Introduction to the Physics of Anyons with Majorana Fermions as an Example

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Lecture 1: Appearance of Anyons

Lecture 2: Majorana Fermions as an example of non-Abelian Anyons
Lecture 1: Appearance of Anyons

Exchange and statistics

Why dimension 2?

Braid group. Abelian and no-Abelian anyons

Formal introduction of anyons: Fusion of anyons and anyon Hilbert space
Exchange and statistics
Particle exchange and statistics

Statistics \iff \text{Behavior of the state (wave function) under the exchange of two identical (quasi)particles}

Exchange of two (quasi)particles

\[ \Psi(\vec{r}_1, \vec{r}_2, \ldots) \rightarrow \Psi(\vec{r}_2, \vec{r}_1, \ldots) = ? \Psi(\vec{r}_1, \vec{r}_2, \ldots) \]
Properties of many-body wave functions:

\[ \psi(\vec{R}_1, \vec{R}_2, \ldots; \vec{r}_1, \vec{r}_2, \ldots) \]

\( \vec{R}_i \) positions of quasiparticles
\( \vec{r}_j \) positions of particles

has to be single-valued with respect to particle coordinates \( \vec{r}_j \)

but not necessarily with respect to quasiparticle coordinates \( \vec{R}_i \)
Exchange as adiabatic dynamical evolution

General statements:

1. Adiabatic theorem: States in a (possibly degenerate) energy subspace separated from others by a gap remain in the subspace when the system is changed adiabatically without closing the gap.

2. Change under adiabatic transport = combination of Berry’s phase/matrix and transformation of instantaneous energy eigenstate (explicit monodromy)

3. Change under adiabatic transport is invariant, but Berry’s phase/matrix and eigenstate transformation depend on choice of gauge (and can be shifted from one to the other)
Exchange as adiabatic dynamical evolution

For a unique ground state $|\Psi\rangle$ (single-valued) separated by a gap from excited states

$$
\iff \quad |\psi\rangle \rightarrow e^{i\varphi} |\psi\rangle,
$$

$$
\varphi = -\frac{1}{\hbar} \int dt \, E(t) + \alpha
$$

**Berry phase**

$$
\alpha = i \int dt \langle \psi | \frac{d}{dt} \psi \rangle = \alpha_g (\text{path}) + \mathcal{G}
$$

$single-valued \ w.f.$

$\alpha_g (\text{path})$ **geometrical phase**

$\mathcal{G}$ **(topology of the path) statistical angle** - of interest!

**Exchange statistics**

$$
|\psi\rangle \rightarrow e^{i\mathcal{G}} |\psi\rangle
$$
Why dimension 2?
Constraint on the exchange

exchange $\hat{\sigma}$

$$\vec{R}_{12}$$ relative position

double exchange $\hat{\sigma}^2$

Is $\hat{\sigma}^2$ an identity?
3D case: \( \hat{\sigma}^2 = 1 \) (identity!)

The contour \( C \) can be deformed to a point \( \vec{R}_{12} = \text{const} = (\vec{R}_{12,i}) \) (i.e., to the case when nothing happens) without crossing the origin \( \vec{R}_{12} = 0 \)

Conclusion: in 3D only bosons (\( \vartheta = 0 \)) or fermions (\( \vartheta = \pi \))

\[ |\psi\rangle \rightarrow \pm |\psi\rangle \]
2D case: $\hat{\sigma}^2 \neq 1$

The contour $C$ cannot be deformed to a point $\vec{R}_{12} = \text{const} (= \vec{R}_{12,i})$
(i.e., to the case when nothing happens)
without crossing the origin $\vec{R}_{12} = 0$

Conclusion: in 2D more possibilities, not only bosons or fermions
Braid group. Abelian and non-Abelian anyons
Particle exchange in 2D: Braid group (for N particles)

Trajectories that wind around starting from initial positions \( \vec{R}_1, \ldots, \vec{R}_N \) to final positions \( \vec{R}_1, \ldots, \vec{R}_N \) (the same set – identical particles)

generated by \( \hat{\sigma}_i \) (braiding of particles \( i \) and \( i+1 \))

defining relations

\[
\hat{\sigma}_i \hat{\sigma}_j = \hat{\sigma}_j \hat{\sigma}_i \quad \text{for} \quad |i - j| \geq 2
\]

\[
\hat{\sigma}_i \hat{\sigma}_{i+1} \hat{\sigma}_i = \hat{\sigma}_{i+1} \hat{\sigma}_i \hat{\sigma}_{i+1}
\]

Note:

1. Braid group is infinite dimensional \( (\hat{\sigma}^2 \neq 1!) \) in contrast to finite-dimensional permutation group \( (\hat{p}^2 = 1) \)

2. Braid group is non-Abelian \( \hat{\sigma}_i \hat{\sigma}_{i+1} \neq \hat{\sigma}_{i+1} \hat{\sigma}_i \)
Representations of the braid group: statistics of particles

Elements of the braid group
(trajectories of particles)

representation

Changes of the states under the evolution
(particle statistics)

1. One-dimensional representations: unique (ground) state

2. Higher-dimensional representations: degenerate (ground) state
1. One-dimensional (Abelian) representations

Unique (ground) state \( |\psi\rangle \)

Transformation under braiding operation \( \hat{\sigma} \)

\[ |\psi\rangle \xrightarrow{\hat{\sigma}} e^{i\vartheta} |\psi\rangle \]

with arbitrary \( \vartheta \) - Abelian anyons

Examples: 1. bosons (\( \vartheta = 0 \)) and fermions (\( \vartheta = \pi \))

2. quasiholes in the at FQHE Laughlin state \( \nu = 1/M \)

\( \vartheta = \pi / M \)
Example 2. quasiholes in the FQHE Laughlin state \( \nu = 1/M \)

Trial wave function for \( N \) fermions at positions \( \vec{r}_i \)
with \( n \) quasiholes at positions \( \vec{R}_\alpha : \)

\[
\psi_\frac{1}{M}(Z_\alpha, z_i) = \prod_{\alpha < \beta}^{n} (Z_\alpha - Z_\beta)^{1/M} e^{-\frac{1}{4M} \sum_{\mu=1}^{n} |Z_\mu|^2} \prod_{\gamma=1}^{n} \prod_{i=1}^{N} (Z_\gamma - z_i) \prod_{k<l}^{N} (z_k - z_l)^{M} e^{-\frac{1}{4} \sum_{j=1}^{N} |z_j|^2} \]

(normalization) \quad (quasihole) \quad (Laughlin w.f.)

\[
\int \prod_{i=1}^{N} d^2 z_i \left| \psi_\frac{1}{M}(Z_\alpha, z_i) \right|^2 = 1 + O(e^{-|Z_\alpha - Z_\beta|})
\]

Berry’s phase = Aharonov-Bohm phase (geometric) of a charge \( q = e/M \) encircling flux of \( \Phi_B = BA \)

\[
\alpha_g(\text{path}) = \frac{e}{M} \frac{\Phi_B}{\hbar c}
\]

Exchanging two quasiholes give a phase of \( \partial = \pi/M \)
from eigenstate transformation (explicit monodromy)

D. Arovas, J.R. Schrieffer, F. Wilczek, 1984
R. Laughlin, 1987
B. Blok, X.G. Wen, 1992

R. Laughlin, 1983
Example 2. quasiholes in the FQHE Laughlin state \( \nu = 1/ M \)

Choice 2 (different “gauge”: single-valued)

\[
\psi_{\frac{1}{M}} (Z_\alpha, z_i) = \prod_{\alpha < \beta} |Z_\alpha - Z_\beta| \left( \frac{1}{M} e^{-\frac{1}{4M} \sum_{\mu=1}^{n} |Z_\mu|^2} \prod_{\gamma=1}^{n} \prod_{i=1}^{N} (Z_\gamma - z_i) \prod_{k<l} (z_k - z_l)^{M} e^{-\frac{1}{4} \sum_{j=1}^{N} |z_j|^2} \right)
\]

Eigenstate transformation (analytic continuation) = trivial

Berry’s phase = Aharonov-Bohm phase + statistical angle \( \vartheta = \pi / M \)
2. Higher-dimensional representations

Degenerate ground state with an orthonormal basis $|\psi_\alpha\rangle$, $\alpha = 1, \ldots, g$

Transformation under braiding operation $\hat{\sigma}$

$$|\psi_\alpha\rangle \xrightarrow{\hat{\sigma}} U(\hat{\sigma})_{\alpha\beta} |\psi_\beta\rangle$$

with matrix $U(\hat{\sigma}) = P \exp(i \int dt \hat{m}) \mathcal{M}$

Berry matrix $\langle \hat{m} \rangle_{\alpha\beta} = i \langle \psi_\alpha | \frac{d}{dt} \psi_\beta \rangle$

explicit monodromy of the w.f.

Particles are non-Abelian anyons if

for at least two $\hat{\sigma}_1$ and $\hat{\sigma}_2$

$$U(\hat{\sigma}_1)_{\alpha\beta} U(\hat{\sigma}_2)_{\beta\gamma} \neq U(\hat{\sigma}_2)_{\alpha\beta} U(\hat{\sigma}_1)_{\beta\gamma}$$

(do not commute!)

Examples: Ising anyons (Majorana fermions, $\nu = 5/2$ qH-state), Fibonacci anyons
Conditions for non-Abelian anyons:

Robust degeneracy of the ground state:

The degeneracy cannot be lifted by local perturbations (which are needed, i.e., for braiding)

Degenerate ground states cannot be distinguished by local measurements

\[ \langle \psi_\alpha | V_{\text{loc}} | \psi_\beta \rangle = C \delta_{\alpha\beta} \]

Braiding of identical particles changes state within the degenerate manifold, but should not be visible for local observer

Nonlocal measurements: parity measurements (for Majorana fermions), etc.

Require topological states of matter with topological degeneracy and long-range entanglement
Comment: In real world (finite systems, etc.): GS degeneracy is lifted

\[ \epsilon \ll \eta \ll \Delta \]

\[ \psi_\alpha \] excited states

\[ \Delta \]

\[ \sim \epsilon \ll \Delta \]

Condition on the time of operations:

\[ \frac{\hbar}{\Delta} \ll T \ll \frac{\hbar}{\epsilon} \]

slow enough to be adiabatic

fast enough to NOT resolve the GS manifold
Formal introduction of Anyons:
Fusion rules and Hilbert space
Set of particles (anyons) \(1, a, b, c \ldots\)

1 - vacuum

Fusion of anyons

Fusion of two Anyons \(a, b:\)

how do they behave as a combined object seen from distances much large than the separation between them \(r \gg l\)
Fusion rules

\[ a \times b = \sum_c N_{ab}^c \cdot c \]

\[ N_{ab}^c \text{ - integers} \]

Non-Abelian anyons if \( \sum_c N_{ab}^c > 1 \) for some \( a \) and \( b \)
Hilbert space – fusion chains \( (N_{ab}^c \leq 1 \text{ for simplicity}) \)

(no creation and annihilations operators!)

\[ \mathcal{H}_n : n \text{ anyons, } a_1, \cdots, a_n \text{ where } a_1, \cdots, a_{n-1} \text{ fuse into } a_n \]

\[(\cdots (a_1 \times a_2) \times a_3) \times \cdots ) \times a_{n-1} = a_n \]

Basis vectors

\[ \begin{array}{ccccccc}
  & a_2 & a_3 & a_4 & \cdots & a_{n-2} & a_{n-1} \\
 a_1 & & & & & & \\
 & e_1 & e_2 & & & e_{n-3} & a_n \\
\end{array} \]

\(a_1 \text{ and } a_2 \text{ fuse into } e_1, e_1 \text{ and } a_3 \text{ into } e_2, \cdots, e_{n-3} \text{ and } a_{n-1} \text{ into } e_n\)

uniquely specified by the intermediate fusion outcomes \(e_1, \cdots, e_{n-3}\)

Dimension

\[ \dim \mathcal{H}_n = \sum_{e_1 \cdots e_{n-3}} N_{a_1 a_2 \cdots}^{e_1} N_{e_{n-3} a_{n-1}}^{e_n} \]
Associativity of braiding: \((a \times b) \times c = a \times (b \times c) = d\)

\[
\begin{align*}
\sum_{e'} (F_d^{abc})_{ee'} =
\end{align*}
\]

- a, b fuse to e;
- e, c fuse to d

- b, c fuse to e';
- e', a fuse to d

F-matrix (basis change)

guarantees the invariance of the Hilbert space construction:

Different fusion ordering is equivalent to the change of the basis vectors
Exchange properties of anyons (in a given fusion channel)

\[ R_{c}^{ab} \]

R-matrix
(braiding in a given fusion channel)
Consistency conditions for F- and R-matrices

\[ \sum_{e'} (F_d^{abc})_{ee'} = \sum_{e'} (F_d^{abc})_{ee'} \]

F- and R-matrices satisfy the pentagon and hexagon consistency relations
Pentagon relation

\[
\left( F_{5}^{12c} \right)_{da} \left( F_{5}^{a34} \right)_{cb} = \sum_{e} \left( F_{d}^{234} \right)_{ce} \left( F_{5}^{1e4} \right)_{db} \left( F_{b}^{123} \right)_{ea}
\]
Hexagon relation

\[
\sum_b \left( F_{4}^{\left.231\right)} \right)_{cd} R_{4}^{1b}\left( F_{4}^{\left.123\right)} \right)_{ba} = R_{c}^{13}\left( F_{4}^{\left.213\right)} \right)_{ca} R_{a}^{12}
\]
If no solution exists the hypothetical set of anyons and fusion rule are incompatible with local quantum physics.

Set of anyons with F- and R-matrices satisfying the pentagon and hexagon equations completely determine an Anyon model.

Alternative approach: topological field theories
Examples:

1: Fibonacci anyons 1, \( \tau \)

Fusion rules: \( \tau \times \tau = 1 + \tau \)
\( 1 \times \{1, \tau\} = \{1, \tau\} \)

2: Ising anyons 1, \( \gamma \), \( \psi \)

\( \gamma \) - non-Abelian anyon \quad \( \psi \) - fermion

Fusion rules: \( \gamma \times \gamma = 1 + \psi \) \quad \( \gamma \times \psi = \gamma \)
\( \psi \times \psi = 1 \) \quad \( 1 \times \{1, \gamma, \psi\} = \{1, \gamma, \psi\} \)

(more in the next lecture)
Hilbert space for $2^N$ Majorana fields $\gamma_m$ ($m=1,\ldots,2N$)

Fusion rules: $\gamma \times \gamma = 1 + \psi$, $\gamma \times \psi = \gamma$
$\psi \times \psi = 1$, $1 \times \{1, \gamma, \psi\} = \{1, \gamma, \psi\}$

1. Split in $N$ pairs $\gamma_{2j-1}, \gamma_{2j}$

2. Specify fusion channels $(1, \psi)_j$ for each pair ($j=1,\ldots,N$)

The fusion tree is now uniquely defined. The resulting "charge" is either 1 or $\psi$ depending on the number $n_\psi$ of the $\psi$ fusion outputs:

- 1 for even $n_\psi$
- $\psi$ for odd $n_\psi$

Equivalent to the Hilbert space for $N$ fermionic modes
End of the Lecture 1

Thank you for your attention!