Fractional statistics: A retrospective view

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Brief history (since the Beginning of Time):

- Rudolf Clausius — notion of entropy (1865);
- James Clerk Maxwell — velocity distribution of molecules in an ideal gas (1867);
- Ludwig Boltzmann — generalization of Maxwell's results (1871);
- Josiah Willard Gibbs — paradox related to indistinguishability of particles (1874);
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Satyendranath Bose (1924)
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Wolfgang Pauli — exclusion principle (1925);
Enrico Fermi (1926)
Paul Adrien Maurice Dirac (1926);
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- quantum-mechanical considerations, namely, analysis of wave function properties, commutators, etc.
- methods from the statistical physics, namely, counting of microstates, generalization of entropy, etc.
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- quantum-mechanical considerations, namely, analysis of wave function properties, commutators, etc.
- methods from the statistical physics, namely, counting of microstates, generalization of entropy, etc.

Such a division is quite conditional as both approaches might be linked tightly sometimes.
Statistics generalizations:

- Giovanni Gentile (jr.) (1940) — intermediate statistics, with maximal occupation of a level limited by some finite $s$. 

- Herbert Sydney Green (1953) — parastatistics: parabosons are anti-symmetric wrt permutation of no more than $k$ particles while parafermions are symmetric in such a case.

- Oscar Wallace Greenberg (1990)

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- $q$-deformed commutators or $q$-mutators

$$[a, a^\dagger]_q = aa^\dagger - qa^\dagger a.$$  
(Oscar Wallace Greenberg, 1990)
Jon Manne Leinaas & Jan Myrheim (1977) showed that in a 2D system the wave-function phase can change arbitrarily at the permutation of two particles.
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Frank Wilczek (1982) — term *anyon* (from English ‘any’).
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- F. Duncan M. Haldane (1991) — generalization of the Pauli exclusion principle; interpolation between the Bose and Fermi limits for the number of microstates.
- Yong-Shi Wu (1994) — distribution function in the Haldane statistics (so called fractional exclusion statistics).
- Constantino Tsallis (1988) — nonextensive statistics (systems with nonadditive entropy, i.e. with long-range interactions, 'memory' effects, strongly non-Markovian processes, etc.).
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Quantum-mechanical approach: Anyons

Anyons

Permutation operator $P_{12}^{}$:

$$P_{12}^{} \psi(1,2) = \psi(2,1) = e^{i \pi \alpha} \psi(1,2).$$ (1)

Repeating its action leads to:

$$P_{21}^{} P_{12}^{} \psi(1,2) = e^{2i \pi \alpha} \psi(1,2) = \psi(1,2),$$ (2)

so that $\alpha = 0$ or 1, corresponding to the symmetric (bosons) or anti-symmetric (fermions) wavefunction.

This is linked to the fact that the double permutation is the identity (unit) operation, $P_{21}^{} P_{12}^{} = I$.

However, this appears to be a property of the 3D space, not in lower dimensions.

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Quantum-mechanical approach: Anyons

Permutation operator $P_{12}$:

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Quantum-mechanical approach: Anyons

The permutation operation can be [clasically] thought as a motion of one particle around another. In the 3D space such a closed path can be shrunk to a point, but this is not the case in the 2D space due to the presence of a hard-core:

![Diagram showing double permutation in 2D (left) is not identity as in 3D (right).](image)

**Figure:** Double permutation in 2D (left) is not identity as in 3D (right).
Quantum-mechanical approach: Anyons

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![Diagram showing double permutation in 2D and 3D space]

Figure: Double permutation in 2D (left) is not identity as in 3D (right).

So, in the 2D space double permutation $P_{12}^2 \neq I$ and there is no restrictions for the wavefunction phase.
Braid group

In the 3D space — permutation group $S_N$.
In the 2D space — so called braid group $B_N$. 
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In the 3D space — permutation group $S_N$.

In the 2D space — so called *braid group* $B_N$.

Its generators $\sigma_i$ are:

![Diagram](image)

**Figure:** Action of the operator $\sigma_1$. The braid connecting lower point 1 with upper point 2 goes *over* the other braid.
Braid group

In the 3D space — permutation group $S_N$.

In the 2D space — so called braid group $B_N$.

Its generators $\sigma_i$ are:

![Braid diagram]

**Figure:** Action of the operator $\sigma_1$. The braid connecting lower point 1 with upper point 2 goes *over* the other braid.

Such an operator corresponds to the permutation of particles 1 and 2 (in a defined direction, say, counterclockwise).
Repeating the action of $\sigma_1$ does not lead to the initial configuration: the braid is “plaited”, i.e. topologically $\sigma_1^2 \neq \sigma_1^{-1}\sigma_1 = I$. 
Braid group

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Figure: Double permutation is not the identity operation:

$$\sigma_2 \neq \sigma_1^{-1}\sigma_1 = I.$$
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Figure: Double permutation is not the identity operation: $\sigma_1^2 \neq \sigma_1^{-1}\sigma_1 = I$. 
Braid group

Generations of the braid group $\sigma_i$ satisfy the following relations (Artin relations):

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},$$

(3)

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad \text{if} \quad |i - j| \geq 2.$$  

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Braid group

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\]

Figure: Graphical interpretation of the property $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$. 
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Figure: Graphical interpretation of the property $\sigma_i \sigma_j = \sigma_j \sigma_i$, $|i - j| \geq 2$. 
The first physical model of anyons as a composite of a magnetic flux tube and a charged particle was suggested by Wilczek (1982).
Quantum-mechanical approach: Anyons

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Camino et al. (2005) reported about the observation of the interference pattern corresponding to the Laughlin quasi-particles (elementary excitations with a fractional charge characteristic to the fractional quantum Hall effect; they are anyon candidates).
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Experimental setup to observe anyons in a system consisting of a superconducting film on a semiconductor heterotransition was suggested by Weeks et al. (2007) and another one involving one-dimensional optical lattices was proposed by Keilmann et al. (2011).
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The fractional quantum Hall effect (FQHE) means a precise quantization of the Hall conductance in 2D electron systems at fractional values of $e^2/h$. 
Quantum-mechanical approach: $q$-deformations

Commutation relations for the creation–annihilation operators $a^\dagger$, $a$ can be interpolated between fermions (anticommutator) and bosons (ordinary commutator) introducing $q$-deformed commutators or $q$-mutators. The simplest generalization is the so-called quon algebra (Greenberg 1990; Mohapatra 1990; Greenberg 1991):

$$[a_j, a_k^\dagger]_q = a_j a_k^\dagger - qa_k^\dagger a_j = \delta_{jk},$$  \hspace{1cm} (5)

where $-1 \leq q \leq 1$ provides a continuous interpolation between $q = -1$ (fermions) and $q = 1$ (bosons).
Quantum-mechanical approach: \( q \)-deformations

Introducing the operator for the “number of particles” \( N \), supplementing the \( q \)-mutator by the respective relations:

\[
\begin{align*}
    aa^\dagger - qa^\dagger a &= 1, \\
    [N, a] &= -a, \\
    [N, a^\dagger] &= a^\dagger,
\end{align*}
\]  

one obtains a modified algebra for new operators

\[
\begin{align*}
    c &= q^{-\lambda N/2} a, \\
    c^\dagger &= a^\dagger q^{-\lambda N/2},
\end{align*}
\]  

where \( \lambda \) is a real number. In this new algebra the commutation relation is:

\[
cc^\dagger - q^{1-\lambda} c^\dagger c = q^{-\lambda N},
\]  

and usually one takes \( \lambda = 1 \) or \( \lambda = 1/2 \).
Quantum-mechanical approach: $q$-deformations

Algebra of $q$-bosonic operators:

\[ aa^\dagger - qa^\dagger a = 1, \quad [a, a] = [a^\dagger, a^\dagger] = 0, \quad (9) \]
\[ [N, a^\dagger] = a^\dagger, \quad [N, a] = -a. \quad (10) \]
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Orthonormalized set of eigenstates is given by:

$$|n\rangle = \frac{1}{\sqrt{[n]_q!}} (a^\dagger)^n |0\rangle, \quad a|0\rangle = 0.$$  \hspace{1cm} (11)
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where the $q$-factorial

\[
[n]_q! = [n]_q [n-1]_q \ldots [1]_q \quad (12)
\]

\[
= (q^{n-1} + \ldots q + 1)(q^{n-2} + \ldots q + 1) \ldots (q + 1)1;
\]

\[
[0]_q! = 1
\]

is defined via so-called $q$-numbers:

\[
[n]_q = \frac{q^n - 1}{q - 1}, \quad [n]_1 = n. \quad (13)
\]
These operators act on eigenstates as follows:

\[ a^\dagger |n\rangle = \sqrt{[n + 1]_q} |n + 1\rangle, \]
\[ a |n\rangle = \sqrt{[n]_q} |n - 1\rangle, \]
\[ N |n\rangle = n |n\rangle, \]

while

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\]

It can be shown that in the grand canonical ensemble occupation numbers for the \( i \)th level with energy \( \varepsilon_i \) are:

\[
n_i = \frac{1}{\ln q} \ln \left( \frac{z^{-1} e^{\beta \varepsilon_i} - 1}{z^{-1} e^{\beta \varepsilon_i} - q} \right),
\]

where \( z = e^{\beta \mu} \) is fugacity, \( \mu \) is chemical potential, and \( \beta = 1/T \).
R.-Monteiro *et al.* (1993) considered the following $q$-bosonic algebra:

\[
\begin{align*}
    a a^\dagger - q a^\dagger a &= q^{-N}, \\
    [N, a^\dagger] &= a^\dagger, \\
    [N, a] &= -a,
\end{align*}
\]

and for the Hamiltonian

\[
H = \hbar \omega A^\dagger A, \quad A^\dagger = a^\dagger q^{N/2}, \quad A = q^{N/2} a,
\]

in the limit of $q \to \infty$ obtained a fermion-like expression for occupation numbers:

\[
\langle N \rangle \simeq \frac{1 + 2 e^{-\hbar \omega \beta q^2}}{1 + e^{\hbar \omega \beta} + e^{-\hbar \omega \beta q^2}}.
\]
Quantum-mechanical approach: $q$-deformations

With a complex parameter $q$ on the unit circle $|q| = 1,$
Quantum-mechanical approach: $q$-deformations

With a complex parameter $q$ on the unit circle $|q| = 1$, Dutt et al. (1994) considered relations between the creation and annihilation operators of $q$-fermions:

$$f_q f_q^\dagger + q^\frac{1}{2} f_q^\dagger f_q = q^{-\frac{1}{2}} N_f,$$

(22)

$$[N_f, f_q] = -f_q, \quad [N_f, f_q^\dagger] = f_q^\dagger.$$

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Yang et al. (1998) suggested the following variant:

$$a_q a_q^\dagger - qa_q^\dagger a_q = 1, \quad a_q^* a_q^\dagger - q^* a_q^\dagger a_q^* = 1. \quad a_q = (a_q^*)^\dagger, \quad (24)$$

where $q = e^{2\pi i/(s+1)}$ is the $(s+1)$th root of unity.
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where $q = e^{2\pi i/(s+1)}$ is the $(s+1)$th root of unity.

As with $f_q$, these operators have the nilpotence property:

$$(a_q)^n = (a_q^*)^n = (a_q^\dagger)^n = (a_q^{\dagger*})^n = 0, \quad \text{if } n \geq s + 1.$$  \hspace{1cm} (25)
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\]

So, such an algebra corresponds to limiting the occupation of a quantum state by some $s$. 
Quantum-mechanical approach: parastatistics

Just in a glance:

\[ [a, a^\dagger] = 1 \quad \text{— bosons;} \quad \{a, a^\dagger\} = 1 \quad \text{— fermions;} \]
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\[ [a^\dagger, a], a^\dagger] = 2a^\dagger \quad \text{— parafermions. . .} \]
Fractional (intermediate) statistics can be obtained by means of interpolation between expressions for the number of microstates $W_i$ in bosonic (B) and fermionic (F) limits:

\begin{align}
W_i^B &= \frac{(G_i + N_i - 1)!}{N_i! (G_i - 1)!}, \\
W_i^F &= \frac{G_i!}{N_i! (G_i - N_i)!},
\end{align}  \tag{26}

where $G_i$ is the degeneration of the $i$th level, and $N_i$ is the number of particles on this level.
Statistical-mechanical generalizations

Fractional (intermediate) statistics can be obtained by means of interpolation between expressions for the number of microstates $W_i$ in bosonic (B) and fermionic (F) limits:

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W_i^B = \frac{(G_i + N_i - 1)!}{N_i! (G_i - 1)!}, \quad W_i^F = \frac{G_i!}{N_i! (G_i - N_i)!},
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(26)

where $G_i$ is the degeneration of the $i$th level, and $N_i$ is the number of particles on this level.

To find the occupation number $n_i = N_i/G_i$ (distribution function) corresponding to a certain type of statistics, the following method might be applied. Entropy $S$ of the system is linked to the number of microstates $W$ via:

\[
S = \ln W, \quad \text{where} \quad W = \prod_i W_i(N_i).
\]

(27)
Statistical-mechanical generalizations

Expression for $n_i$ can be obtained by finding an extremum of this functional at additional constraints fixing the number of particles in the system $N = \sum_i N_i$, \( \text{(28)} \) and the total energy $E = \sum_i \varepsilon_i N_i$, \( \text{(29)} \) where $\varepsilon_i$ is the energy of the $i$th level.

Thus, we face the conditional extremum problem:

$$\delta S + \alpha \delta N - \beta \delta E = 0,$$

\( \text{(30)} \) where variations are wrt $N_i$ and Lagrange multipliers are linked to temperature $T$ and chemical potential $\mu$:

$$\alpha = \frac{\mu}{T}, \quad \beta = \frac{1}{T}.$$
Expression for $n_i$ can be obtained by finding an extremum of this functional at additional constraints fixing the number of particles in the system

$$N = \sum_i N_i,$$  \hspace{1cm} (28)

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where variations are wrt $N_i$ and Lagrange multipliers are linked to temperature $T$ and chemical potential $\mu$: $\alpha = \mu / T$, $\beta = 1 / T$. 
Gentile statistics

Let us postulate an intermediate distribution, in which the maximal occupation of a level is limited by a finite number $s$. Such a statistics is known as the Gentile statistics (Gentile 1940). It is clear that the $s = 1$ limit corresponds to the Fermi distribution, and the $s = \infty$ describes the Bose distribution.

Expression for occupation numbers in the Gentile statistics can be obtained using the expression for the number of possibilities to distribute particles over all possible energy levels in the form:

$$W = \prod_i \frac{G_i!}{n_i(0)!n_i(1)! \ldots n_i(s)!},$$

where

$$G_i = \sum_{j=0}^{s} n_i(j)$$

is the weighting coefficient of the $i$th state.
Gentile statistics

The number of particles as the $i$th level is

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We thus have variational problem (30) with the number of particles and energy given by (28) and (29), respectively. Solving this problem, one derives the occupation numbers:

$$n_i^G = \frac{1}{z^{-1}e^{\varepsilon_i/T} - 1} - \frac{s + 1}{z^{-(s+1)}e^{(s+1)\varepsilon_i/T} - 1},$$  \hspace{1cm} (33)$$

where $z = e^{\mu/T}$ is fugacity. After a simple exercise we do obtain the Fermi distribution at $s = 1$ and the Bose distribution at $s = \infty$. 

Gentile statistics

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Gentile statistics

High-temperature behavior is similar to that in the Bose-statistics; in the low-temperature limit similarities with the Fermi-statistics are evident, in particular, there exists an analog of the Fermi level $\varepsilon_G$. \[pVNT = \sum_{j=1}^{\infty} \left( N_A D, V T_D / d \right)_{j-1} b_G j, (34)\]

where $A_{D, d}$ is a constant, and virial coefficients $b_G j$ are linked to that of the Bose-system $b_B j$:

\[b_G j = b_B j, \text{ if } j \leq s,\]

\[b_G s + 1 = b_B s + 1 + s(s + 1) D / d.\] (35)
Gentile statistics

High-temperature behavior is similar to that in the Bose-statistics; in the low-temperature limit similarities with the Fermi-statistics are evident, in particular, there exists an analog of the Fermi level $\varepsilon_G$.

Equation of state of an ideal $D$-dimensional gas of $N$ particles with the dispersion law $\varepsilon_p = a p^d$ obeying the Gentile statistics is given by:

$$\frac{pV}{NT} = \sum_{j=1}^{\infty} \left( \frac{N}{A_{D,d}VT^{D/d}} \right)^{j-1} b_j^G,$$

where $A_{D,d}$ is a constant, and virial coefficients $b_j^G$ are linked to that of the Bose-system $b_j^B$:

$$b_j^G = b_j^B, \quad \text{if } j \leq s,$$

$$b_{s+1}^G = b_{s+1}^B + \frac{s}{(s + 1)^{D/d}}. \quad (35)$$
In the high-temperature limit, the heat capacity of such a system $C_V$ is also linked to that of the Bose-gas $C_V^B$:

$$\frac{C_V}{N} = \frac{C_V^B}{N} - \frac{1}{(s+1)^{D/d}} \frac{Ds}{d} \left( \frac{Ds}{d} - 1 \right) \left( \frac{N}{A_{D,d} VT^{D/d}} \right)^s + \ldots .$$
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Gentile statistics was used to solve the problem of restricted partitions in number theory (Srivatsan et al. 2006).
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One can show that with a special choice of the statistics parameter $s$ a finite Bose-system can be modeled (Rovenchak 2009).
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One can show that with a special choice of the statistics parameter $s$ a finite Bose-system can be modeled (Rovenchak 2009).

A certain (incomplete) equivalence of the Gentile statistics and the anyonic statistics was established by Shen et al. (2010).
Haldane–Wu statistics

Haldane (1991) proposed to introduce the parameter of statistical interaction

\[ g = -\frac{d_{N+\Delta N} - d_N}{\Delta N}, \quad (36) \]

where \( d_N \) is the dimensionality of the space of single-particle states for the system of \( N \) particles provided that coordinates of the remaining \( N - 1 \) particles are fixed.
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It is clear that \(g = 1\) corresponds to fermions (adding one particle removes one state according the Pauli principle) and \(g = 0\) corresponds to bosons (no limitations for state occupation). In fact, Haldane’s proposal consists in postulating a certain generalized Pauli principle concerning several states.
Haldane–Wu statistics

The number of microstates $W_i$ can be interpolated using the following formula (Wu 1994):

$$W_i = \frac{[G_i + (N_i - 1)(1 - g)]!}{N_i! [G_i - gN_i - (1 - g)]!}$$

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reducing to $W_i^B$ in the limit of $g = 0$ and to $W_i^F$ as $g = 1$, respectively.
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Mean occupation numbers $n_i = N_i/G_i$ are given by:

$$n_i = \frac{1}{w \left( e^{(\varepsilon_i - \mu)/T} \right) + g},$$ (38)

where the function $w(x)$ solves such a transcendental equation

$$w^g(x) [1 + w(x)]^{1-g} = x \equiv e^{(\varepsilon_i - \mu)/T}.$$ (39)
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where the function $w(x)$ solves such a transcendental equation

$$w^g(x) [1 + w(x)]^{1-g} = x \equiv e^{(\varepsilon_i - \mu) / T}.$$  \hspace{1cm} (39)

At $g = 0$ one obtains $w(x) = x - 1$ (Bose distribution), and at $g = 1$ the function $w(x) = x$ leading to the Fermi distribution.
Haldane–Wu statistics

It is quite straightforward to show that at $T = 0$ the behavior of occupation numbers resembles the Fermi statistics:

\[
n_i = \begin{cases} 
\frac{1}{g}, & \text{if } \varepsilon_i < \mu_0, \\
0, & \text{if } \varepsilon_i > \mu_0, 
\end{cases}
\]  

(40)

where $\mu_0$ is an analog of the Fermi energy.
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where $\mu_0$ is an analog of the Fermi energy.

Eq. (39) can be solved analytically for some values of $g$ beyond the Bose and Fermi limits, namely: $g = \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4, \frac{2}{3}, \frac{3}{2}, \frac{3}{4}, \frac{4}{3}$. The simplest result is obtained for $g = 1/2$ corresponding to so called semions:

$$ n_i = \frac{1}{\sqrt{1/4 + e^{2(\varepsilon_i - \mu)/T}}} \quad \text{(41)} $$
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(41)

Generally, for particles obeying the Haldane–Wu statistics the terms ‘excluson’) or ‘$g$-on’ are used.
Haldane–Wu statistics

It appears that interacting fermions in the Calogero–Sutherland model with the Hamiltonian ($\hbar = m = 1$)

$$H = \sum_{i=1}^{N} \left( -\frac{1}{2} \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} \omega^2 x_i^2 \right) + \sum_{1 \leq i < j \leq N} \frac{\lambda}{(x_i - x_j)^2}, \quad \lambda = \frac{g(g - 1)}{2},$$

correspond the the ideal exclusion gas (Murthy & Shankar 1994).
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correspond the the ideal exclusion gas (Murthy & Shankar 1994). This statistics also can be applied to describe 2D electron gas with short-range interactions (Bhaduri et al. 1996). Three-particle Calogero model with \(-1/4 < \lambda < 0\) can emulate the anyonic statistics (Sree Ranjani et al. 2009).
Haldane–Wu statistics

Equation of state of a 2D ideal gas of $N$ particles with dispersion $\varepsilon_k = \hbar^2 k^2 / 2m$ in the limit of $e^{\mu/T} \ll 1$, when the function $w(x) = x + g - 1$ (Wu 1994):

$$\frac{p}{T} = \rho_2 \left( 1 + \frac{2g - 1}{4} \rho_2 \lambda^2 \right),$$

where $p$ is pressure, $T$ is temperature, $\rho_2 = N/V_2$ is the 2D density. This is nothing but the virial expansion.
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“Statistical interaction” is repulsive at $g > 1/2$ and attractive at $g < 1/2$.
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“Statistical interaction” is repulsive at $g > 1/2$ and attractive at $g < 1/2$.

High-temperature expansion of the specific heat of $D$-dim. ideal exclusion gas with dispersion $\varepsilon_p = ap^s$:

$$\frac{C_V}{N} = \frac{D}{s} \left[1 + \frac{g - 1/2}{2^{D/s}} \frac{\rho_D}{A_{D,s} T^{D/s}} \left(1 - \frac{D}{s}\right) + \ldots\right],$$

where $\rho_D$ is the $D$-dim density, $A_{D,s}$ is some constant.
Haldane–Wu statistics

Thermodynamics of the ideal Haldane–Wu gas was studied by several groups (Isakov et al. 1996; Joyce et al. 1996).
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Anghel et al. (2012) used the fractional exclusion statistics to calculate thermodynamic properties of the relativistic nuclear matter.
Polychronakos statistics

Polychronakos (1996): let the first particle in a system can occupy one of $G$ states, the second particle has $(G - \gamma)$ states to choose, the third one has $(G - 2\gamma)$ states to choose, etc.

Total number of microstates:

$$W = \prod_{i} G_{i} \left( G_{i} - \gamma \right) \left( G_{i} - 2\gamma \right) \ldots \left( G_{i} - (N_{i} - 1)\gamma \right) N_{i}!.$$  

(44)

or

$$W_{i} = \gamma_{i} N_{i} \left( \frac{G_{i}}{\gamma_{i}} \right)! \frac{N_{i}!}{\left( \frac{G_{i}}{\gamma_{i}} - N_{i} \right)!}.$$  

(45)

One can show in a standard way that mean occupation numbers $n_{i} = N_{i} / G_{i}$ in this statistics are given by a simple expression:

$$n_{i} = 1 \ e^{\left( \varepsilon_{i} - \mu \right) / T + \gamma}.$$  

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$$n_i = \frac{1}{e^{(\varepsilon_i - \mu)/T} + \gamma}. \quad (46)$$
Polychronakos statistics

Acharya & Narayana Swamy (1994) considered the abovementioned expression as a simple variant of statistical-mechanical description of anyons with a correct limiting behavior in both fermionic and bosonic limits.
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Virial coefficients of the $D$-dim. gas with dispersion $\varepsilon_p = a p^\gamma$ in the Polychronakos statistics:

$$b^P_j (\gamma) = |\gamma|^{j-1} b^B_j, \quad \text{if} \quad \gamma < 0,$$

$$b^P_j (\gamma) = \gamma^{j-1} b^F_j, \quad \text{if} \quad \gamma > 0,$$

where $b^B_j, b^F_j$ is the $j$th virial coefficient of the Bose- or Fermi-system, respectively.
Polychronakos statistics

Distribution function (46) of the Polychronakos statistics can occur in the context of $q$-deformed algebras.
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Mirza & Mohammadzadeh (2010) studied so called thermodynamic geometry of several fractional statistics types and reported a phenomenon similar to the Bose-condensation in a gas obeying the Polychronakos statistics.
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Distribution function (46) of the Polychronakos statistics can occur in the context of $q$-deformed algebras. Mirza & Mohammadzadeh (2010) studied so called thermodynamic geometry of several fractional statistics types and reported a phenomenon similar to the Bose-condensation in a gas obeying the Polychronakos statistics. Zare et al. (2012) considered the Bose-gas on a stretched horizon of Schwarzschild and Kerr black holes using the Polychronakos statistics to model interactions in the graviton gas.
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With the parameter $\gamma$ on the unit circle $\gamma = e^{i\pi \nu}$, $\nu = 0 \div 1$, it is possible to show that in the bosonic limit such a statistics corresponds to the Bose-gas with a spectrum containing a small dissipative part $\epsilon_p = \epsilon_p + i \kappa_p$, which is linked to the statistics parameter $\nu$: $\kappa_p \simeq \pi \nu T \epsilon_p$ (Rovenchak 2012).
Nonextensive statistics

Traditionally, entropy is defined as the logarithm of the number of microstates:

$$S = \ln W,$$  \hspace{1cm} (48)

It is additive:

$$S(A + B) = S(A) + S(B),$$ \hspace{1cm} (49)

where $A$ and $B$ denote subsystems.

Boltzmann–Gibbs entropy can be expressed via probabilities $p_j$ of the $j$th state:

$$S = - \sum_j p_j \ln p_j.$$ \hspace{1cm} (50)
Nonextensive statistics

Additivity of the entropy can be violated for various systems, including:

- fractal structures;
- systems with long-range interactions;
- essentially non-Markovian processes (systems with “memory”);
- such approaches are applicable in social sciences and humanities (models of financial markets, linguistic laws, etc.).

Nonextensive Tsallis statistics (1988), generalized entropy:

\[ S_q = \frac{1}{q-1} (1 - W \sum_{n=1}^{\infty} p_n^q) \]

\[ W \sum_{n=1}^{\infty} p_n = 1 \]

\[ q \in \mathbb{R} \]
Nonextensive statistics

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Nonextensive Tsallis statistics (1988), generalized entropy:

\[
S_q = \frac{1}{q-1} \left( 1 - \sum_{n=1}^{W} p_n^q \right), \quad \sum_{n=1}^{W} p_n = 1. \quad q \in \mathbb{R} \tag{51}
\]
Nonextensive statistics

In the $q \to 1$ limit, the Boltzmann–Gibbs entropy is obtained:

$$p_n^{q-1} = e^{(q-1) \ln p_n} \approx 1 + (q - 1) \ln p_n$$

and entropy becomes

$$S_q = \frac{1}{q-1} \left( 1 - \sum_{n=1}^{W} p_n^q \right) = \ldots = - \sum_{n=1}^{W} p_n \ln p_n,$$

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coinciding with the expected result.

New condition of additivity (extensivity):

$$S_q(A + B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B), \quad (52)$$

that is, $S_q$ is a nonextensive/nonadditive quantity. The $q$ index is in fact a nonextensivity measure.
Nonextensive statistics

As the ordinary entropy, $S_q$ reaches maximum at equal probabilities $p_n = 1/W \ \forall n$ (so called Laplace principle):

$$S_q = \frac{W^{1-q} - 1}{1 - q}. \quad (53)$$

In the limit of $q \to 1$ this leads to the known Boltzmann relation $S = \ln W$. Using the $q$-logarithm $\ln_q x \equiv \frac{x^{1-q} - 1}{1 - q}$, one can write the Tsallis entropy in a Boltzmann-like form $S_q = \ln_q W$. \quad (54)
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Using the $q$-logarithm

$$\ln_q x \equiv \frac{x^{1-q} - 1}{1 - q}, \quad \ln_1 x = \ln x,$$  

(54)

one can write the Tsallis entropy in a Boltzmann-like form

$$S_q = \ln_q W.$$  

(55)
Nonextensive statistics

The $q$-exponential (Tsallis $q$-exponential) is inverse to the $q$-logarithm:

$$
\exp_q(x) = [1 + (1 - q)x]^{1/(1-q)},
$$

becoming an ordinary exponential as $q \to 1$. 

\[ (56) \]
Nonextensive statistics

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$$\exp_q(x) = [1 + (1 - q)x]^{1/(1-q)}, \quad (56)$$

becoming an ordinary exponential as $q \to 1$.

Other $q$-exponentials appearing in some other problems involving in particular $q$-deformed commutators, can be defined as follows:

$$\exp_q x \equiv e_q^x = \sum_{j=0}^{\infty} \frac{x^j}{[j]_q!}, \quad \text{Exp}_q x \equiv E_q^x = \sum_{j=0}^{\infty} q^{j(j-1)/2} \frac{x^j}{[j]_q!},$$

where $[j]_q!$ is the $q$-factorial. It is clear that

$$e_1^x = E_1^x = e^x. \quad (57)$$
Nonextensive statistics

Applying the standard method of Lagrange multipliers, one can obtain for probabilities $p_n$ the Gibbs-like expression:

$$ p_n = \frac{1}{Z_q} \exp_q \left( -\frac{\varepsilon_n}{T} \right), $$

(58)

Additional coefficient in the relation between $T$ and $\beta$ complicates the description of systems using the Tsallis entropy.

Some modifications of the described approach exist, known as statistics of (Tsallis–)Mendez–Plastino (1998), Curado(–Tsallis) (1991), Bashkirov (2006), etc.
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$$T = \frac{q}{Z_q^{q-1}} \frac{1}{\beta}, \quad \text{not just} \quad T = \frac{1}{\beta}.$$
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$$p_n = \frac{1}{Z_q} \exp_q \left( -\frac{\varepsilon_n}{T} \right),$$  \hspace{2cm} (58)$$

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q-generalizations of the Fermi-Dirac and Bose–Einstein distributions are also known:

\[
q_i = \frac{1}{\left\{1 + (q - 1)\beta(\varepsilon_i - \mu)\right\}^{\frac{1}{q-1}} \pm 1}
\]  

\[
= \frac{1}{\exp_q[\beta(\varepsilon_i - \mu)] \pm 1}.
\]
Nonextensive statistics

In the so-called incomplete information theory, the following incomplete normalization ad respective entropy are considered:

\[ \sum_{i=1}^{W} p_i^q = 1, \quad S_q = - \sum_{i=1}^{W} p_i^q \ln p_i. \]  

(61)
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\]

Occupation numbers of the Bose- and Fermi-like excitations in the so-called extensive incomplete statistics are:

\[
n_i = \frac{1}{e^{q(\varepsilon_i - \mu)/T} \pm 1}. \tag{62}
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Nonextensive statistics

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\[ n_i = \frac{1}{e^{q(\varepsilon_i - \mu)/T} \pm 1}. \] (62)

Nonextensive analogs are obtained is the fashion similar to the Tsallis statistics:

\[ n_i = \frac{1}{\{1 + (q - 1)\beta(\varepsilon_i - \mu)\}^{q/\beta - 1} \pm 1} = \frac{1}{\exp_q[q\beta(\varepsilon_i - \mu)] \pm 1}. \]
Virial expansion for the equation of state of a 2D system reads:

\[
\frac{p}{T} = \rho_2 \left( 1 + b_2 \rho_2 \lambda^2 + b_3 (\rho_2 \lambda^2)^2 + \ldots \right),
\]  

(63)

where \( p \) is pressure, \( T \) is absolute temperature, \( \rho_2 = N/V_2 \) is a 2D density (concentration), and

\[
\lambda = \left( \frac{2\pi \hbar^2}{mT} \right)^{1/2}
\]  

(64)

is the thermal de Broglie wavelength. Factors \( b_j \) are dimensionless \( j \)th virial coefficients.
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\]  

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is the thermal de Broglie wavelength. Factors \( b_j \) are dimensionless \( j \)th virial coefficients.

The second virial coefficients of the Bose- and Fermi-systems:

\[
b_2^F = +\frac{1}{4}, \quad b_2^B = -\frac{1}{4}.
\]  

(65)
The second virial coefficient of the ideal anyon gas:

\[ b_2(\alpha) = -\frac{1}{4}(1 - 4\alpha + 2\alpha^2). \] (66)
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For higher virial coefficients exact analytical results are not known due to the complexity of the problem of \( N \geq 3 \) anyons.
The second virial coefficient of the ideal anyon gas:

$$b_2(\alpha) = -\frac{1}{4}(1 - 4\alpha + 2\alpha^2).$$  \hspace{1cm} (66)

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Exact symmetry relation for the third virial coefficient:

$$b_3(\alpha) = b_3(1 - \alpha).$$  \hspace{1cm} (67)
Expressions for virial coefficients can be applied to establish correspondences between anyonic statistics and other types of fractional statistics.
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Ideal 2D gas obeying the Gentile statistics with maximal occupation $s \geq 2$ has the 2nd virial coefficient

$$b_2^G = b_2^B = -\frac{1}{4},$$

which is impossible to link with that of anyons $b_2(\alpha) = -\frac{1}{4}(1 - 4\alpha + 2\alpha^2)$, except for the trivial case $\alpha = 0$. 


The 2nd virial coefficient of the ideal 2D Haldane–Wu gas equals

\[ b_{2}^{\text{HW}} = \frac{1}{4}(2g - 1). \]  

(69)
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Comparing it to that of anyons, the following relation between the parameters of statistics \( g \) and \( \alpha \) is obtained:

\[ g = 2\alpha - \alpha^2. \]  

(70)
The 2nd virial coefficient of the ideal 2D Polychronakos gas

\[ b_2^P = -\frac{1}{4} |\gamma|, \quad (71) \]

(boson-like statistics type for \( \gamma < 0 \)) leads to the following connection between \( \gamma \) and \( \alpha \):

\[ \gamma = 4\alpha - 2\alpha^2 - 1. \quad (72) \]
The 2nd virial coefficient of the ideal 2D Polychronakos gas

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(boson-like statistics type for \( \gamma < 0 \)) leads to the following connection between \( \gamma \) and \( \alpha \):

\[ \gamma = 4\alpha - 2\alpha^2 - 1. \quad (72) \]

However, none of the abovementioned results allows establishing a complete correspondence of anyons neither with the Haldane–Wu statistics nor with the Polychronakos statistics: the 3rd virial coefficients are different.
No complete analogy can be established between the Gentile, Haldane–Wu, and Polychronakos statistics. In particular:
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In the high-$T$ limit, the equation of state in the Gentile statistics contains the Bose-like correction only, which is not the case for the Haldane–Wu or Polychronakos statistics.
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Approximate correspondence can be achieved on the basis of the Fermi level analog, but this cannot be extended to the whole temperature domain.
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In a certain approximation, correspondence between the Haldane–Wu and Gentile statistics can be seen from the behavior of the occupation numbers.
Let us consider small deviations from some traditional statistics (Bose or Fermi).

\[ N = \sum_j G_j \frac{e^{(\varepsilon_j - \mu_B)/T}}{1} \]
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Let the analyzed system has spectrum $\varepsilon_j$ and $G_j$ is the degeneration of the $j$th level. Chemical potential of the respective Bose-system $\mu_B$ is linked to the number of particles $N$ and temperature $T$ as follows:

$$N = \sum_j \frac{G_j}{e^{(\varepsilon_j - \mu_B)/T} - 1}. \quad (73)$$
Let us now consider a system obeying the Polychronakos statistics with the parameter $\gamma = a - 1$, where $a \to 0$. 

The chemical potential of such a system $\mu_P$ is defined by

$$N = \sum_j G_j e^{\left(\epsilon_j - \mu_P\right)/T - (a - 1)} + a,$$

(74)

and it can be written using a small deviation from that of the Bose-system:

$$\mu_P = \mu_B + \Delta \mu_P.$$ 

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and it can be written using a small deviation from that of the Bose-system:

$$\mu_P = \mu_B + \Delta \mu_P.$$
Expand further the expression under summation in Eq. (74) into series wrt small $a$ and $\Delta \mu_P$ to the linear corrections:

$$N = \sum_j \frac{G_j}{e^{(\varepsilon_j - \mu_P)/T} - 1 + a} = \sum_j \frac{G_j}{e^{(\varepsilon_j - \mu_B)/T} - 1} +$$

$$+ \sum_j \frac{G_j}{[e^{(\varepsilon_j - \mu_B)/T} - 1]^2} \left\{ \frac{\Delta \mu_P}{T} e^{(\varepsilon_j - \mu_B)/T} - a \right\}. \quad (76)$$
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$$+ \sum_j \frac{G_j}{[e^{(\epsilon_j - \mu_B)/T} - 1]^2} \left\{ \frac{\Delta \mu_P}{T} e^{(\epsilon_j - \mu_B)/T} - a \right\}. \quad (76)$$

Taking into account (73), one obtains:

$$\frac{\Delta \mu_P}{T} = a \frac{P}{N + P}, \quad \text{where} \quad P = \sum_j \frac{G_j}{[e^{(\epsilon_j - \mu_B)/T} - 1]^2}, \quad (77)$$
In a similar fashion, one can define the correction to the chemical potential in a Bose-system analog under within the Tsallis approach:

\[
N = \sum_j \frac{G_j}{e^{(\varepsilon_j - \mu_{Ts})/T} - 1} = \sum_j \frac{G_j}{e^{(\varepsilon_j - \mu_B)/T} - 1} + \\
+ \sum_j \frac{G_j e^{(\varepsilon_j - \mu_B)/T}}{\left[e^{(\varepsilon_j - \mu_B)/T} - 1\right]^2} \left\{ \frac{\Delta \mu_{Ts}}{T} - \left(\frac{\varepsilon_j - \mu_B}{T}\right)^2 \frac{q - 1}{2} \right\},
\]

where \( q \to 1 \) and chemical potential

\[
\mu_{Ts} = \mu_B + \Delta \mu_{Ts}.
\]
Similar expansions can be written for a Bose-system with the spectrum $\varepsilon_j + \Delta \varepsilon_j$, where a small correction $\Delta \varepsilon_j$ is, for instance, caused by interactions. For chemical potential $\mu$ we have $\mu_B$:

$$\mu = \mu_B + \Delta \mu,$$  \hspace{1cm} (80)

while

$$
N = \sum_j \frac{G_j}{e^{(\varepsilon_j + \Delta \varepsilon_j - \mu)/T} - 1} = \sum_j \frac{G_j}{e^{(\varepsilon_j - \mu_B)/T} - 1} + \\
+ \sum_j \frac{G_j e^{(\varepsilon_j - \mu_B)/T}}{[e^{(\varepsilon_j - \mu_B)/T} - 1]^2} \Delta \mu - \Delta \varepsilon_j.$$

(81)
It means that, to a certain degree, an interacting Bose-system can be modeled by systems obeying various types of fractional statistics with parameters being defined by the spectrum correction $\Delta \varepsilon_j$. 

To find such correspondences one can require, e.g., that energies in different systems in the respective statistics coincide, that is:

$$E = \sum_j \varepsilon_j G_j n_j = E_B + \Delta E,$$

(82)

where $E_B$ is the energy of the Bose-system with spectrum $\varepsilon_j$:

$$E_B = \sum_j \varepsilon_j G_j e^{(\varepsilon_j - \mu_B)/T} - 1,$$

(83)

here, $\varepsilon_j$ is the excitation spectrum (equal to $\varepsilon_j$ or $\varepsilon_j + \Delta \varepsilon_j$ in the considered examples), and $\Delta E$ is a correction.
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Main approaches to define a fractional (intermediate) statistics generalizing conventional quantum Bose–Einstein and Fermi–Dirac distributions are analyzed. Introductory information is given on anyons and some $q$-deformed algebras of creation–annihilation operators are briefly reviewed. Special attention is paid to statistical-mechanical approaches by Gentile, Haldane–Wu, and Polychronakos. Nonextensive/nonadditive generalizations of the Boltzmann–Gibbs entropy and respective fractional statistics are considered as well. Some possibilities to find connections between different types of statistics are briefly analyzed.
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Summary

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Thank you for your attention!

Дякую за увагу!

Vielen Dank für Ihre Aufmerksamkeit!