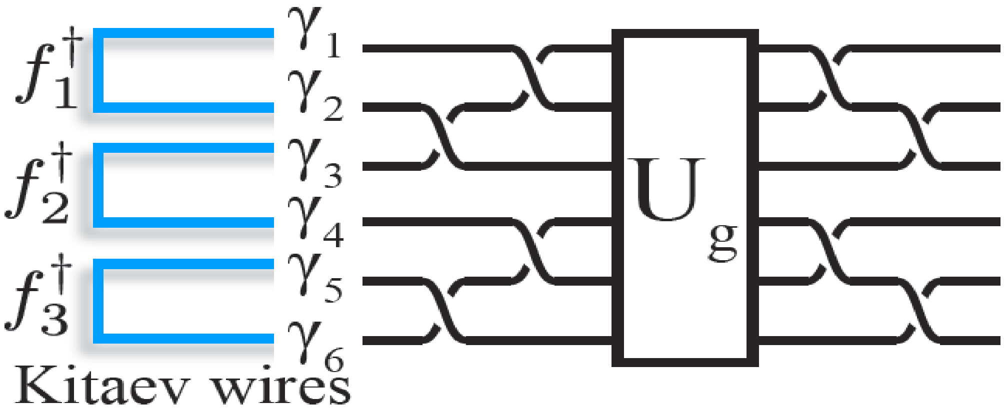
Unitary for an oracle



$$U_{g_0} = 1 \qquad U_{g_1} = U_{12}{}^2$$

$$U_{g_2} = U_{34}{}^2 \qquad U_{g_3} = U_{56}{}^2$$

Realization of the algorithm via braiding (optimum sequence)



$$U_{D-J}(g_i) = \boxed{U_{45}U_{56}}\ \boxed{U_{23}U_{12}}\ U_{g_i}\ \boxed{U_{56}U_{45}}\ \boxed{U_{12}U_{23}}$$

can be done in parallel

can be done in parallel

Realization of the algorithm in five steps!

Results:

$$U_{D-J}(g_0)\left|00\right\rangle = \left|00\right\rangle \qquad U_{D-J}(g_0)\left|00\right\rangle = \left|11\right\rangle$$

$$U_{D-J}(g_1)\left|00\right\rangle = i\left|10\right\rangle \qquad U_{D-J}(g_1)\left|00\right\rangle = i\left|01\right\rangle$$

Read out: measuring parities (particle numbers) in the wires

Conclusion

Majorana fermions provide an example of non-Abelian anyons

- fundamental physical interest

- applications for quantum computation

Cold atomic/molecular systems provides a possibility
to implement and to manipulate Majorana fermions

Thank you for your attention!