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Introduction to the Physics of Anyons with Majorana Fermions as an Example

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Anyon Physics of Ultracold Atomic Gases
Kaiserslautern 10 – 15 December 2018

Lecture 1: Appearance of Anyons

Lecture 2: Majorana Fermions as an example of non-Abelian Anyons

Lecture 1: Appearance of Anyons

Exchange and statistics

Why dimension 2?

Braid group. Abelian and no-Abelian anyons

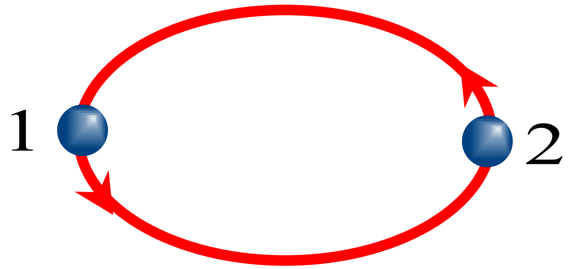
Formal introduction of anyons:
Fusion of anyons and anyon Hilbert space

Exchange and statistics

Particle exchange and statistics

Statistics $\Longleftrightarrow$ Behavior of the state (wave function) under the exchange of two identical (quasi)particles

Exchange of two (quasi)particles



$$\Psi(\vec{r}_1,\vec{r}_2,\ldots) \rightarrow \Psi(\vec{r}_2,\vec{r}_1,\ldots) = ?\ \Psi(\vec{r}_1,\vec{r}_2,\ldots)$$

Properties of many-body wave functions:

$$\psi(\vec{R}_1, \vec{R}_2, \ldots; \vec{r}_1, \vec{r}_2, \ldots)$$

$\vec{R}_i$ positions of quasiparticles

$\vec{r}_j$ positions of particles

has to be single-valued with respect to particle coordinates $\vec{r}_j$

but not necessarily with respect to quasiparticle coordinates $\vec{R}_i$

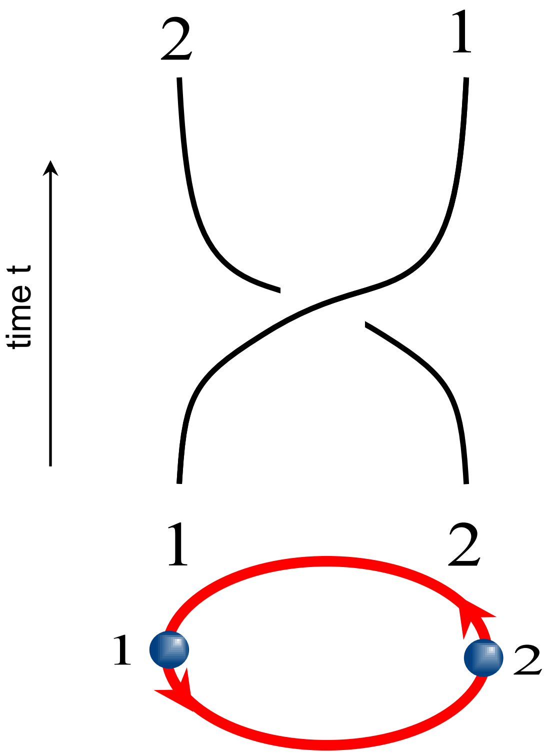
Exchange as adiabatic dynamical evolution
General statements:

1. Adiabatic theorem: States in a (possibly degenerate) energy subspace separated from others by a gap remain in the subspace when the system is changed adiabatically without closing the gap.

2. Change under adiabatic transport = combination of Berry's phase/matrix and transformation of instantaneous energy eigenstate (explicit monodromy)

3. Change under adiabatic transport is invariant, but Berry's phase/matrix and eigenstate transformation depend on choice of gauge (and can be shifted from one to the other)

Exchange as adiabatic dynamical evolution



For a unique ground state $|\Psi\rangle$ (single-valued)

separated by a gap from excited states

$$\Rightarrow \quad |\psi\rangle \to e^{i\varphi}|\psi\rangle, \qquad \varphi = -\frac{1}{\hbar}\underbrace{\int dt\, E(t)}_{\text{dynamical phase}} + \alpha$$

Berry phase $\alpha = i\int dt \langle \psi | \frac{d}{dt}\psi\rangle = \alpha_g(\text{path}) + \vartheta$

(single-valued w.f.)

α_g (path) geometrical phase

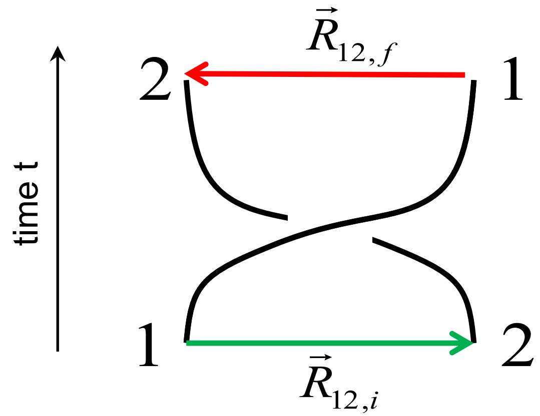
ϑ (topology of the path) **statistical angle** - of interest!

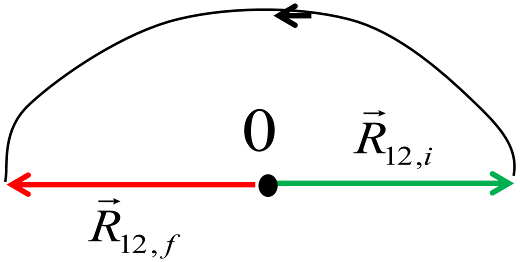
Exchange statistics $\quad |\psi\rangle \to e^{i\vartheta}|\psi\rangle$

Why dimension 2?

Constraint on the exchange $\vec{R}_{12}$ relative position

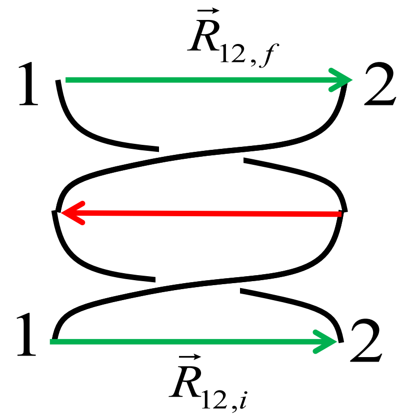
exchange $\hat{\sigma}$

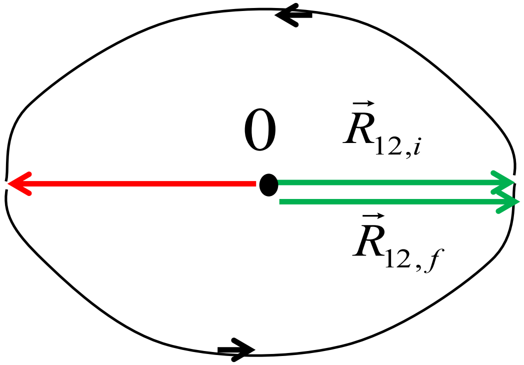




$$\vec{R}_{12,f} = -\vec{R}_{12,i}$$

double exchange $\hat{\sigma}^2$

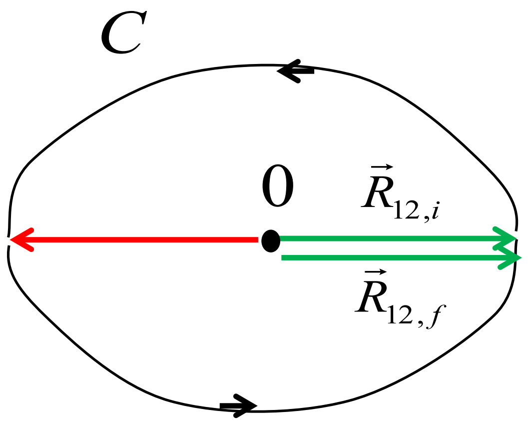




$$\vec{R}_{12,f} = \vec{R}_{12,i}$$

Is $\hat{\sigma}^2$ an identity?

3D case: $\hat{\sigma}^2 = 1$ (identity!)

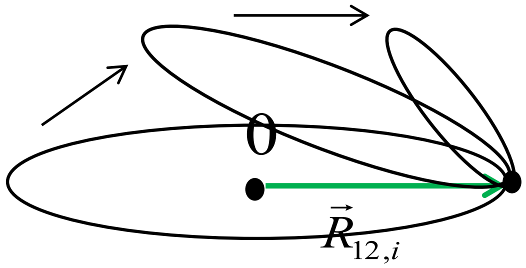


The contour C **can** be deformed

to a point $\vec{R}_{12} = \text{const}\ (= \vec{R}_{12,i})$

(i.e., to the case when nothing happens)

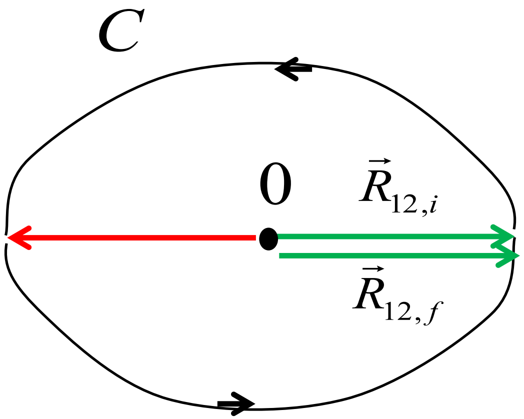
without crossing the origin $\vec{R}_{12} = 0$



Conclusion: in 3D only bosons ($\vartheta = 0$) or fermions ($\vartheta = \pi$)

$$|\psi\rangle \to \pm|\psi\rangle$$

2D case: $\hat{\sigma}^2 \neq 1$



The contour C **cannot** be deformed

to a point $\vec{R}_{12} = \text{const}\ (= \vec{R}_{12,i})$

(i.e., to the case when nothing happens)

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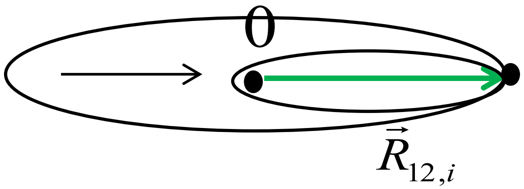


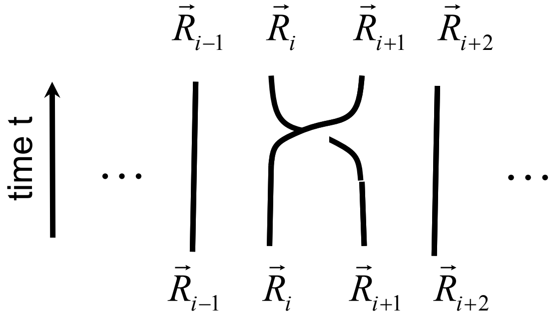
Conclusion: in 2D more possibilities, not only bosons or fermions

Braid group. Abelian and non-Abelian anyons

Particle exchange in 2D: Braid group (for N particles)

Trajectories that wind around starting from initial positions $\vec{R}_1,\ldots,\vec{R}_N$ to final positions $\vec{R}_1,\ldots,\vec{R}_N$ (the same set – identical particles)

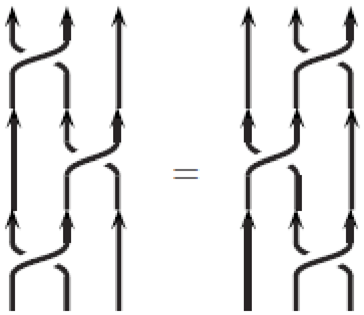
generated by $\hat{\sigma}_i$
(**braiding** of particles i and i+1)



defining relations

$$\hat{\sigma}_i\hat{\sigma}_j = \hat{\sigma}_j\hat{\sigma}_i \quad \text{for } |i-j| \geq 2$$

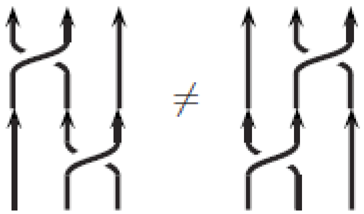
$$\hat{\sigma}_i\hat{\sigma}_{i+1}\hat{\sigma}_i = \hat{\sigma}_{i+1}\hat{\sigma}_i\hat{\sigma}_{i+1}$$



Note:

1. Braid group is infinite dimensional ($\hat{\sigma}^2 \neq 1!$) in contrast to finite-dimensional permutation group ($\hat{p}^2 = 1$)

2. Braid group is non-Abelian $\hat{\sigma}_i\hat{\sigma}_{i+1} \neq \hat{\sigma}_{i+1}\hat{\sigma}_i$



Representations of the braid group: statistics of particles

Elements of the braid group
(trajectories of particles)

$$\Big\Downarrow \text{ representation}$$

Changes of the states under the evolution
(particle statistics)

1. One-dimensional representations: unique (ground) state

2. Higher-dimensional representations: degenerate (ground) state

1. One-dimensional (Abelian) representations

Unique (ground) state $|\psi\rangle$

Transformation under braiding operation $\hat{\sigma}$

$$|\psi\rangle \xrightarrow{\hat{\sigma}} e^{i\vartheta}|\psi\rangle$$

with arbitrary ϑ - Abelian anyons

Examples: 1. bosons ($\vartheta = 0$) and fermions ($\vartheta = \pi$)

2. quasiholes in the at FQHE Laughlin state $\nu = 1/M$

$$\vartheta = \pi / M$$

Example 2. quasiholes in the FQHE Laughlin state $\nu = 1/M$

R. Laughlin, 1983

Trial wave function for N fermions at positions $\vec{r}_i$
with n quasiholes at positions $\vec{R}_\alpha$:

$$z_j = (x_j + iy_j)/l_B \quad Z_\alpha = (X_\alpha + iY_\alpha)/l_B$$

Choice 1

$$\psi_{\frac{1}{M}}(Z_\alpha, z_i) = \underbrace{\prod_{\alpha<\beta}^{n} (Z_\alpha - Z_\beta)^{\frac{1}{M}} e^{\frac{-1}{4M}\sum_{\mu=1}^{n}|Z_\mu|^2}}_{\text{normalization}} \underbrace{\prod_{\gamma=1}^{n}\prod_{i=1}^{N}(Z_\gamma - z_i)}_{\text{quasihole}} \underbrace{\prod_{k<l}^{N}(z_k - z_l)^M e^{-\frac{1}{4}\sum_{j=1}^{N}|z_j|^2}}_{\text{Laughlin w.f.}}$$

$$\int \prod_{i=1}^{N} d^2 z_i \, |\psi_{\frac{1}{M}}(Z_\alpha, z_i)|^2 = 1 + O(e^{-|Z_\alpha - Z_\beta|})$$

Berry's phase = Aharonov-Bohm phase (geometric) $\alpha_g(\text{path}) = \dfrac{e}{M}\dfrac{\Phi_B}{\hbar c}$
of a charge $q = e/M$ encircling flux of $\Phi_B = BA$

Exchanging two quasiholes give a phase of $\vartheta = \pi / M$
from eigenstate transformation (explicit monodromy)

D. Arovas, J.R. Schrieffer,F.Wilczek,1984
R. Laughlin, 1987
B. Blok, X.G. Wen, 1992

Example 2. quasiholes in the FQHE Laughlin state $\nu = 1/M$

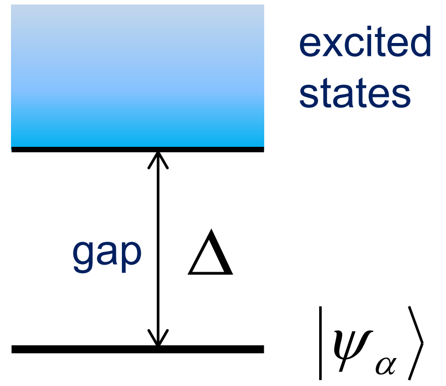
Choice 2 (different "gauge": single-valued)

$$\psi_{\frac{1}{M}}(Z_\alpha, z_i) = \prod_{\alpha<\beta}^{n} \left|Z_\alpha - Z_\beta\right|^{\frac{1}{M}} e^{\frac{-1}{4M}\sum_{\mu=1}^{n}\left|Z_\mu\right|^2} \prod_{\gamma=1}^{n}\prod_{i=1}^{N}(Z_\gamma - z_i)\prod_{k<l}^{N}(z_k - z_l)^M e^{-\frac{1}{4}\sum_{j=1}^{N}\left|z_j\right|^2}$$

Eigenstate transformation (analytic continuation) = trivial

Berry's phase = Aharonov-Bohm phase + statistical angle $\vartheta = \pi / M$

2. Higher-dimensional representations



Degenerate ground state
with an orthonormal basis $|\psi_\alpha\rangle, \alpha = 1, \ldots, g$

Transformation under braiding operation $\hat{\sigma}$

$$|\psi_\alpha\rangle \xrightarrow{\hat{\sigma}} U(\hat{\sigma})_{\alpha\beta} |\psi_\beta\rangle$$

with **matrix** $U(\hat{\sigma}) = P\exp(i \int dt\hat{m})\mathcal{M}$ $\quad (\hat{m})_{\alpha\beta} = i\langle \psi_\alpha | \frac{d}{dt} \psi_\beta \rangle$

Berry matrix $\quad$ explicit monodromy of the w.f.

Particles are non-Abelian anyons if $U(\hat{\sigma}_1)_{\alpha\beta} U(\hat{\sigma}_2)_{\beta\gamma} \neq U(\hat{\sigma}_2)_{\alpha\beta} U(\hat{\sigma}_1)_{\beta\gamma}$
for at least two $\hat{\sigma}_1$ and $\hat{\sigma}_2$

(do not commute!)

Examples: Ising anyons (Majorana fermions, $\nu = 5/2$ qH-state),
Fibonacci anyons

Conditions for non-Abelian anyons:

Robust degeneracy of the ground state:

The degeneracy cannot be lifted by local perturbations
(which are needed, i.e., for braiding)

Degenerate ground states cannot be distinguished
by local measurements

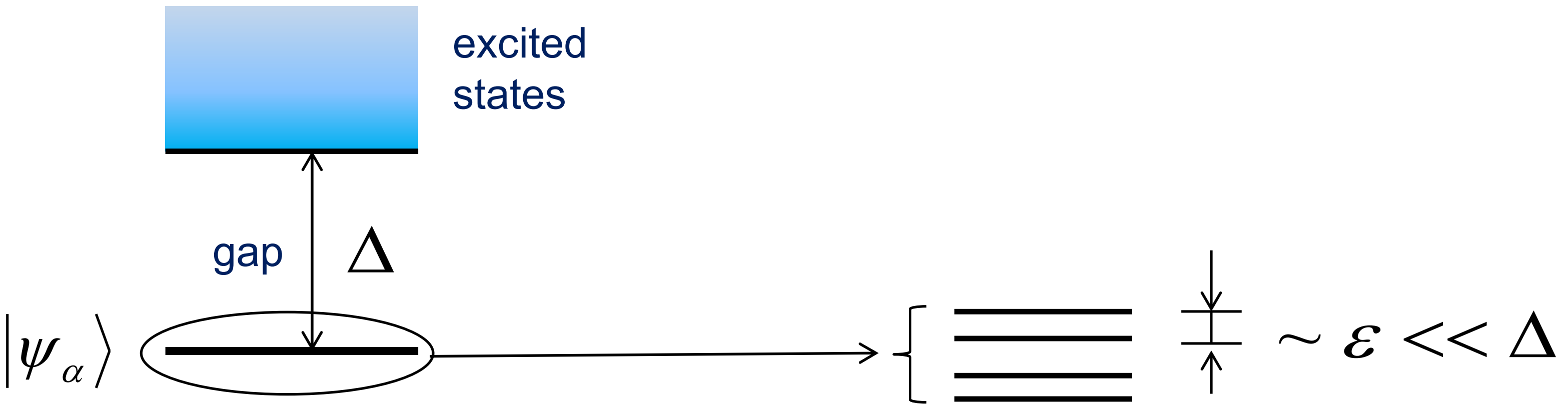
$$\left\langle \psi_\alpha \middle| V_{\mathrm{loc}} \middle| \psi_\beta \right\rangle = C\delta_{\alpha\beta}$$

Braiding of identical particles changes state within the degenerate manifold,
but should not be visible for local observer

Nonlocal measurements: parity measurements (for Majorana fermions), etc.

Require topological states of matter with
topological degeneracy and long-range entanglement

Comment: In real world (finite systems, etc.): GS degeneracy is lifted



Condition on the time of operations:

$$\frac{\hbar}{\Delta} << T << \frac{\hbar}{\varepsilon}$$

slow enough to be adiabatic

fast enough to NOT resolve the GS manifold

Formal introduction of Anyons:

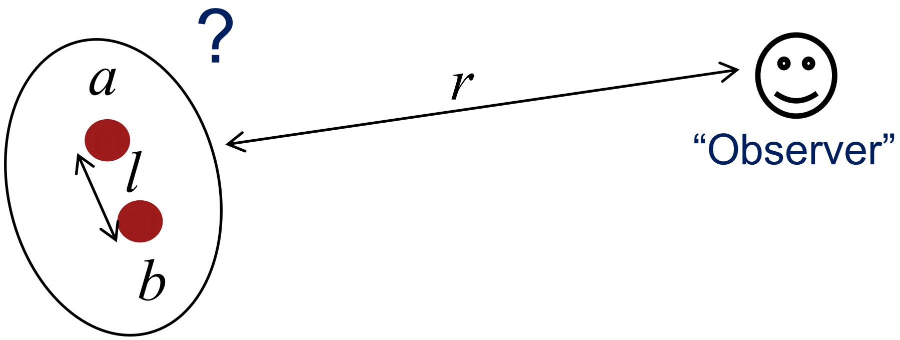
Fusion rules and Hilbert space

Set of particles (anyons) $1, a, b, c \ldots$ 1 - vacuum

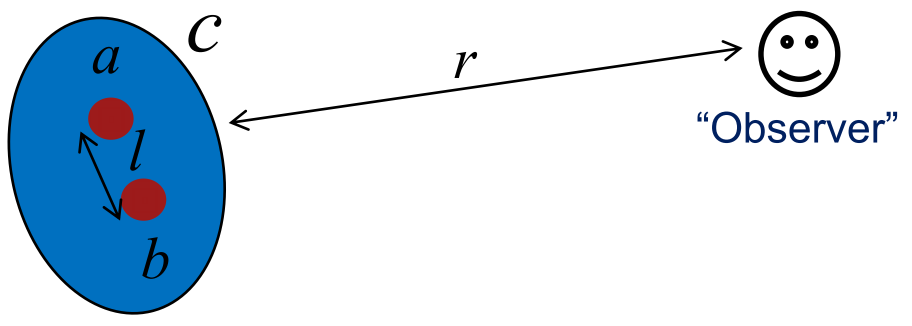
Fusion of anyons

Fusion of two Anyons a, b:

how do they behave as a combined object seen from distances much large than the separation between them $r >> l$

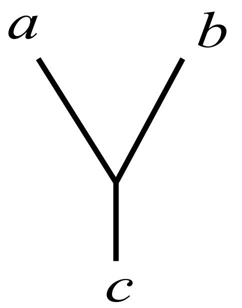


Fusion rules



$$a \times b = \sum_c N_{ab}^c \; c$$

N_{ab}^c - integers



Non-Abelian anyons if $\sum_c N_{ab}^c > 1$ for some a and b

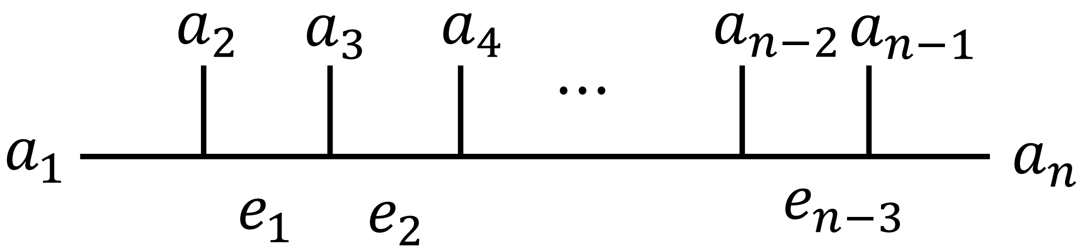
Hilbert space – fusion chains ($N_{ab}^{c} \leq 1$ for simplicity)

(no creation and annihilations operators!)

$\mathcal{H}_n$: n anyons, $a_1, \cdots, a_n$ where $a_1, \cdots, a_{n-1}$ fuse into a_n

$$(\cdots(a_1 \times a_2) \times a_3) \times \cdots) \times a_{n-1} = a_n$$

Basis vectors

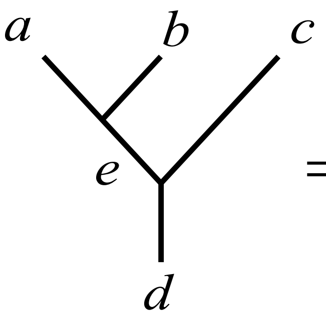


a_1 and a_2 fuse into e_1 , e_1 and a_3 into e_2, …, e_{n-3} and a_{n-1} into e_n

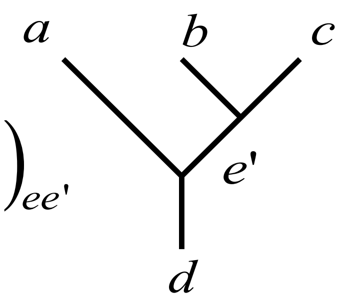
uniquely specified by the intermediate fusion outcomes $e_1, \cdots, e_{n-3}$

Dimension $\quad \dim \mathcal{H}_n = \sum_{e_1 \ldots e_{n-3}} N_{a_1 a_2}^{e_1} \cdots N_{e_{n-3} a_{n-1}}^{e_n}$

Associativity of braiding: $(a \times b) \times c = a \times (b \times c) = d$



$$= \sum_{e'} \left(F_d^{abc} \right)_{ee'}$$



a,b fuse to e;
e,c fuse to d

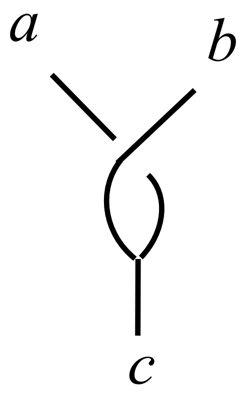
b,c fuse to e';
e',a fuse to d

F-matrix (basis change)

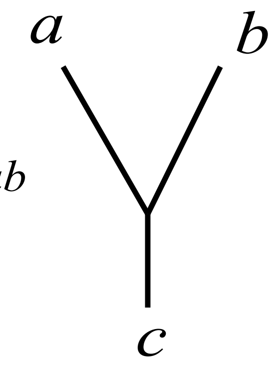
guaranties the invariance of the Hilbert space construction:

Different fusion ordering is equivalent to the change of the basis vectors

Exchange properties of anyons (in a given fusion channel)



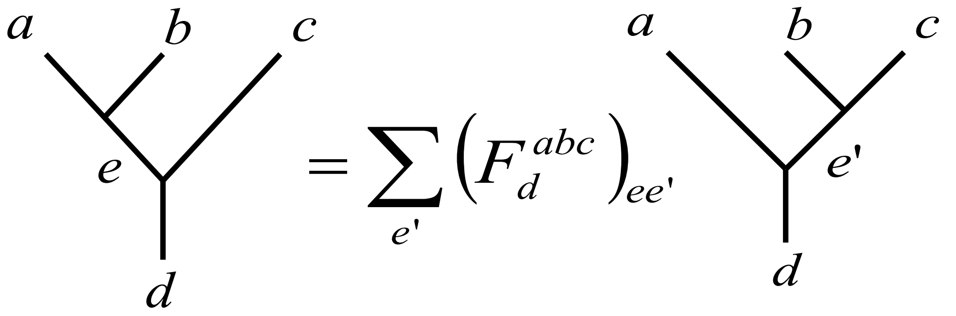
$$= R^{ab}_{c}$$

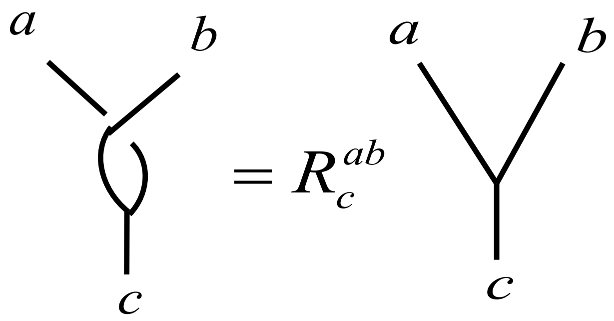


R-matrix

(braiding in a given fusion channel)

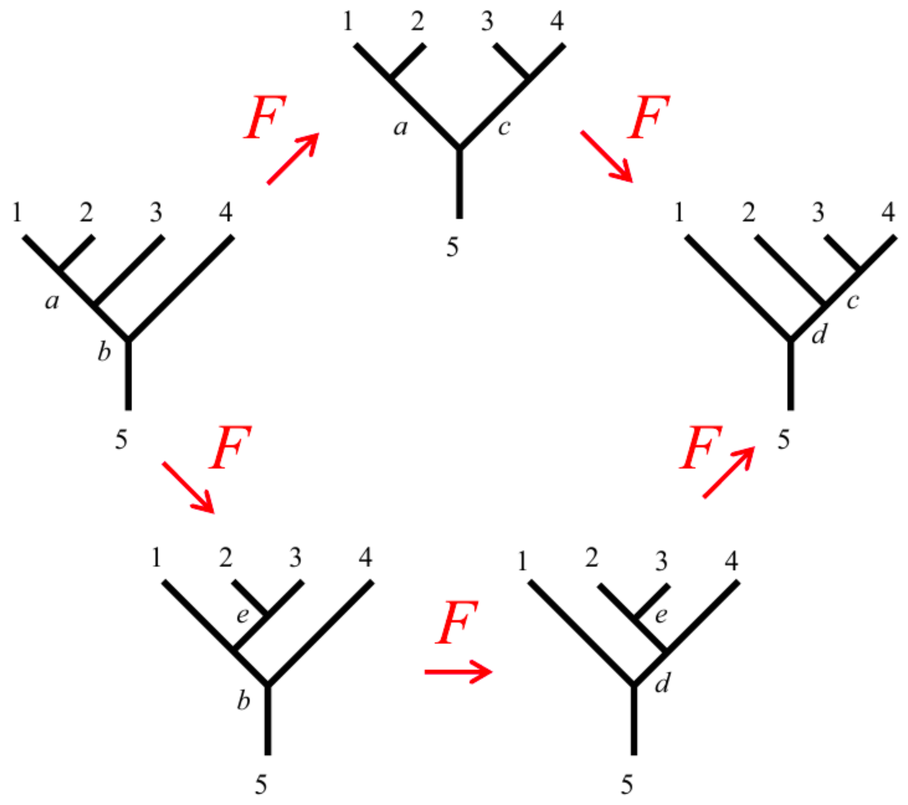
Consistency conditions for F- and R-matrices





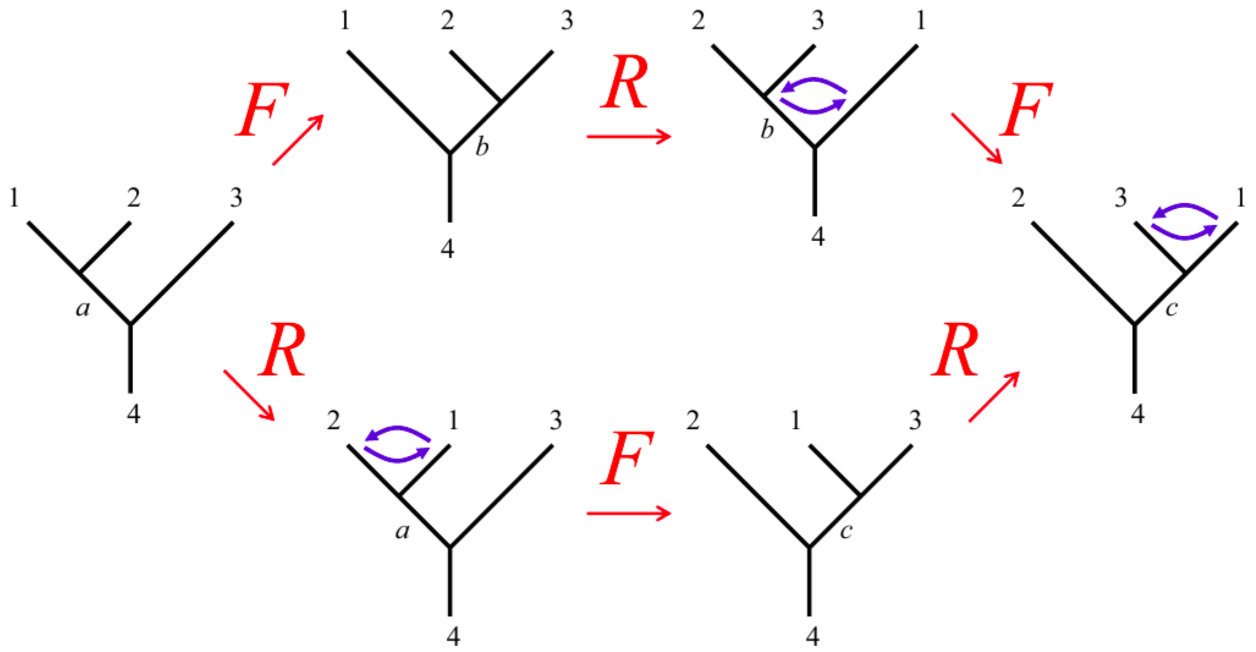
F- and R-matrices satisfy the pentagon and hexagon consistency relations

Pentagon relation



$$\left(F_5^{12c}\right)_{da}\left(F_5^{a34}\right)_{cb}=\sum_e\left(F_d^{234}\right)_{ce}\left(F_5^{1e4}\right)_{db}\left(F_b^{123}\right)_{ea}$$

Hexagon relation



$$\sum_b \left(F_4^{231}\right)_{cd} R_4^{1b} \left(F_4^{123}\right)_{ba} = R_c^{13} \left(F_4^{213}\right)_{ca} R_a^{12}$$

Set of anyons with F- and R-matrices satisfying the pentagon and hexagon equations completely determine an Anyon model.

If no solution exists the hypothetical set of anyons and fusion rule are incompatible with local quantum physics.

Alternative approach: topological field theories

Examples:

1: Fibonacci anyons $1, \tau$

Fusion rules: $\tau \times \tau = 1 + \tau$

$$1 \times \{1, \tau\} = \{1, \tau\}$$

2: Ising anyons $1, \gamma, \psi$

γ - non-Abelian anyon ψ - fermion

Fusion rules: $\gamma \times \gamma = 1 + \psi$ $\gamma \times \psi = \gamma$

$$\psi \times \psi = 1 \qquad 1 \times \{1, \gamma, \psi\} = \{1, \gamma, \psi\}$$

(more in the next lecture)

Hilbert space for $2N$ Majorana fields γ_m ($m=1,\ldots,2N$)

Fusion rules: $\gamma\times\gamma=1+\psi$ $\gamma\times\psi=\gamma$

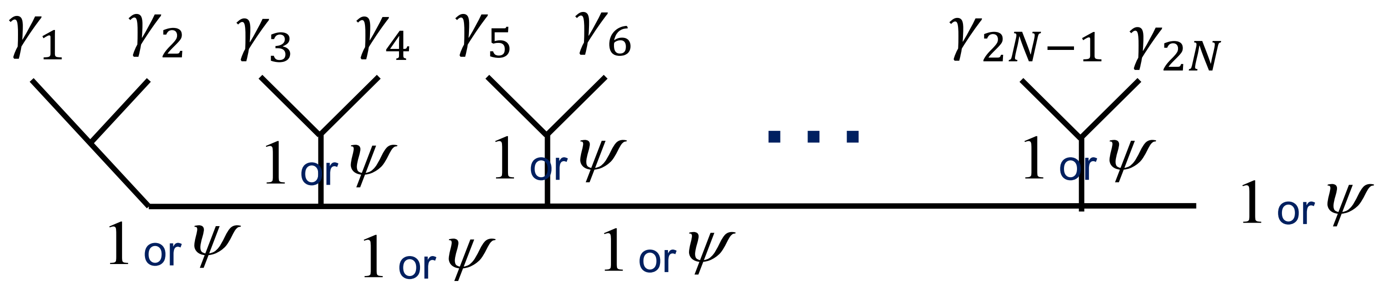
$\psi\times\psi=1$ $1\times\{1,\gamma,\psi\}=\{1,\gamma,\psi\}$

1. Split in N pairs $\gamma_{2j-1},\gamma_{2j}$

2. Specify fusion channels $(1,\psi)_j$ for each pair ($j=1,\ldots,N$)

The fusion tree is now uniquely define. The resulting “charge” is either 1 or ψ depending on the number n_ψ of the ψ fusion outputs:

1 for even n_ψ ψ for odd n_ψ



Equivalent to the Hilbert space for N fermionic modes

End of the Lecture 1

Thank you for your attention!