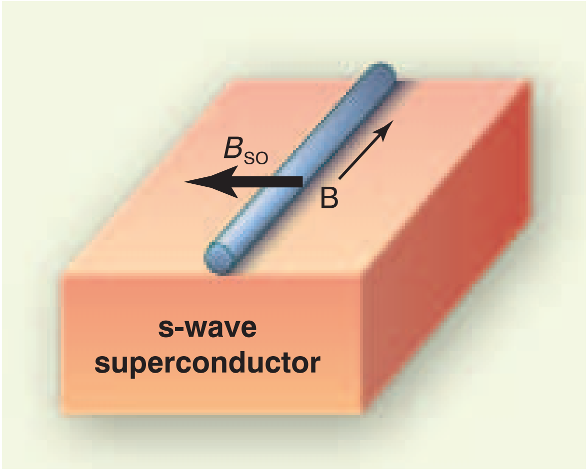
Anyonic defects: A new paradigm for non-Abelian Statistics

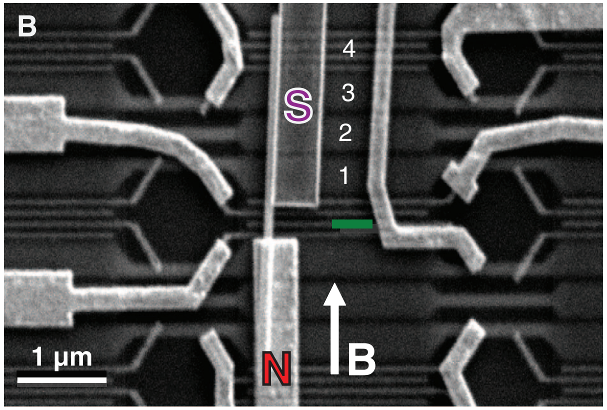
Erez Berg

Weizmann Institute of Science

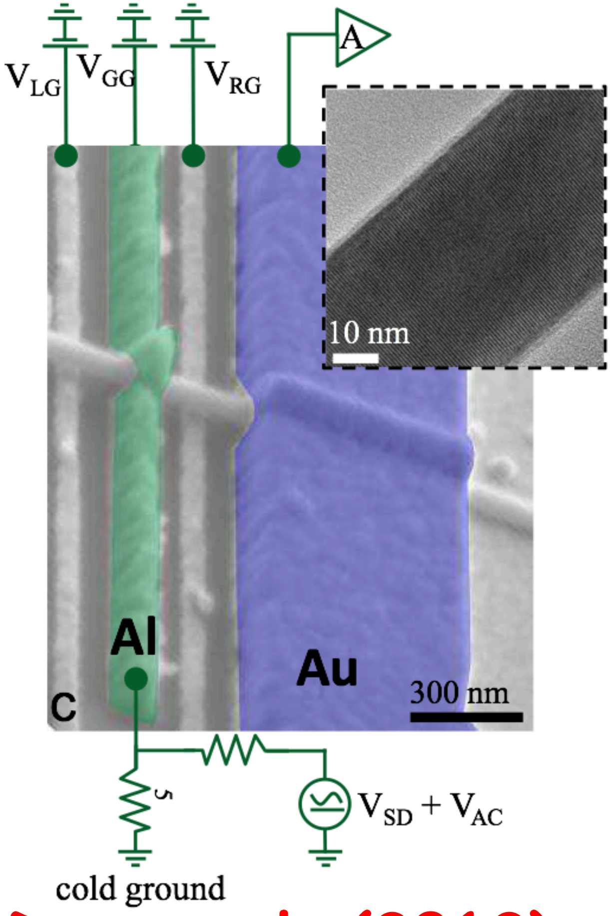
In collaboration with:
Netanel Lindner (Technion)
Gil Refael (Caltech)
Ady Stern (Weizmann)
Frank Pollmann (MPI Dresden)
Ari Turner (Johns Hopkins)

Majorana zero modes





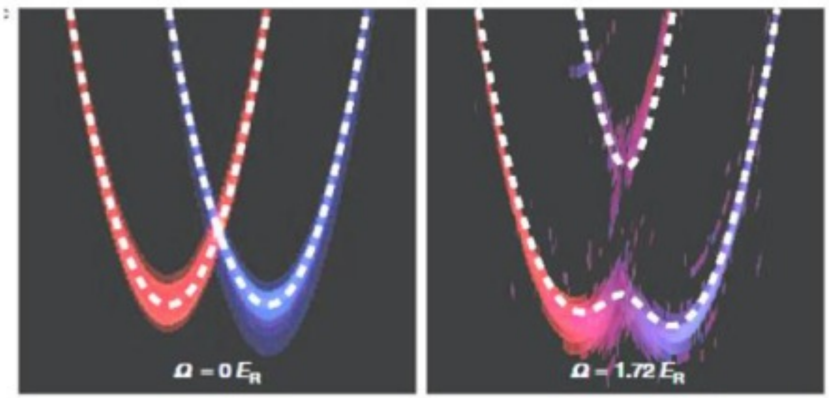
Mourik et al. (2012)



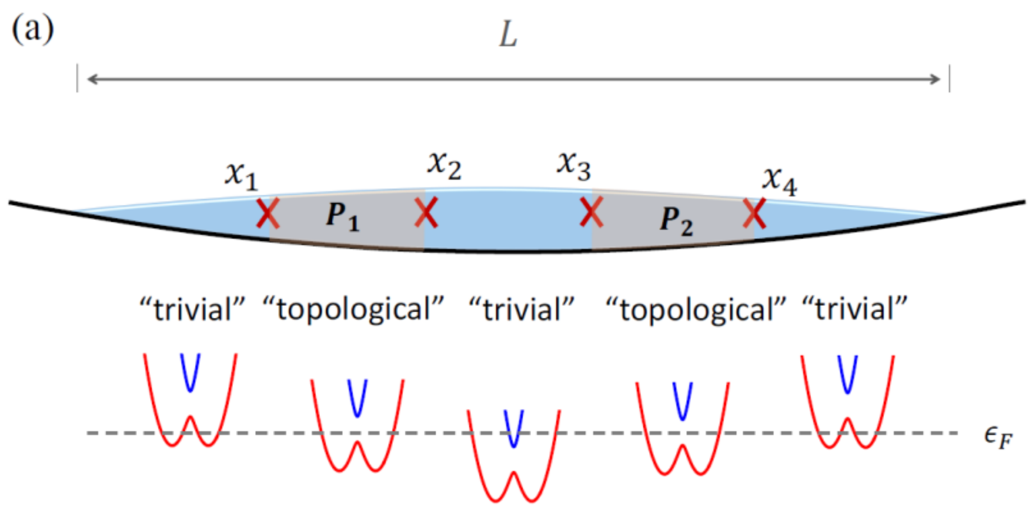
Das et al. (2012)

Short advertisement: Majoranas in cold atom systems without proximity

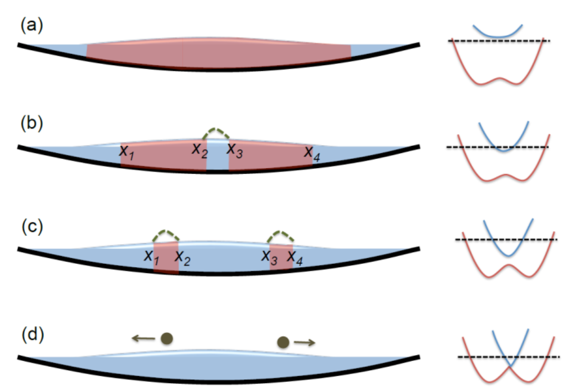
Ruhman, EB, and Altman, arXiv:1412:3444



Wang, Yu, Fu, Miao, Huang, Chai, Zhai, and Zhang, PRL (2012).
Cheuk, Sommer, Hadzibabic, Yefsah, Bakr, and Zwierlein, PRL (2012).



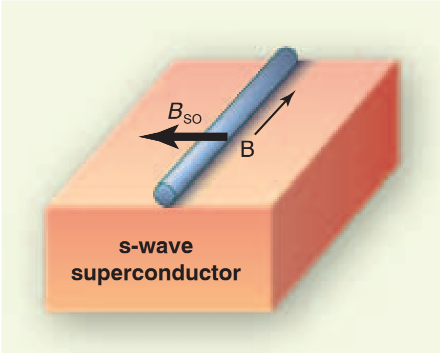
Probing scheme: ramp down the Zeeman field and SOC!

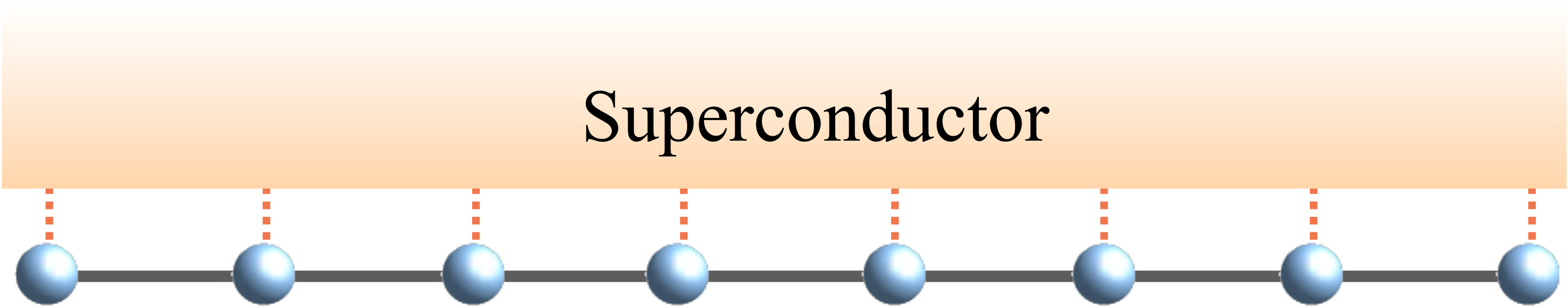


Outline

- Brief review of Majorana fermions
- "Fractionalized Majoranas" on fractional quantum Hall edges
  - Fractionalized 1D superconductors
  - Twist defects
- Anyonic defects in non-Abelian systems

Majorana fermions in a superconducting wire

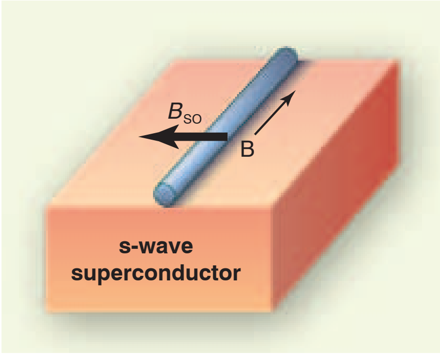


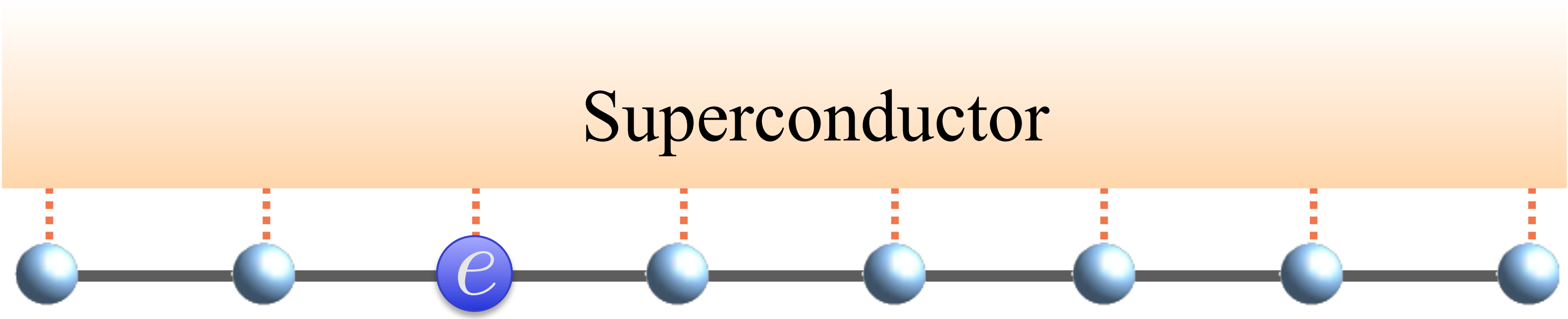


$$H = \sum_{i,j} \left[-t_{ij} \left(c_i^\dagger c_j + H.c. \right) + \Delta_{ij} \left(c_i^\dagger c_j^\dagger + H.c. \right) \right] - \sum_i \mu c_i^\dagger c_i$$

Kitaev (2002), Sau et al. (2010), Oreg et al. (2010), ...

Majorana fermions in a superconducting wire





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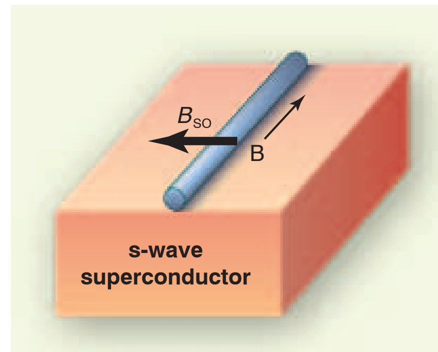

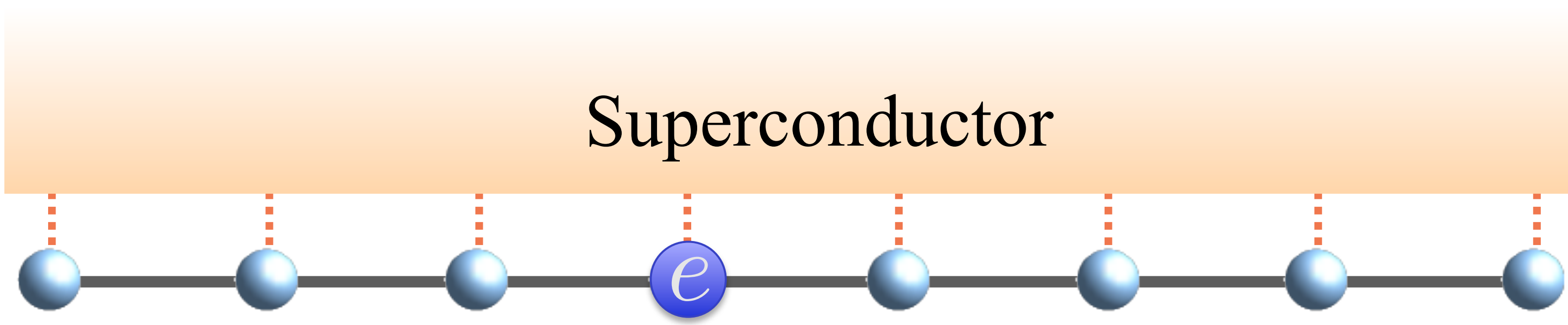

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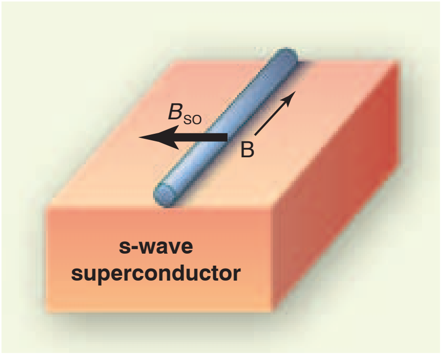

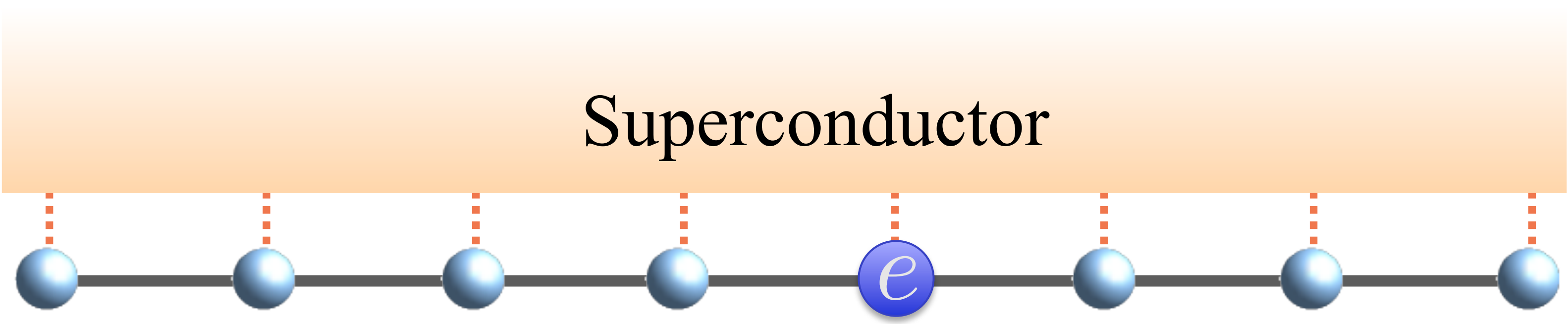

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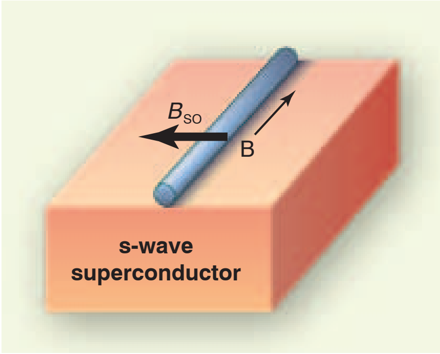

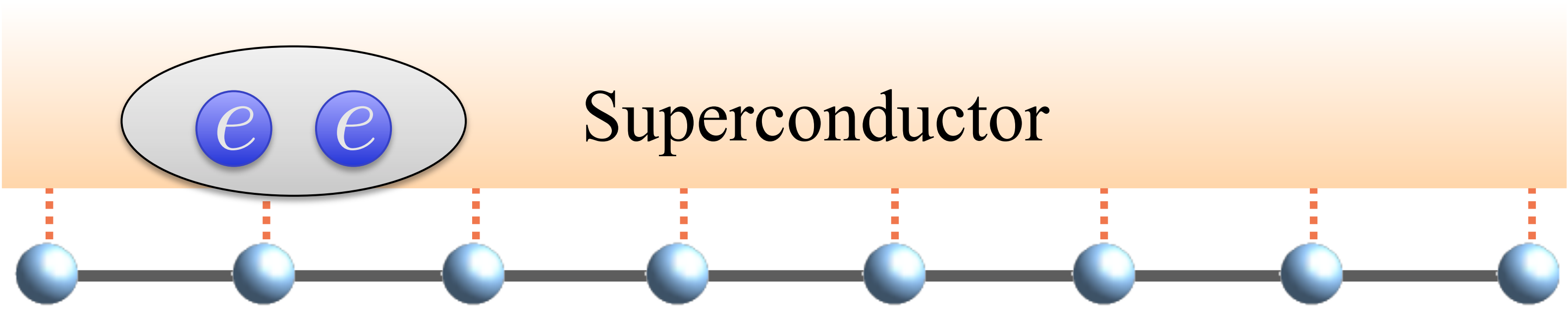

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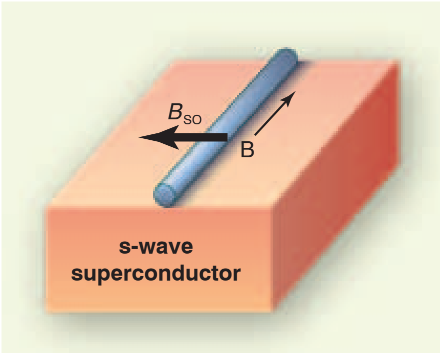
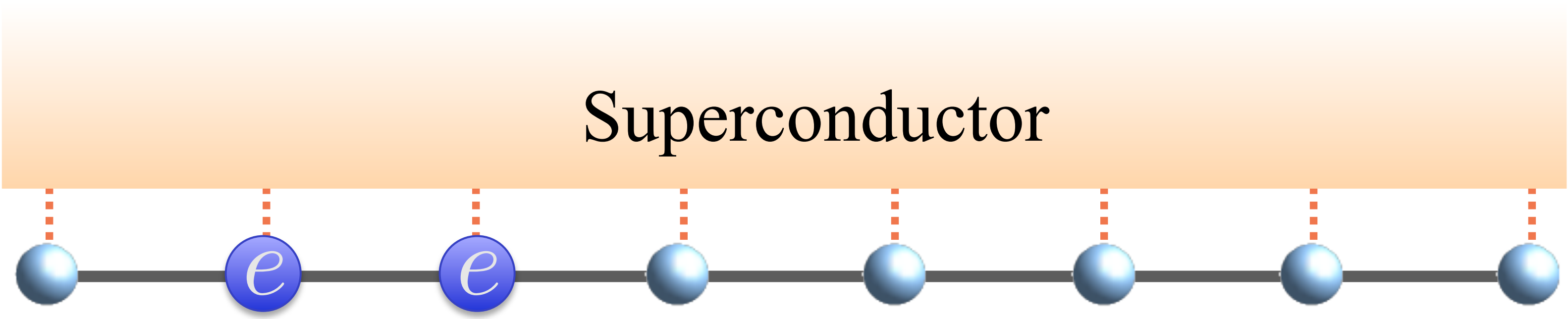
Kitaev (2002), Lutchyn et al. (2010), Oreg et al. (2010), ...

Majorana fermions in a superconducting wire





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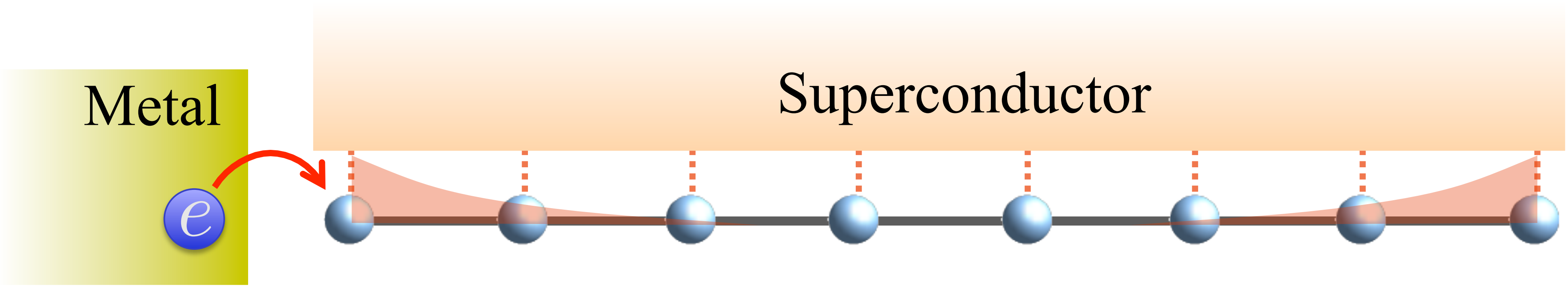
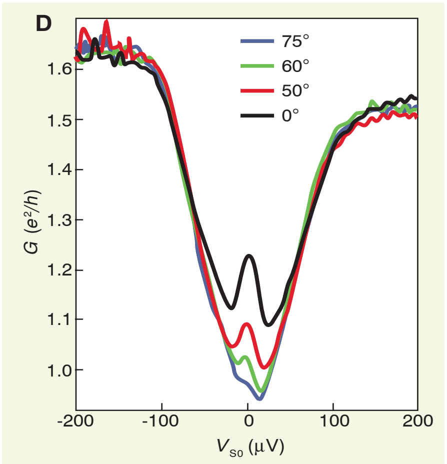
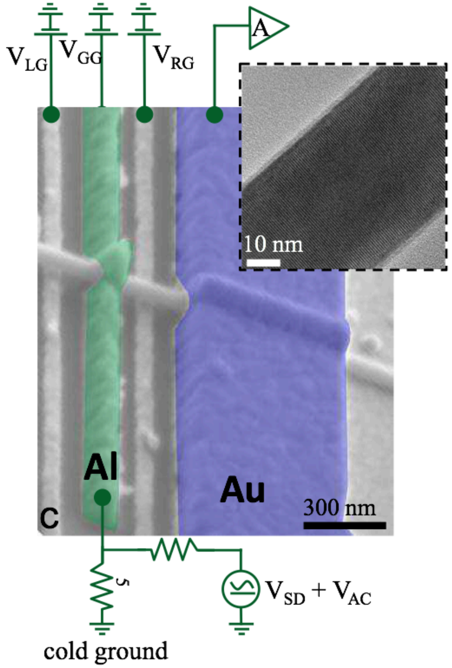
Kitaev (2002), Sau et al. (2010), Oreg et al. (2010), ...

Adding electrons to the chain

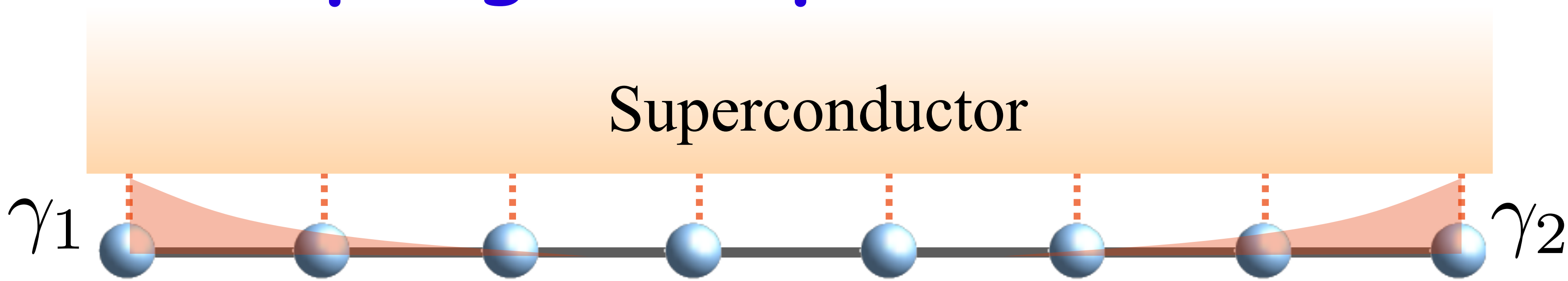


"Majorana Zero modes" at both ends!





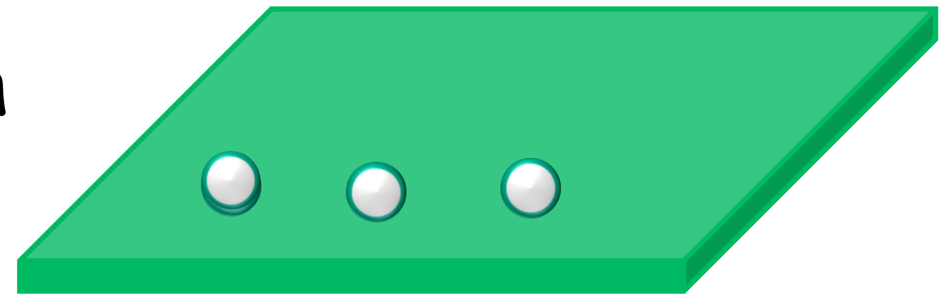
Lessons from the one dimensional topological superconductor



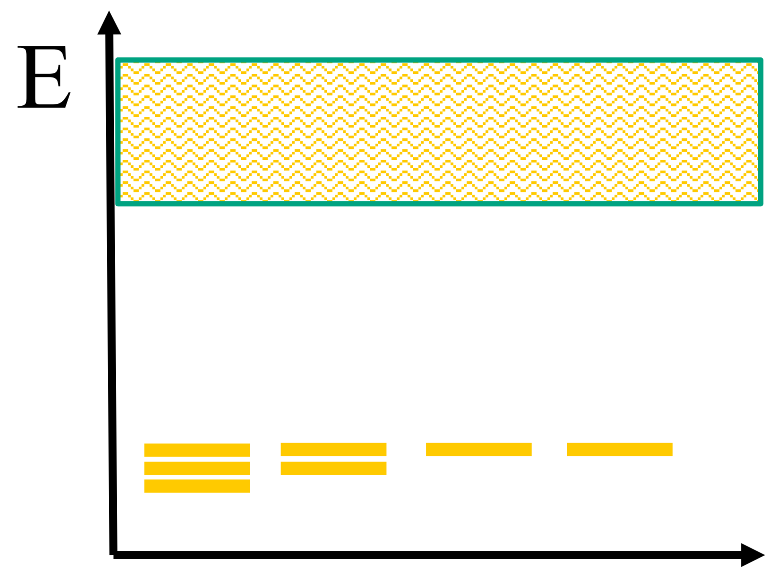
- Gapped system, two degenerate ground states, characterized by having a different fermion parity
- Defects (in this case, the edges of the system) carry protected zero modes described by anti-commuting operators: $\gamma_1\gamma_2 = -\gamma_2\gamma_1$
- Ground state degeneracy is "topological": no local measurement can distinguish between the two states!
- Useful as a "quantum bit"?

Non-Abelian Anyons

Degeneracy increases with
number of anyons



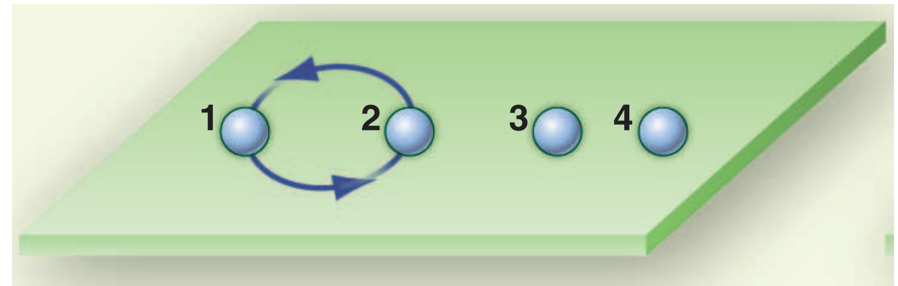
"quantum dimension" $\dim H_{GS} \sim \lambda^N$



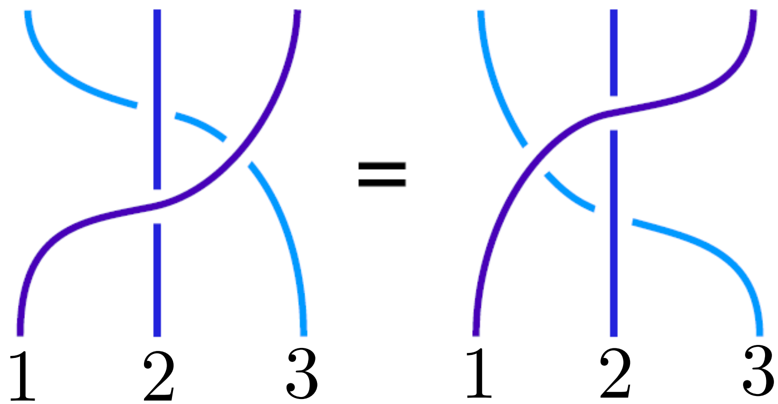
Robust to local perturbations!

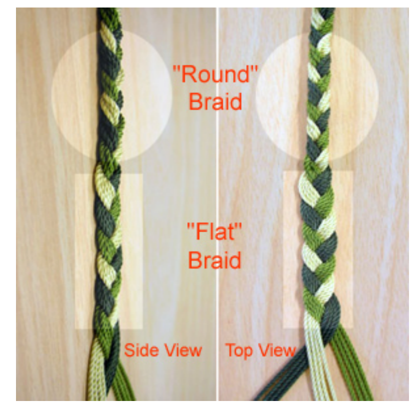
Non-Abelian Statistics: Braiding

$$|\psi_i\rangle \to \sum_j U_{ij}|\psi_j\rangle$$



Braid group:

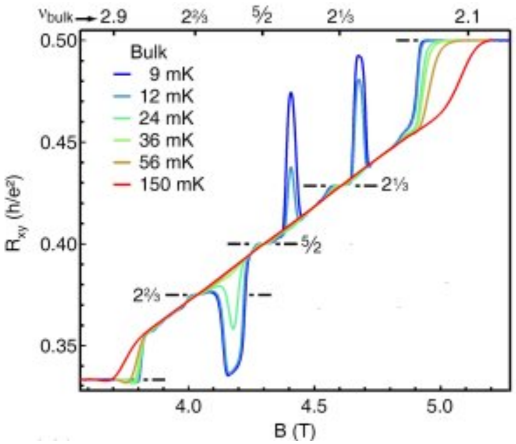




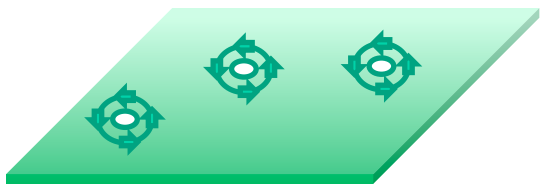
Majorana Fermions: $$e^{(\pi/4)\gamma_1\gamma_2} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

2D vortices: Ivanov, Read & Green,…
1D wire network: Alicea et.al (2010)

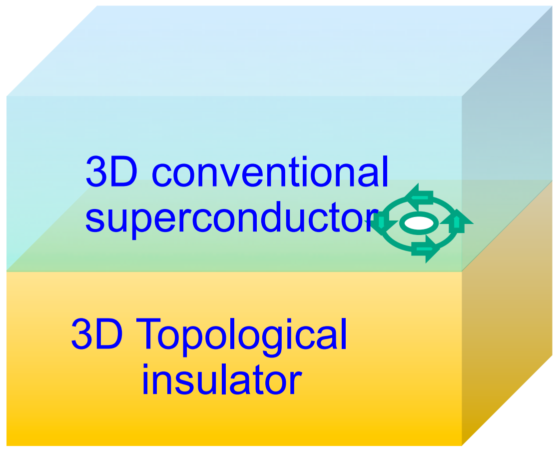
The quest for non-Abelian systems



Fractional
QH



2D p+ip
superconductors



Superconductor - 3D Top. Insulator
(Semiconductor) heterostructures

All of these realize Ising anyons
(i.e. Majoranas)

Can we get something richer?

- The braiding of Majorana zero modes are non-universal: a general unitary transformation cannot be performed in a protected way
- Can we get something richer than Majorana fermions in 1D ?

"Theorem" (Fidkowsky, 2010; Turner, Pollmann, and EB, 2010): Gapped, local Hamiltonians of fermions or bosons in 1D, can give (at best) Majorana zero modes.

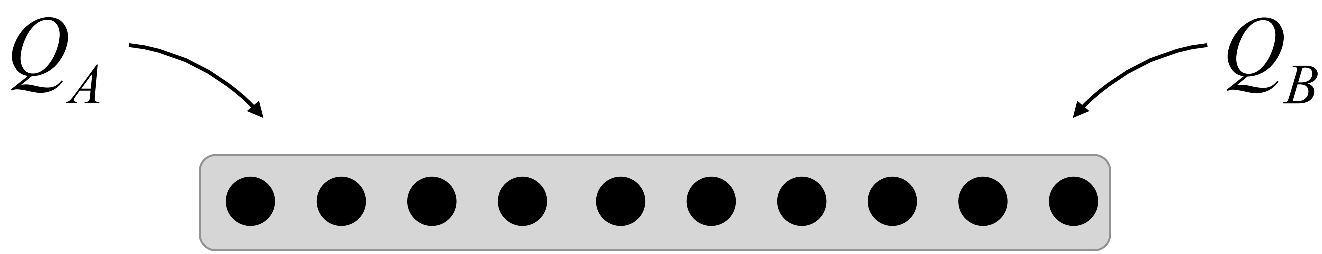
Gapped phases of fermions

A. Turner, F. Pollmann, EB (2010)

No symmetries: only one gapped phase in 1D

(N. Shuch et al., 2011)

- Conservation of **fermion parity** with $Q = (-1)^{N_{\mathrm{total}}}$



"Fractionalization" of the parity operator (in low-energy subspace) $Q = Q_A Q_B$

Q_A, Q_B either fermionic or bosonic!

$$Q^A Q^B = e^{i\mu} Q^B Q^A$$

➡ two distinct phases with $\mu = 0, \pi$

Beyond Majorana fermions

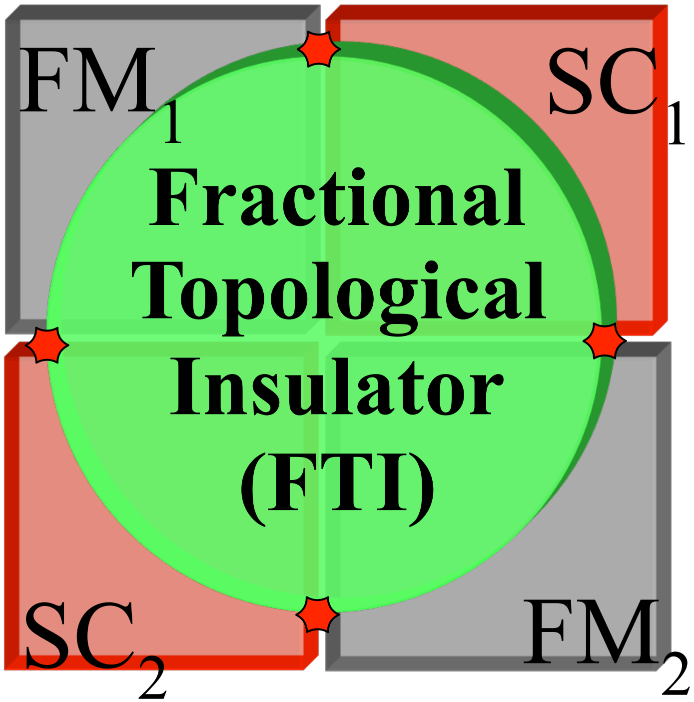
Consider the effectively 1D boundaries
of 2D a topological phase
which supports (abelian) anyons.

"Fractional topological insulator":

Laughlin Quantum Hall state with:

$\nu = 1/m$ for spin up

$\nu = -1/m$ for spin down (m odd)



Stable phase: Levin and Stern (2010)

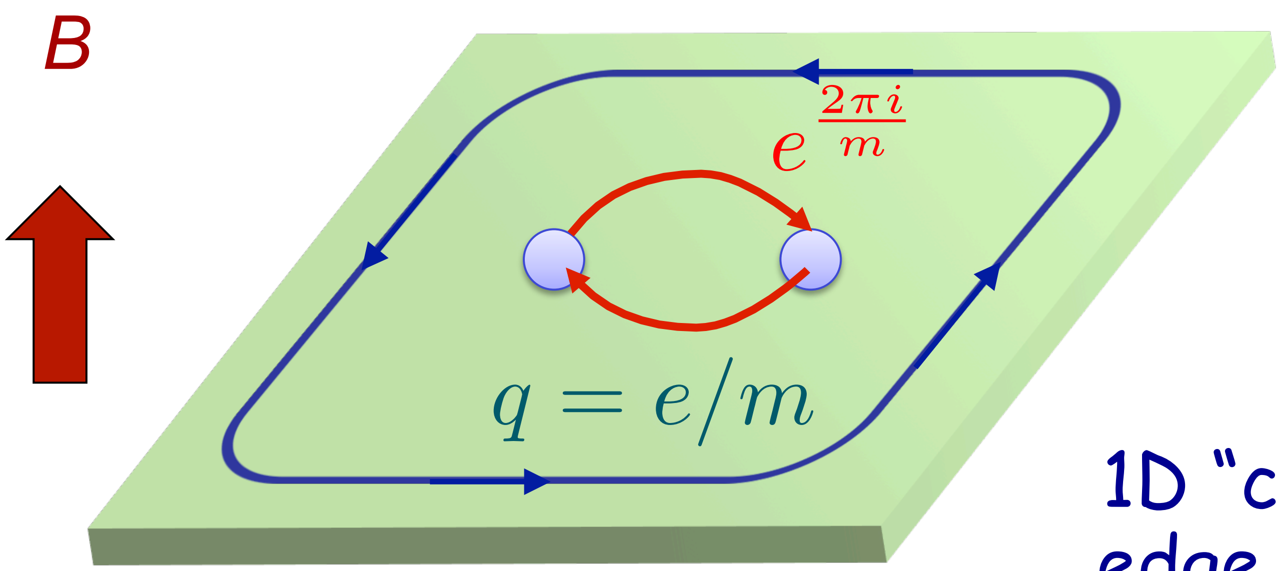
Majorana fermions at SC/FM interfaces: Fu and Kane (2009)

Fractional Quantum Hall effect

Electrons in two dimensions, high magnetic field

Special density (number of electrons/flux quantum), ultra clean, low temperature:
$\nu = 1/m$ Fractional Quantum Hall (Laughlin) state

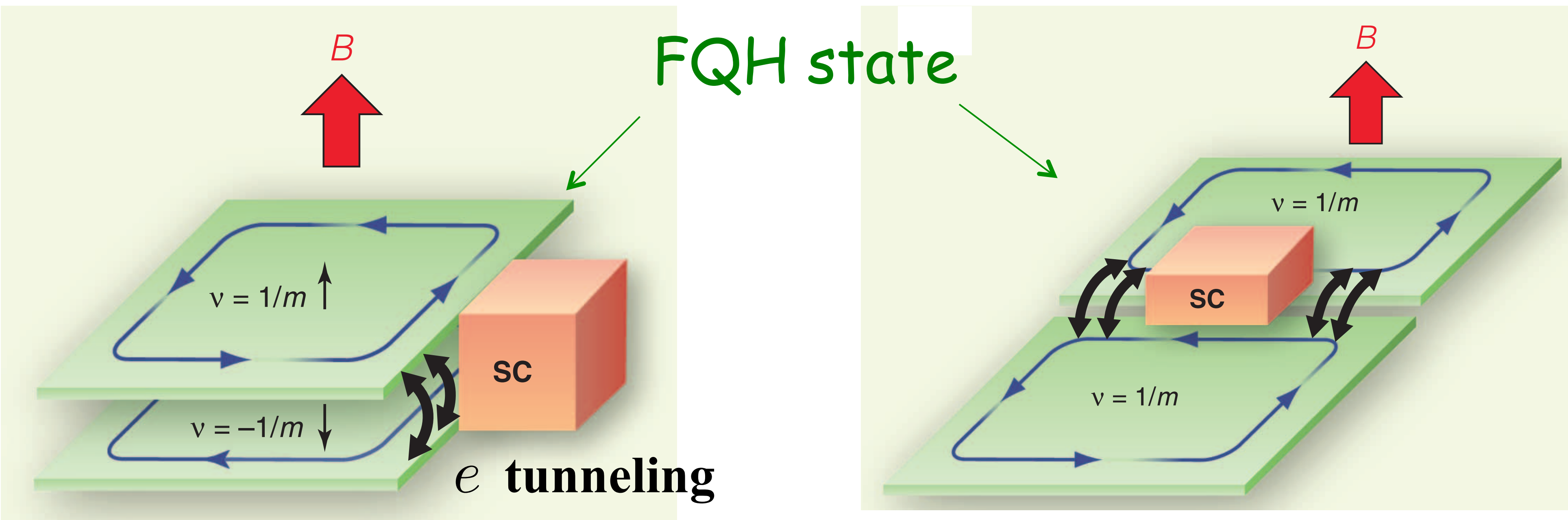
Gapped bulk, gapless chiral edge states
Excitations: fractional charge, fractional statistics!



1D "chiral" edge state

Beyond Majorana fermions

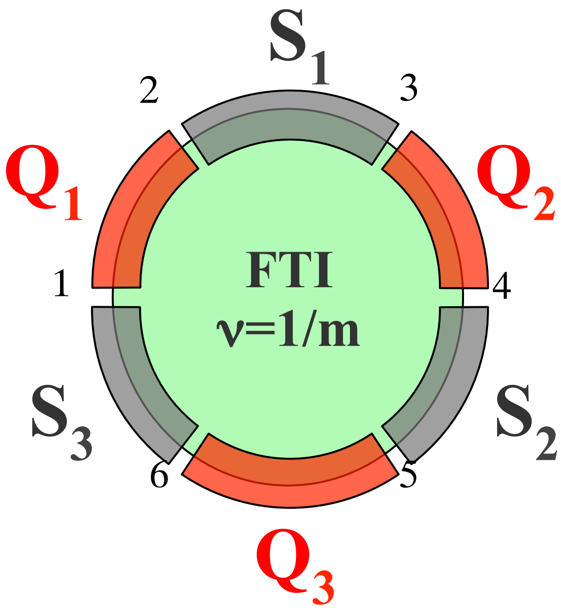
Fractional quantum Hall "realizations"
of a Fractional Topological Insulator



Lindner, EB, Stern, Refael (2013);
Clarke, Alicea, Shtengel (2013);
Cheng (2013)

Ground state degeneracy

Physical picture:



Charges in SC conserved mod(2)

Q_j = n/m, n = 0,...,2m-1

Spins in FM conserved mod(2)

(el. spin=1)

S_j = n/m, n = 0,...,2m-1

Spin and charge are conjugate variables:

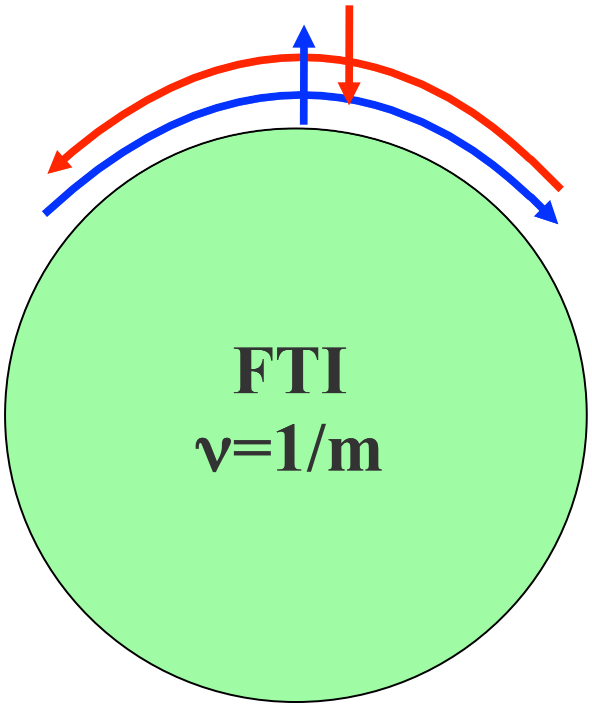
$$e^{i\pi S_i} e^{i\pi Q_j} = e^{\frac{i\pi}{m}(\delta_{i,j+1}-\delta_{i,j})} e^{i\pi Q_j} e^{i\pi S_i}$$

2N domains, fixed total Q, S: $(2m)^{N-1}$

approximately degenerate ground states

Interface "anyon" with quantum dimension $\sqrt{2m}$

Effective Model for Fractional Topological Insulator Edge States

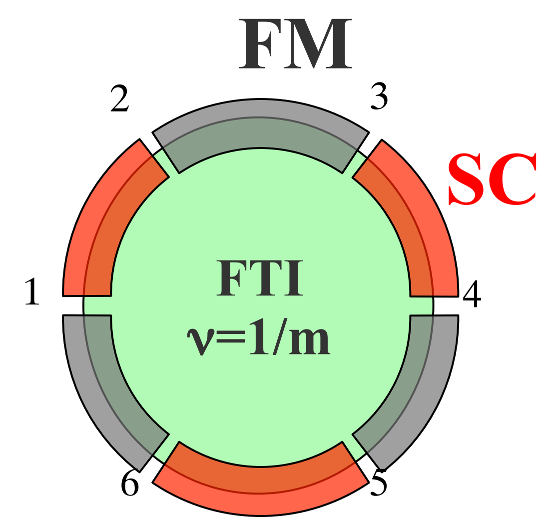


Non-chiral Luttinger liquid edge state:

$$H = \frac{u}{2\pi\nu}\int dx \left[K(x)\left(\partial_x \phi\right)^2 + \frac{1}{K(x)}\left(\partial_x \theta\right)^2 \right]$$

Effective Model for Fractional Topological Insulator Edge States

$$H = \frac{u}{2\pi\nu}\int dx\left[K(x)(\partial_x\phi)^2 + \frac{1}{K(x)}(\partial_x\theta)^2\right]$$
$$-\int dx\,[g_S(x)\underbrace{\cos(2m\phi)}_{\psi_R\psi_L + H.c.} + g_F(x)\underbrace{\cos(2m\theta)}_{\psi_R^\dagger\psi_L + H.c.}]$$



Comm. Relations: $[\phi(x),\theta(x')] = i\frac{\pi}{m}\Theta(x'-x)$

2N domains

Charge density: $\rho = \frac{1}{\pi}\partial_x\theta$

Spin density: $s^z = \frac{1}{\pi}\partial_x\phi$

Electron: $\psi_{R,L} \propto e^{im(\phi\pm\theta)}$

Laughlin q.p.: $\chi_{R,L} \propto e^{i(\phi\pm\theta)}$

Ground state degeneracy

Large cosine terms (strong coupling to SC/FM)

$$-\int dx\left[g_S(x)\cos(2m\phi)+g_F(x)\cos(2m\theta)\right]$$

ϕ, θ pinned near the minima of the cosines:

$$\phi_n=\frac{\pi}{m}n,\ \ n\in 0,1,\ldots,2m-1$$

$$\theta_k=\frac{\pi}{m}n,\ \ k\in 0,1,\ldots,2m-1$$

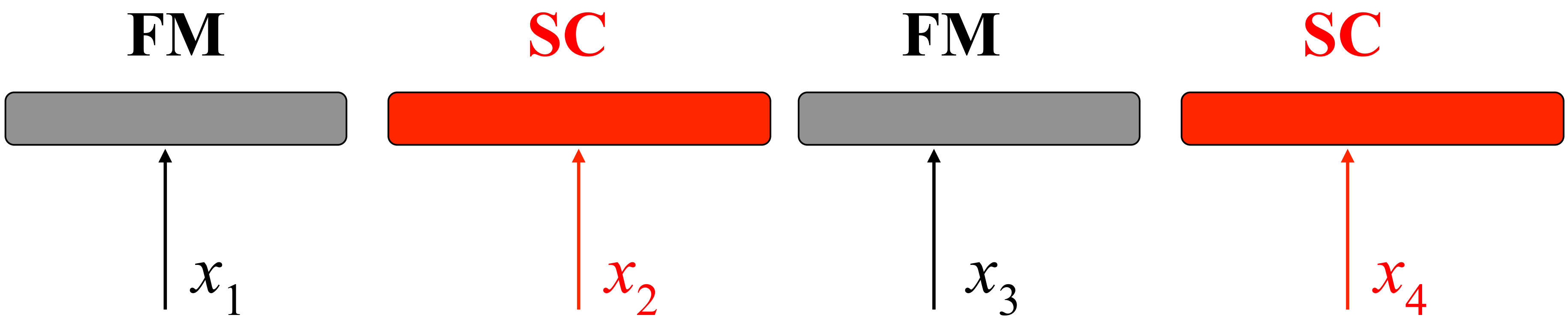
But... ϕ, θ are dual vararíables: cannot be "localized" simultaneously

$$e^{i\theta(x)}e^{i\phi(x')}=e^{i\frac{\pi}{m}\Theta(x-x')}e^{i\phi(x)}e^{i\theta(x)}$$

2N domains: ~$(2m)^N$ approximately degenerate ground states

Q and S operators

In terms of the φ, θ fields, one can define the Q, S operators:

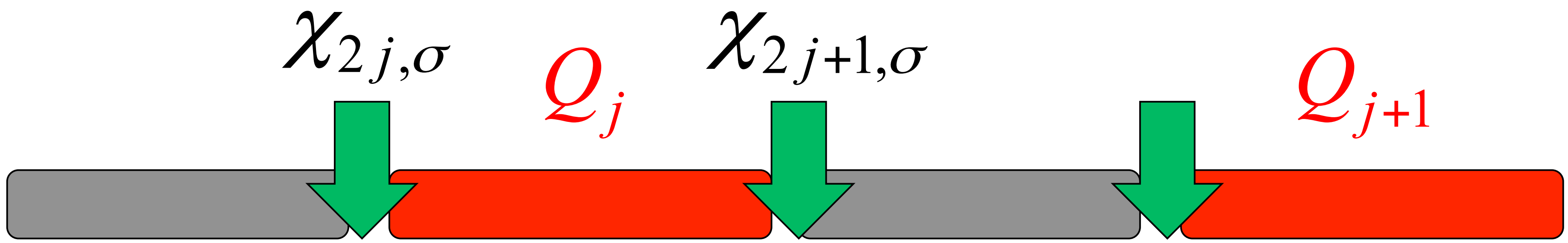


$$e^{i\pi Q_2} = e^{i\int_{x_1}^{x_3} dx \partial_x \theta} = e^{i[\theta(x_3)-\theta(x_1)]}$$

$$e^{i\pi S_3} = e^{i\int_{x_2}^{x_4} dx \partial_x \phi} = e^{i[\phi(x_4)-\phi(x_2)]}$$

$$e^{i\pi S_i} e^{i\pi Q_j} = e^{i\frac{\pi}{m}(\delta_{i,j+1}-\delta_{i,j-1})} e^{i\pi Q_j} e^{i\pi S_i}$$

"Fractionalized Majorana operators"



$$\chi_{r,\sigma}\left|q_1,\ldots,q_j,\ldots;s\right\rangle \propto \left|q_1,\ldots,q_j+1,\ldots;s+\sigma\right\rangle$$

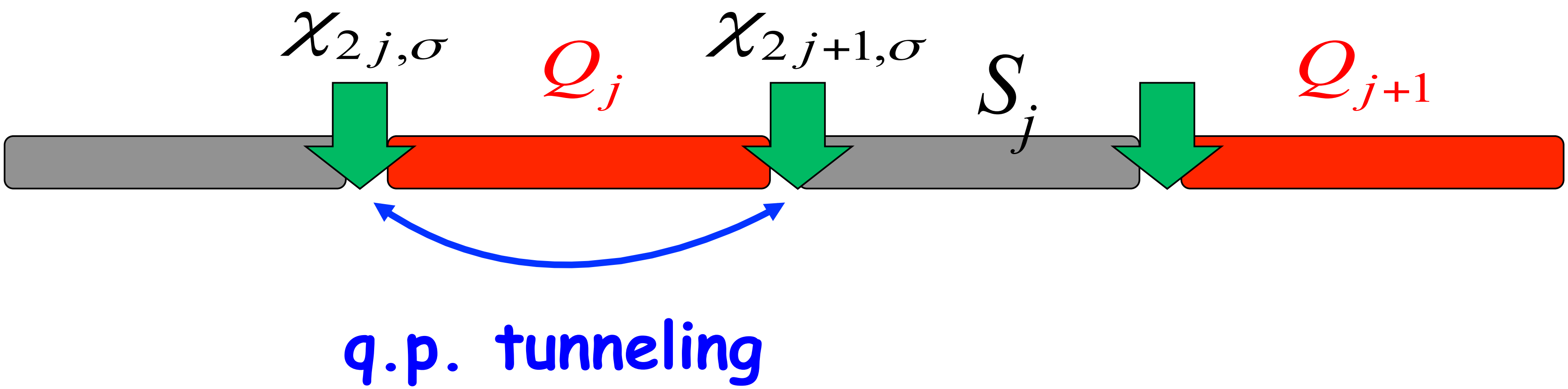
$$\left[H,\chi_{r\sigma}\right]=0 \qquad \left(\chi_{r\sigma}\right)^{2m}=1$$

$\chi_{r\sigma}$ have q.p commutation relations

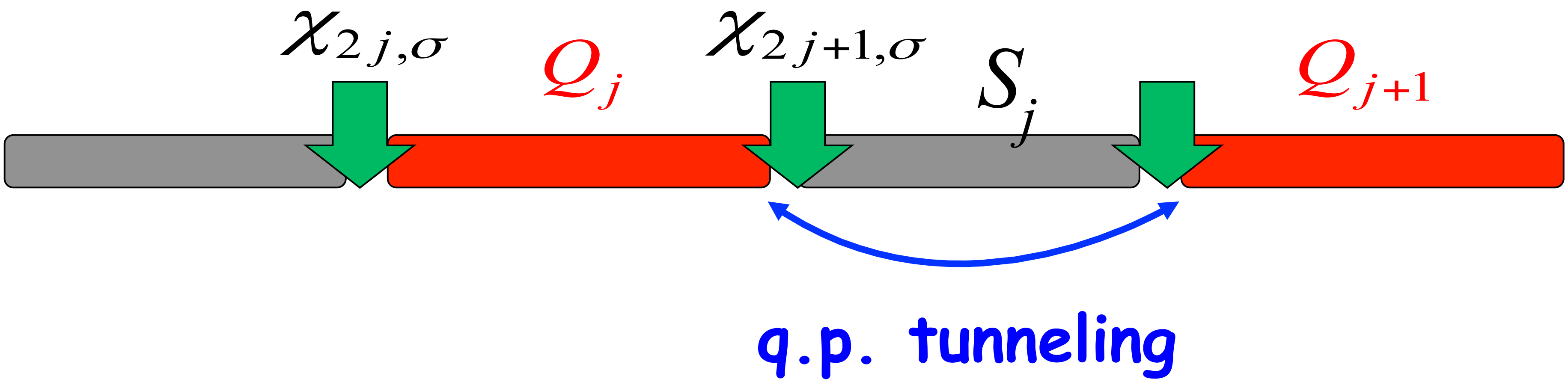
$$\chi_{j,\sigma}\chi_{k,\uparrow}=e^{i\pi/m}\chi_{k,\uparrow}\chi_{j,\sigma}$$
$$\chi_{j,\sigma}\chi_{k,\downarrow}=e^{-i\pi/m}\chi_{k,\downarrow}\chi_{j,\sigma}$$

1D model of "Parafermions": P. Fendley, arXiv:1209.0472

Coupling of interfaces



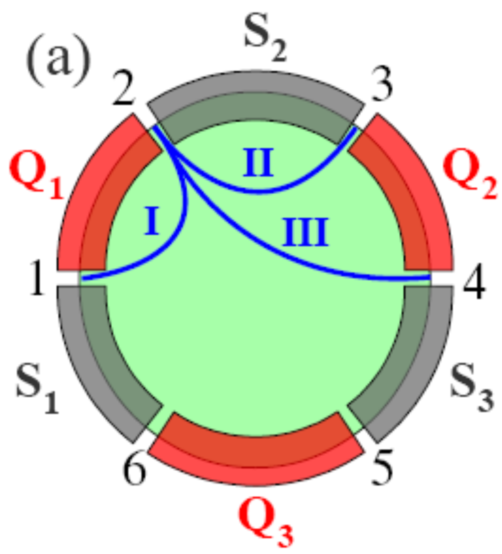
$$H_Q = -t\chi_{2j,\sigma}\chi^{\dagger}_{2j+1,\sigma} + h.c. = -2t\cos\left(\pi Q_j\right)$$



$$H_S = -t\chi_{2j+1,\sigma}\chi^{\dagger}_{2j+2,\sigma} + h.c. = -2t\cos\left(\pi S_j\right)$$

Braiding

Braiding domain walls 3 and 4:

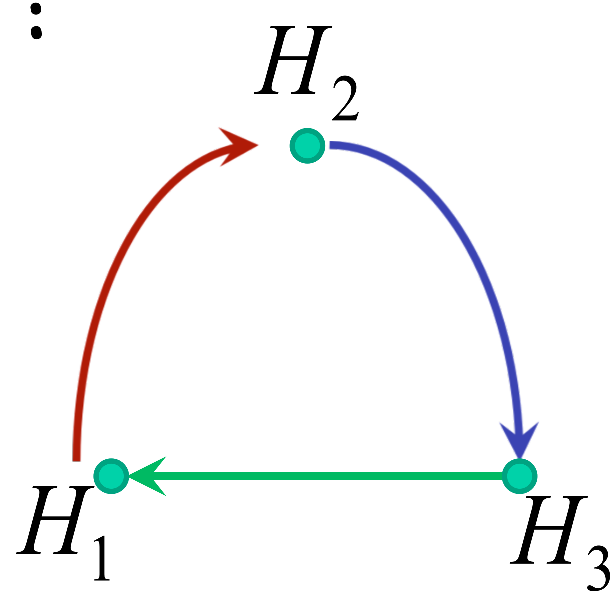


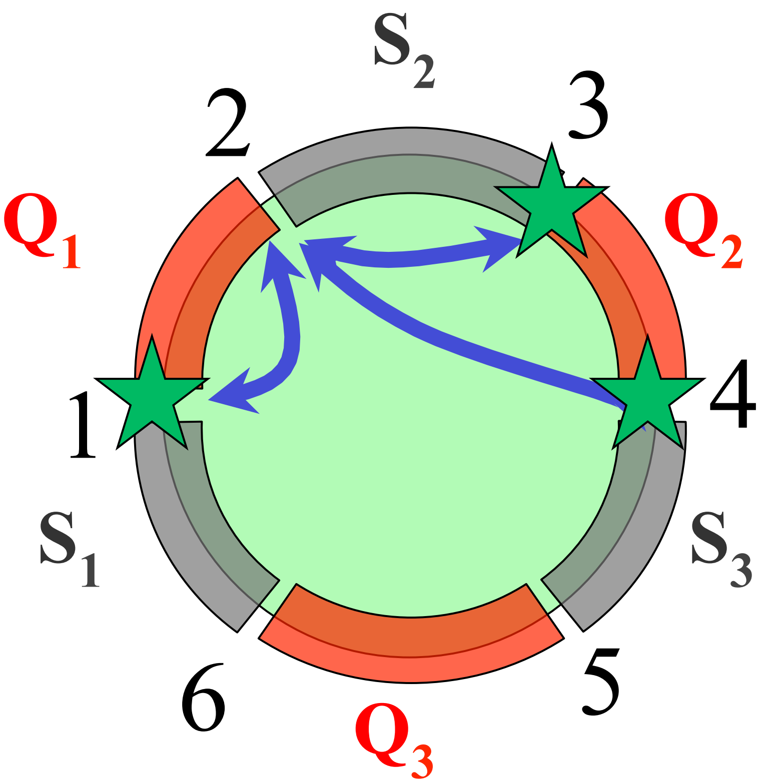
For an arbitrary coupling of any three domain walls, the ground state degeneracy remains $(2m)^2$ as long as only one spin species is allowed to tunnel.

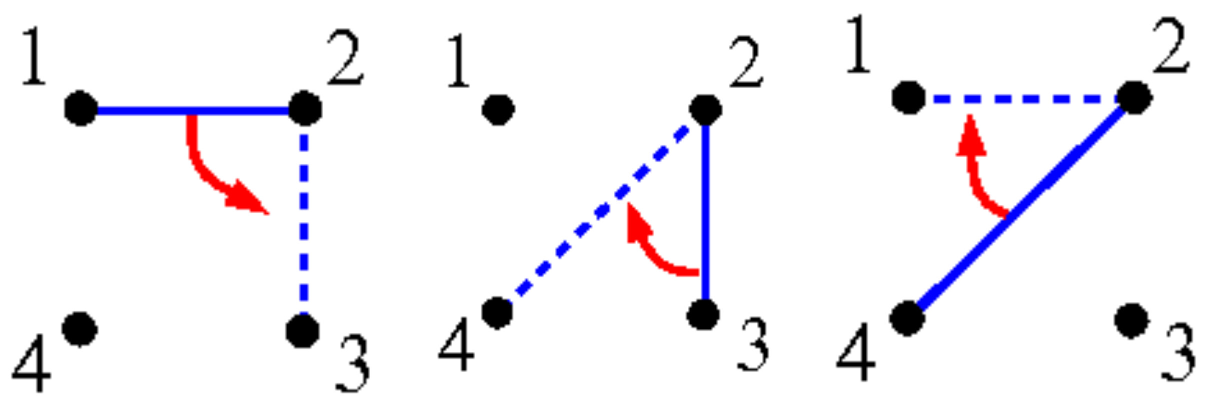
Braiding

Braiding interfaces ★ :

$$H(t) = \sum_{ij} \lambda_{ij}(t) H_{ij}$$







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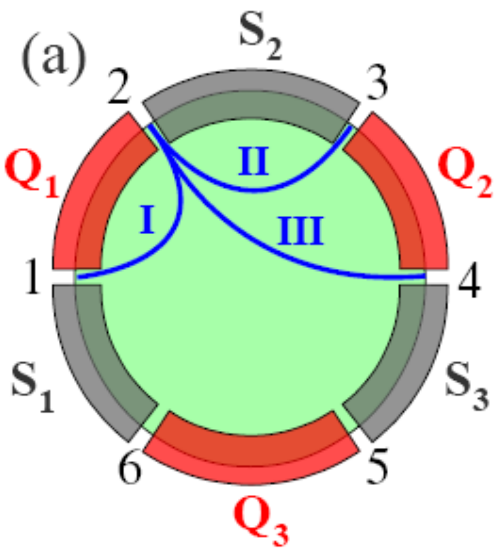


$$U_{34} = \exp\left(i\frac{\pi m}{2}\hat{Q}_2^2\right) = \exp\left(i\frac{\pi}{2m}q_2^2\right)$$

$$Q_2 = \frac{1}{m}q_2, \qquad q_2 = 0, \ldots, 2m-1$$

Example: m=3 $\quad q_2 = 2p + 3q \quad (p = 0, 1, 2, \;\; q = 0, 1)$

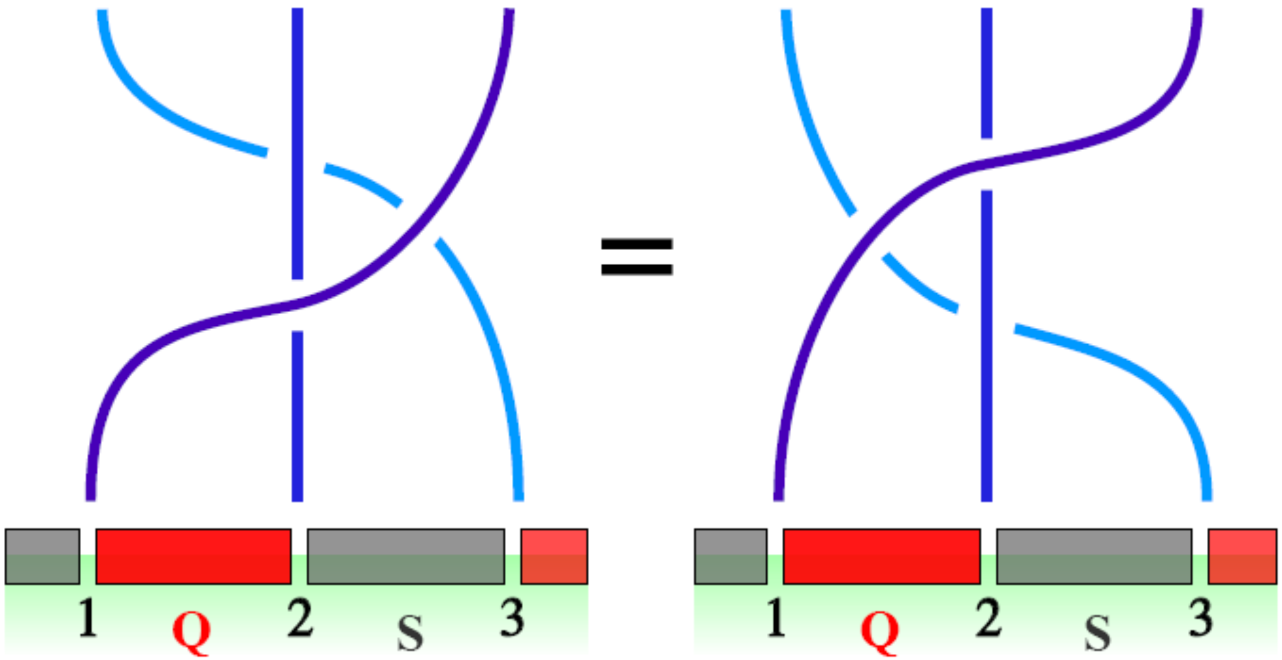
$$U_{34} = \exp\left(i\frac{\pi}{6}q_2^2\right) = \exp\left(-i\frac{\pi}{2}q^2\right)\exp\left(i\frac{2\pi}{3}p^2\right)$$

(Majorana) ⊗ (Something new!)

The Braid Group

$$\left[\hat{U}_{i,i+1}, \hat{U}_{j,j+1}\right] = 0 \;\; (|i-j| > 1),$$
$$\hat{U}_{j,j+1}\hat{U}_{j+1,j+2}\hat{U}_{j,j+1} = \hat{U}_{j+1,j+2}\hat{U}_{j,j+1}\hat{U}_{j+1,j+2}$$

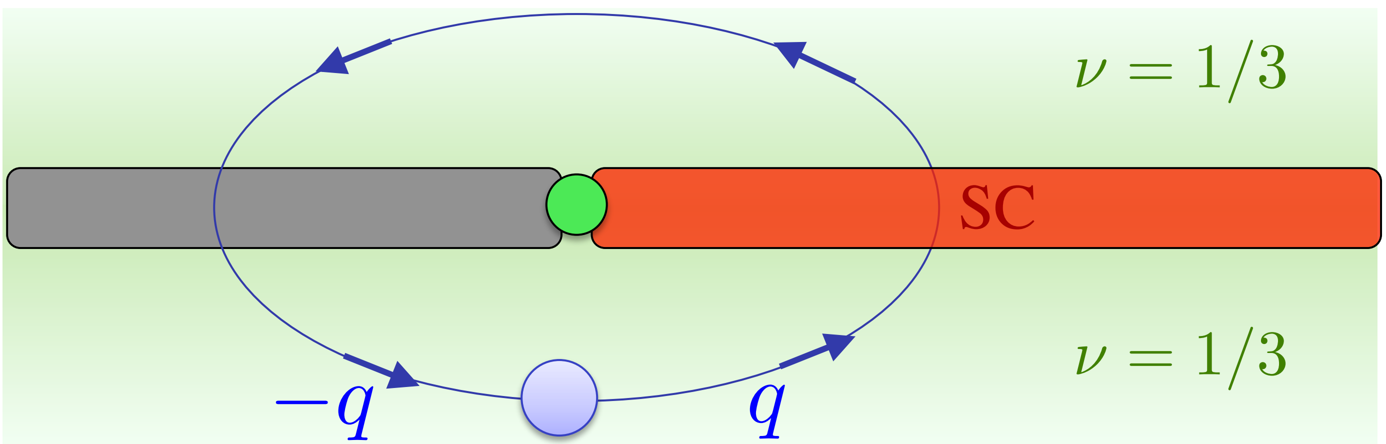
(Yang-Baxter equation)



Both equations hold: rep. of the braid group

Fractionalized zero modes at "twist defects" in topological phases

Ends of line defects that interchange anyon types ("topological symmetry")



The "defect line" can permute anyon types.

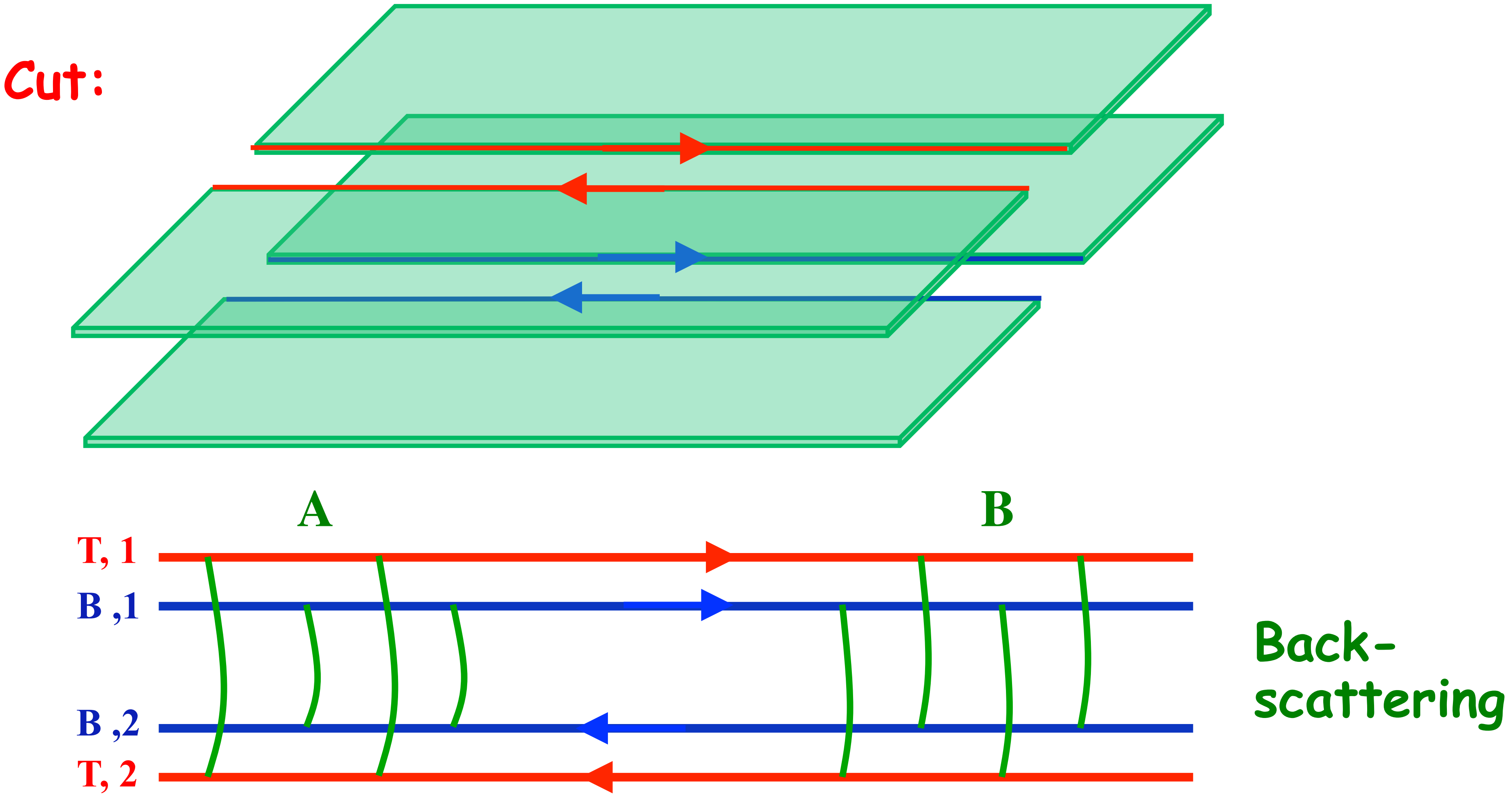
Barkeshli, Jian, Qi (2013); Fidkowski, Lindner, Kitaev (unpublished)

Fractionalized zero modes at "twist defects" in topological phases

Another example: $\nu=1/3$ bilayer

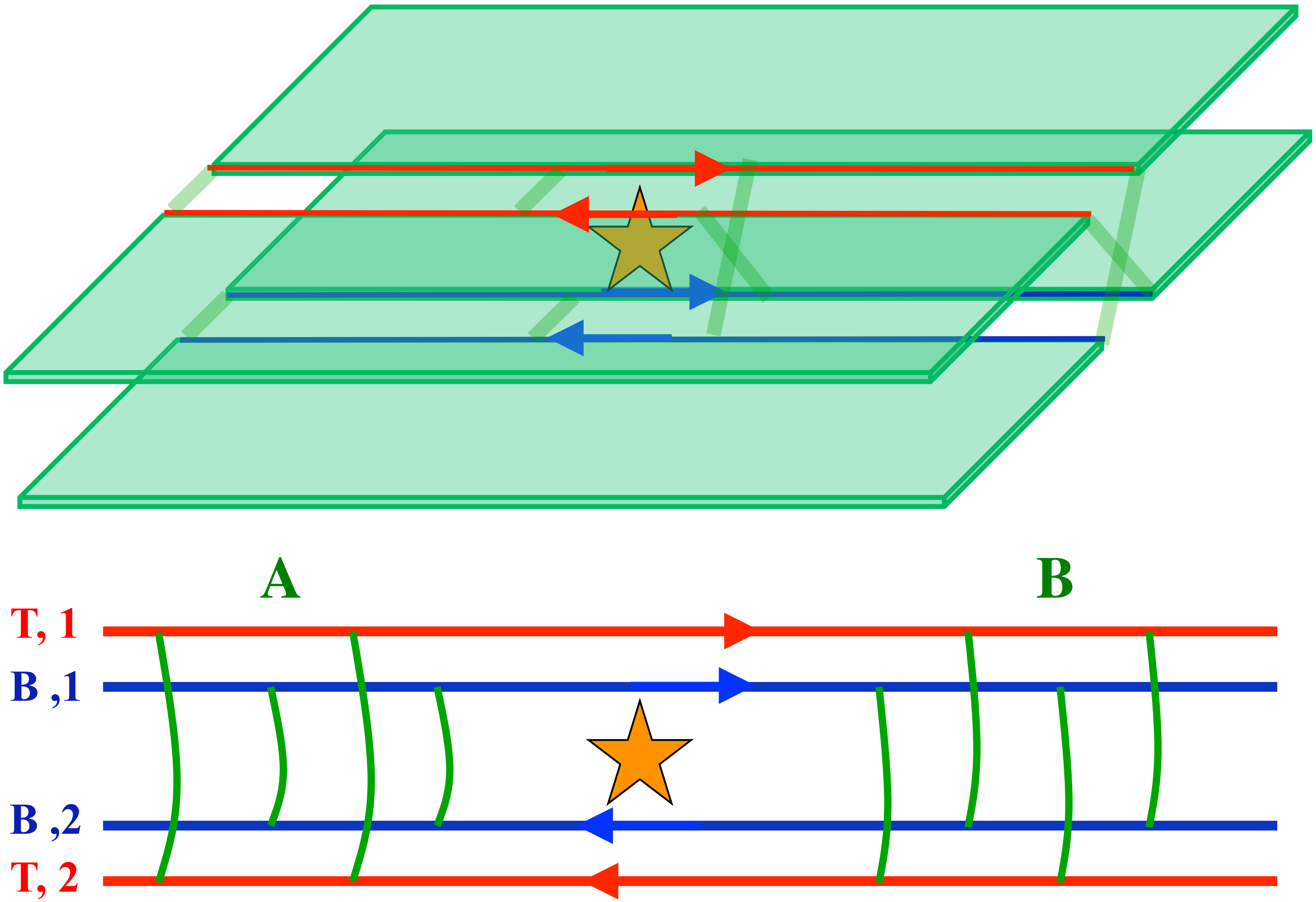
Fractionalized zero modes at "twist defects" in topological phases

Another example: ν=1/3 bilayer



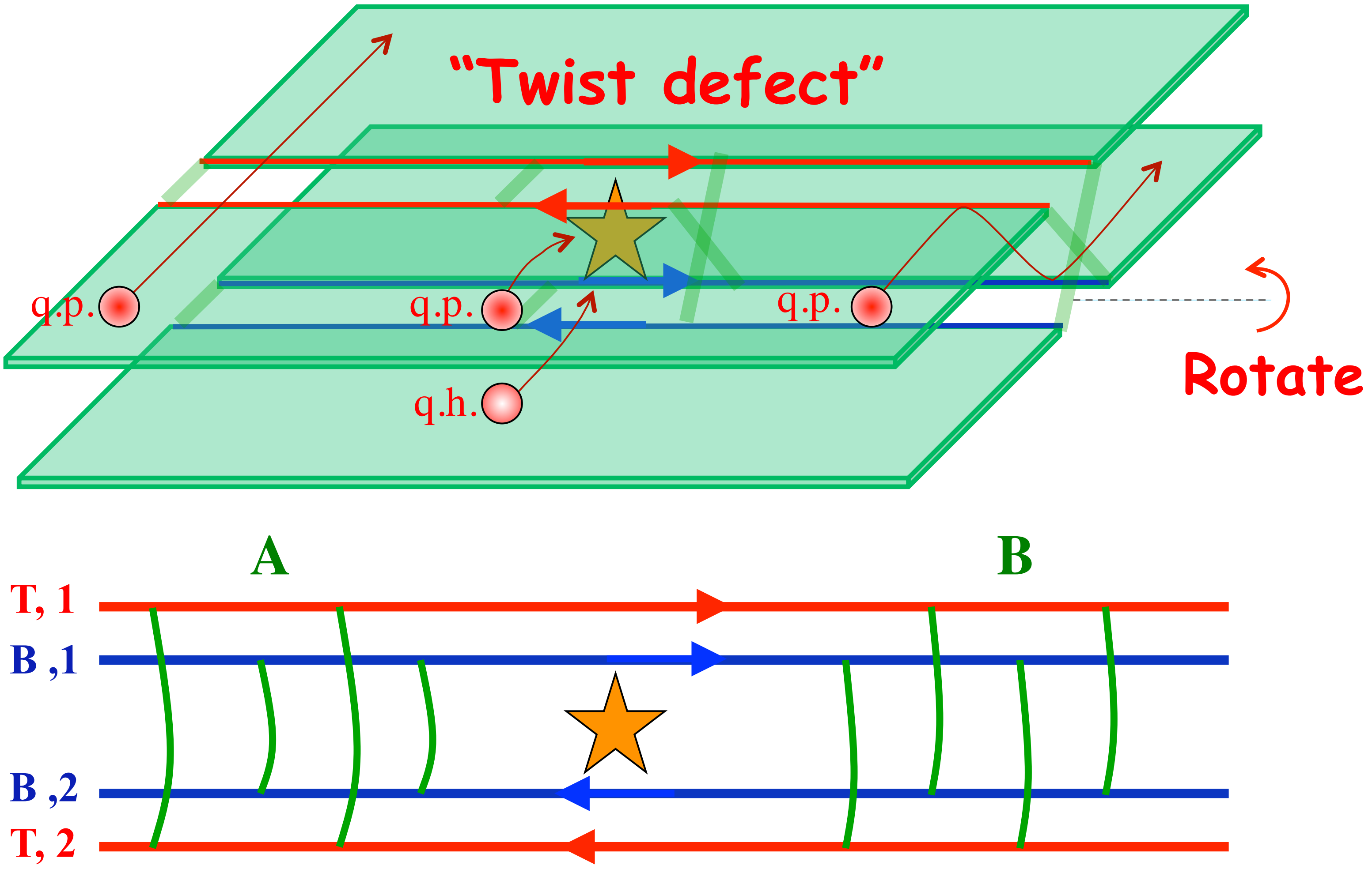
Fractionalized zero modes at "twist defects" in topological phases

$\nu=1/3$ bilayer

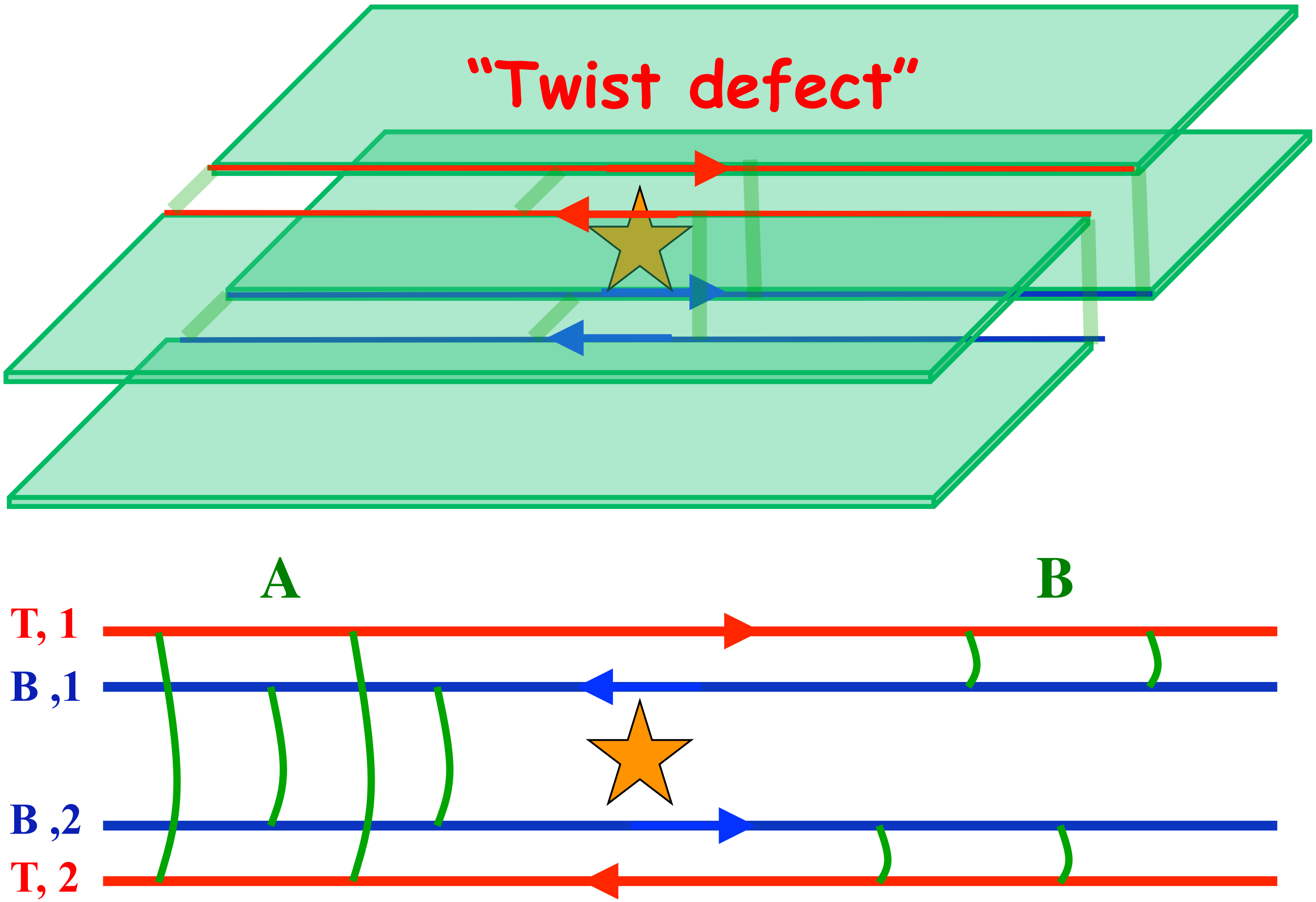


Fractionalized zero modes at "twist defects" in topological phases

ν=1/3 bilayer



Fractionalized zero modes at "twist defects" in topological phases

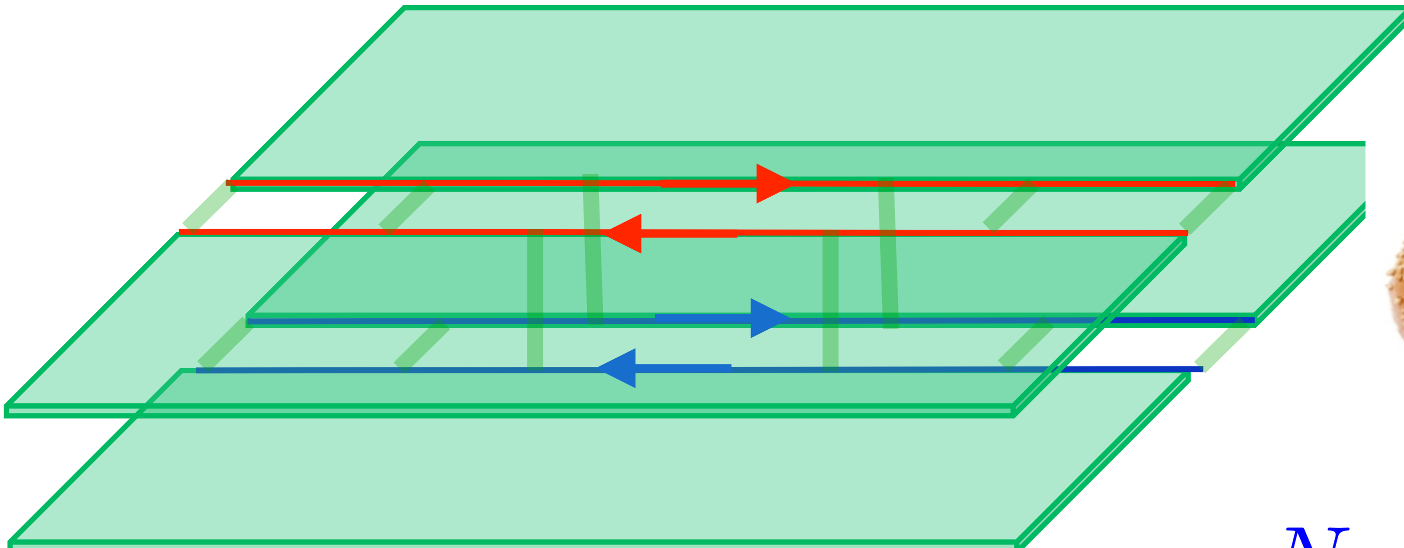


Fractionalized zero modes at "twist defects" in topological phases

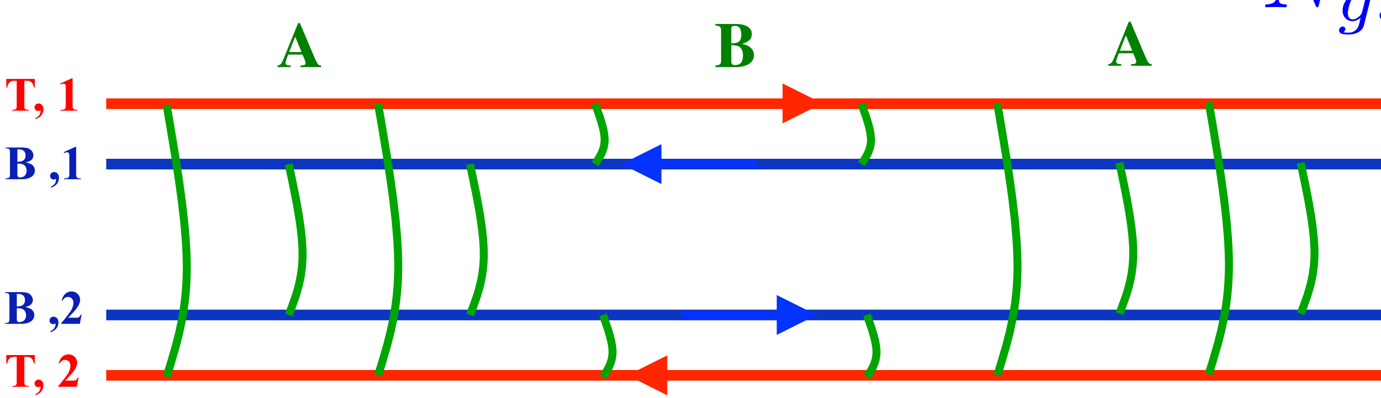
Alternate A,B domains:

Parafermions without superconductivity!

High genus surface



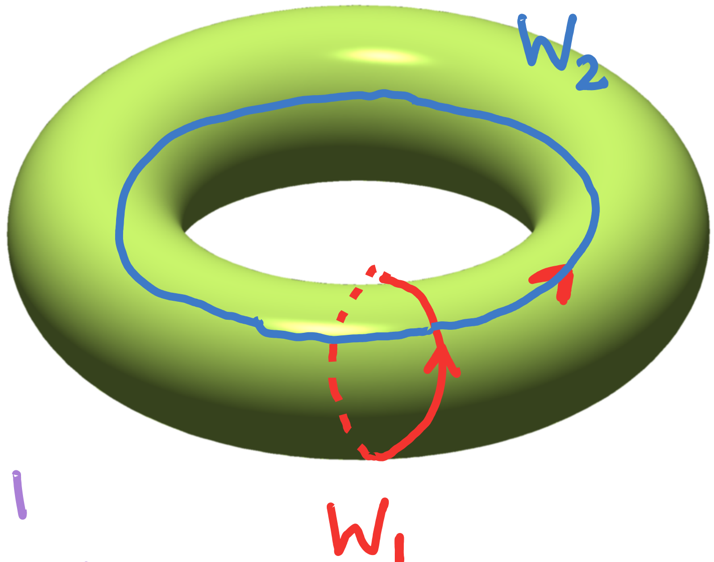
$$N_{gs} = 3^{N_{\mathrm{holes}}}$$



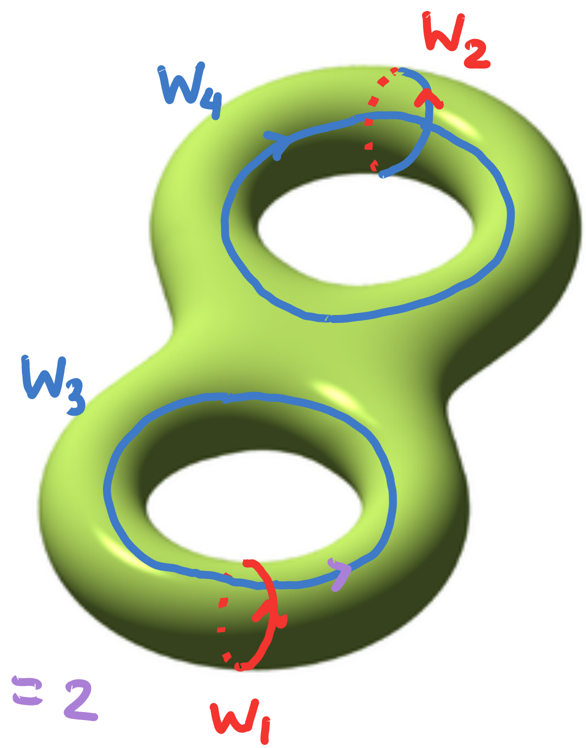
Origin of topological degeneracy: Degeneracy on high genus surfaces

$$W_1 W_2 = e^{\frac{2\pi i}{m}} W_2 W_1$$

$$[W_1, H] = 0 \quad [W_2, H] = 0$$



$g = 1$
(genus)



$g = 2$

$$N_{g.s.} = m^g$$

X. G. Wen (1991)

Outline

- “Fractionalized Majoranas” on fractional quantum Hall edges
  - Fractionalized 1D superconductors
  - Twist defects

- Anyonic defects in non-Abelian systems

Enriching non-Abelian phases by defects

Defects in Abelian phases (e.g. FQH) have non-Abelian properties.

However, the non-Abelian statistics of defects in Abelian phases is never universal for TQC.

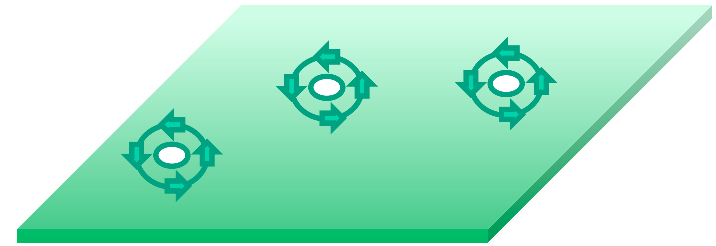
Begin with a non-Abelian phase and "enrich" its properties by defects?

Ising anyons

ν = 5/2 QHE

px+ipy Superconductors

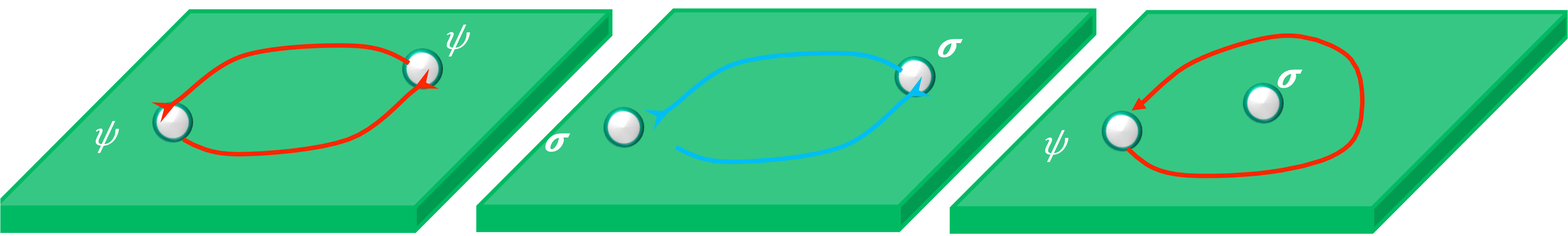
Kitaev's hexagonal spin model



Three types of particles: I (vacuum), ψ (fermion), σ (vortex)

Fusion rules: $\psi \times \psi = I \qquad \sigma \times \psi = I$

$$\sigma \times \sigma = I + \psi$$



$$R^I_{\psi\psi} = -1$$

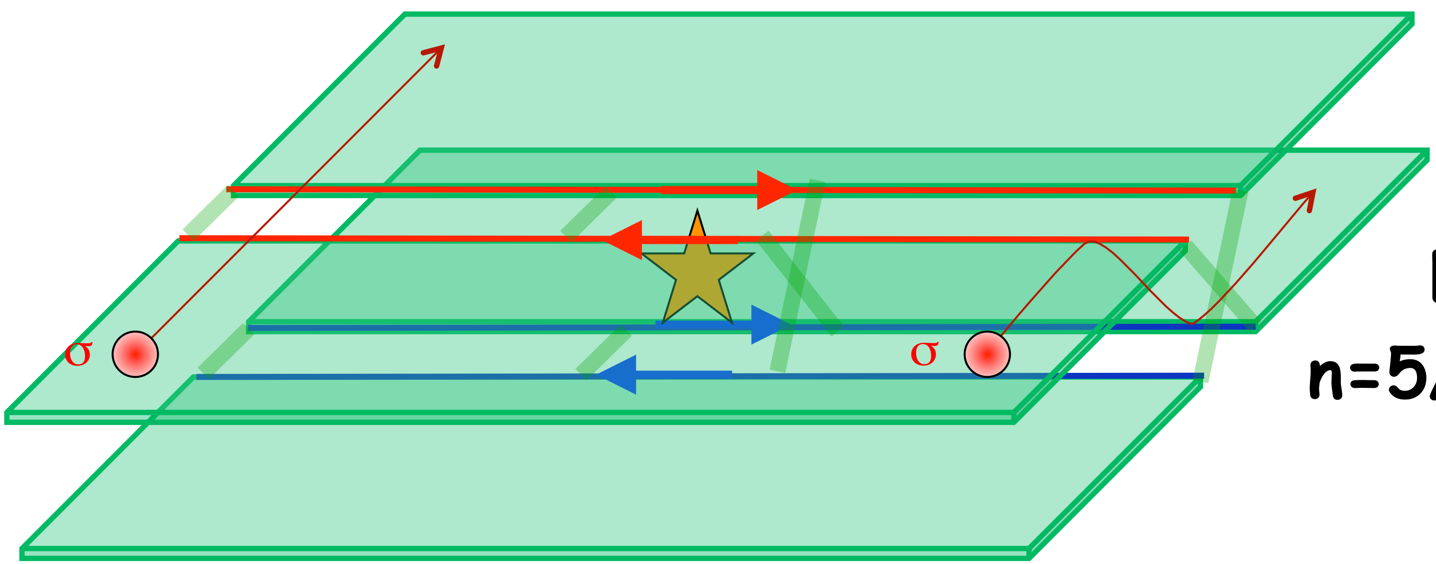
$$R^I_{\sigma\sigma} = e^{-i\pi/8}$$

$$R^\psi_{\sigma\sigma} = e^{3i\pi/8}$$

$$\left(R_{\sigma\psi}\right)^2 = -1$$

Defects in a bilayer Ising phase

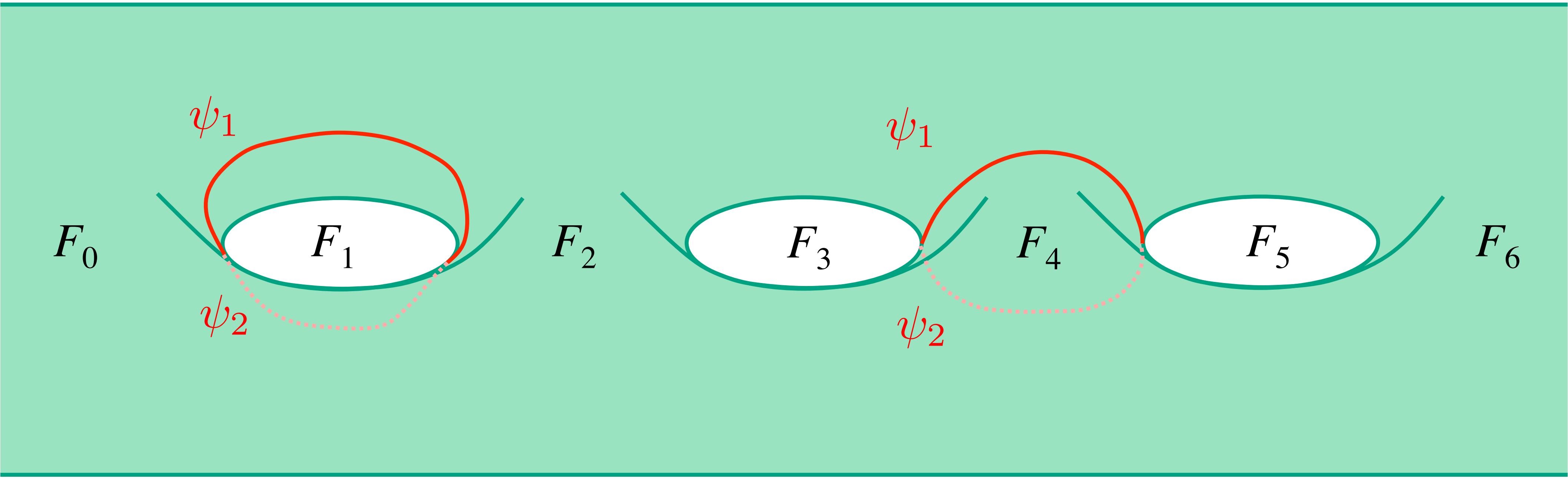
- What is the mathematical description of the zero modes associated with the defects?
- Can the zero modes realize universal TQC even though the host Ising phase is not universal?



Bilayer of
n=5/2/p+ip SC/…

Ground states

States can be described fluxes of holes, and measured by fermion loop operators

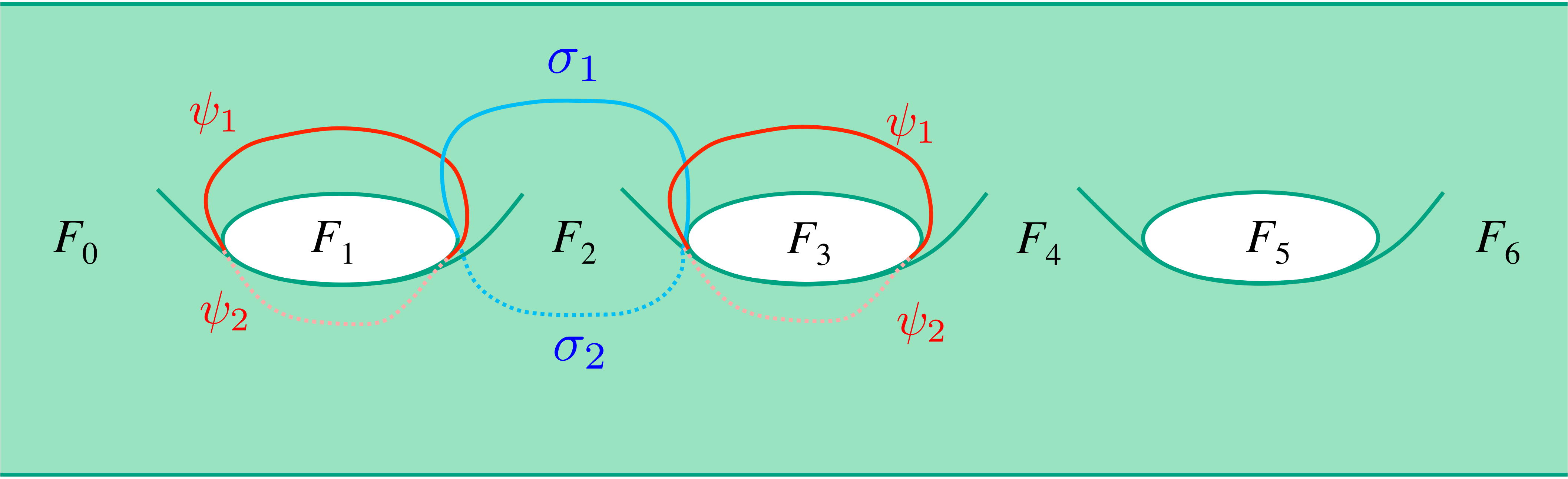


$F_j = 0, 1$ **Z_2 flux: represent as** $F_j = (1 + \sigma_j^z)/2$

Not all flux states are ground states

Creating flux states

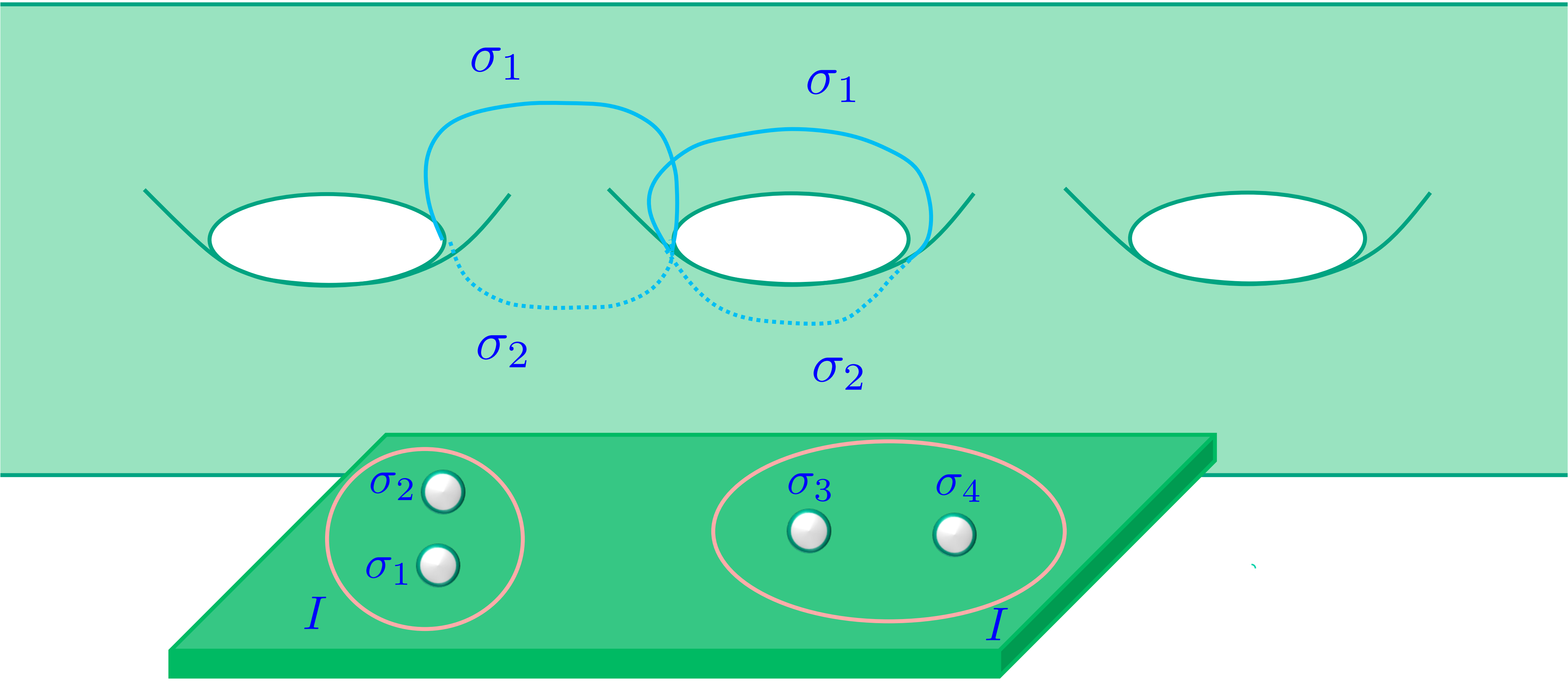
Flux states can be created by σ loops



σ loop operator $W_{2,3}$ flips F_1 and F_2

Blocking rules

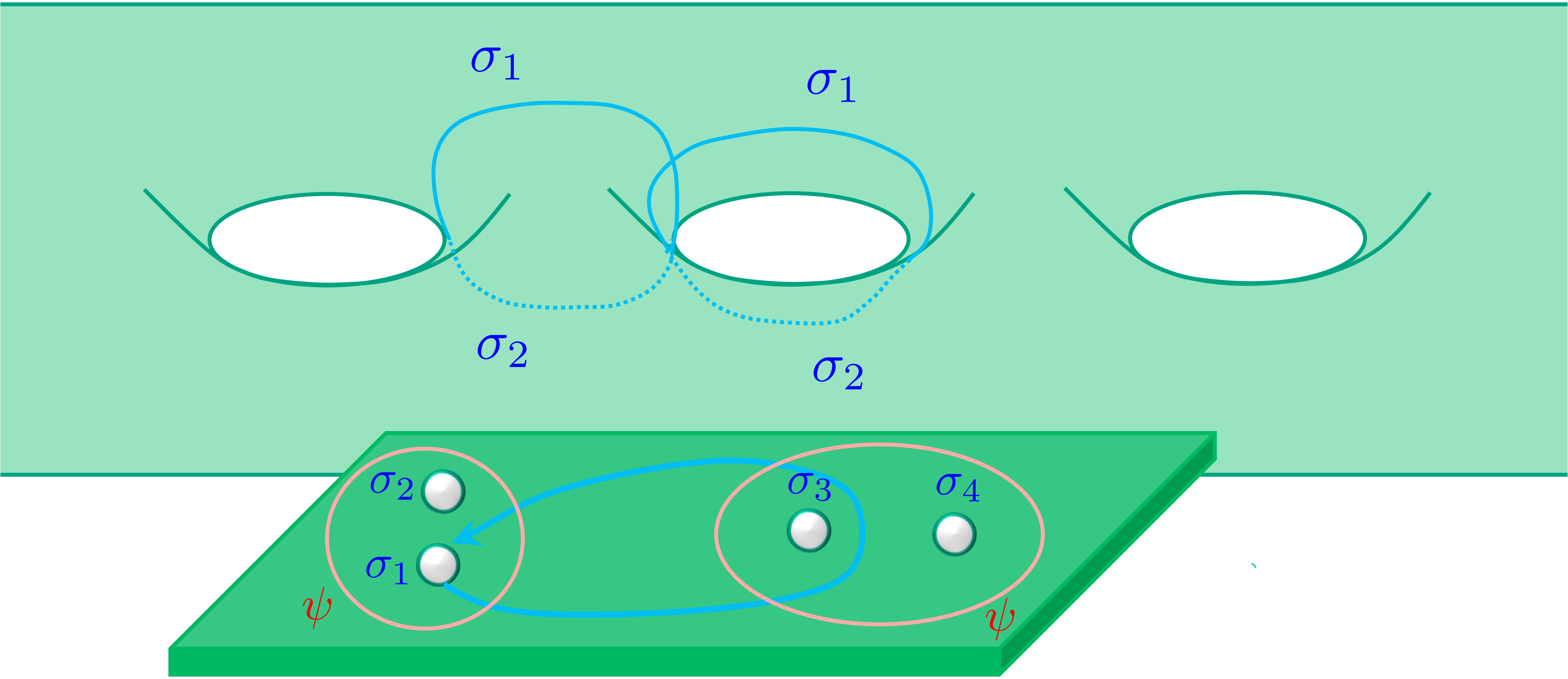
Act with two neighbor W operators:



Blocking rules

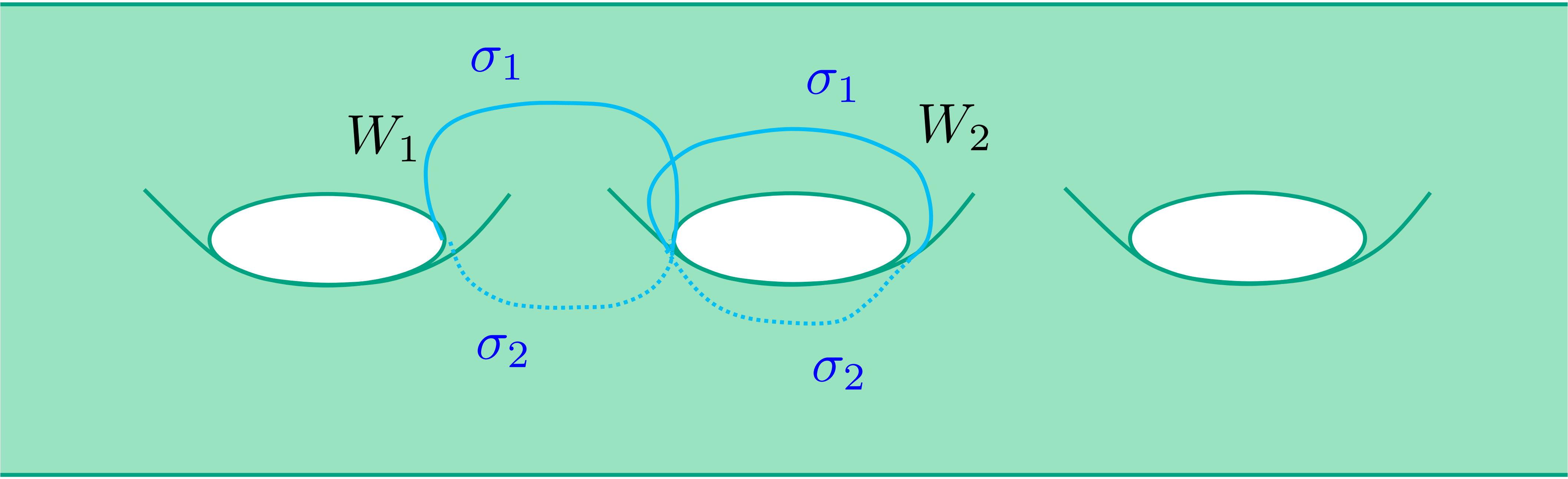
Act with two neighbor W operators:

A ψ excitation is created!



Blocking rules

Final state has ψ excitation!

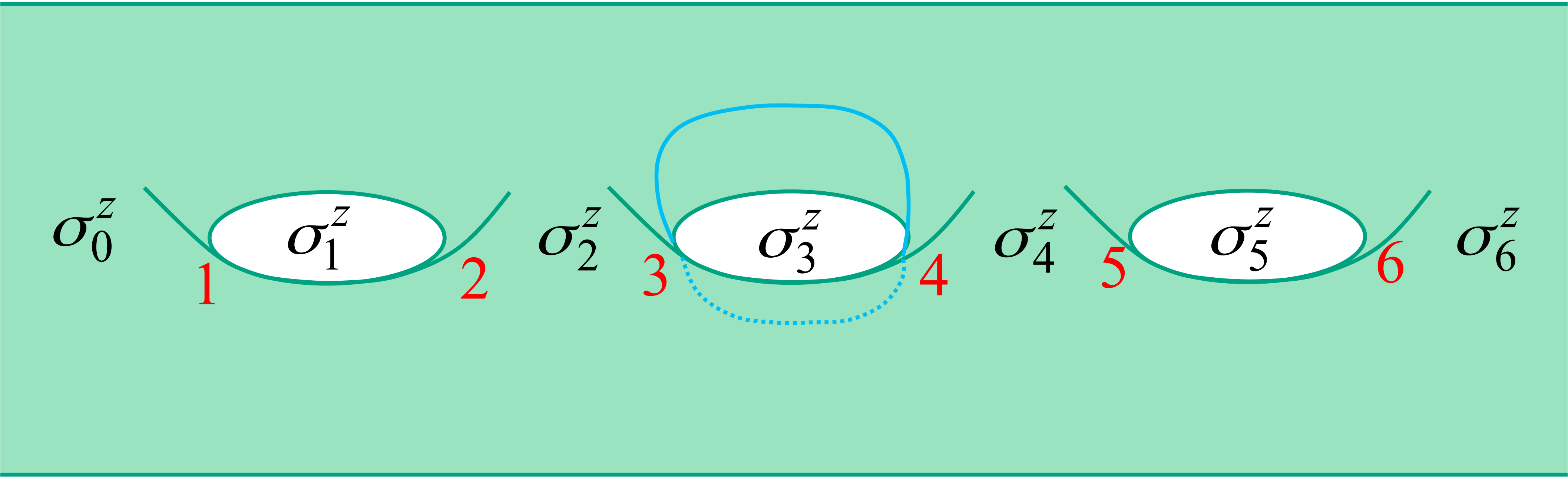


$$W_1 W_2 W_1^{-1} W_2^{-1} = 0$$

(projected to
the ground state subspace)

Tunneling operators

Nearest neighbors: form in a **<u>convenient gauge</u>**:

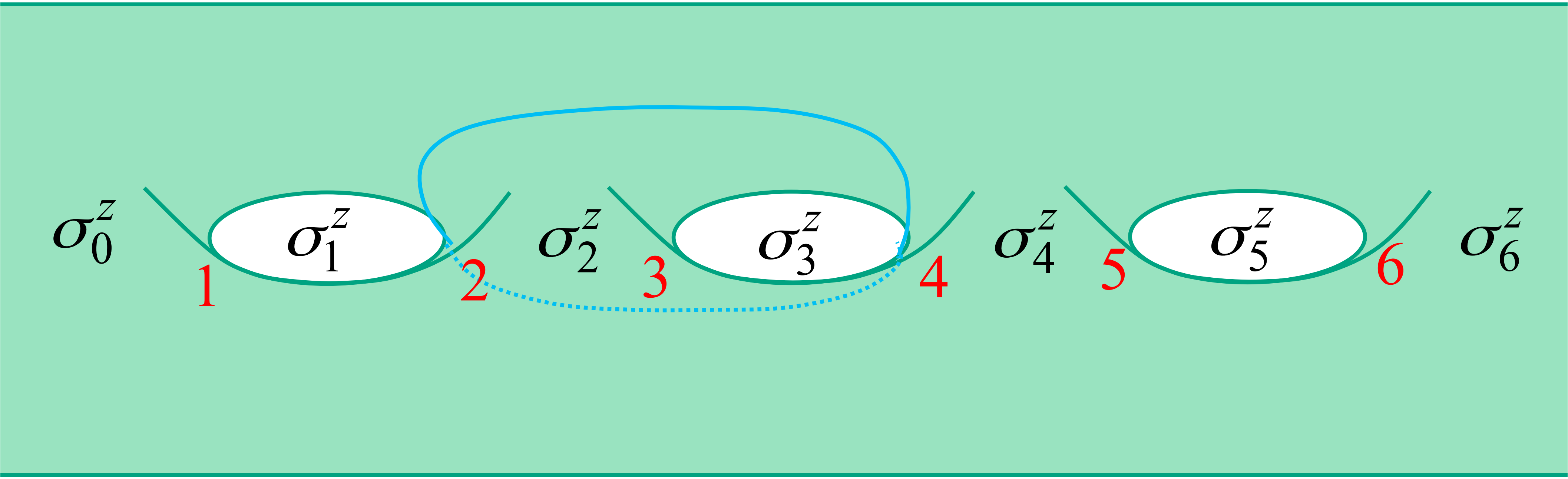


$$W_{i,i+1} = \sigma_{i-1}^x \left(\frac{1+\sigma_i^z}{2} \right) \sigma_{i+1}^x$$

Hermitian
-not unitary
(projected)

Tunneling operators

General form: defined by tri-algebra

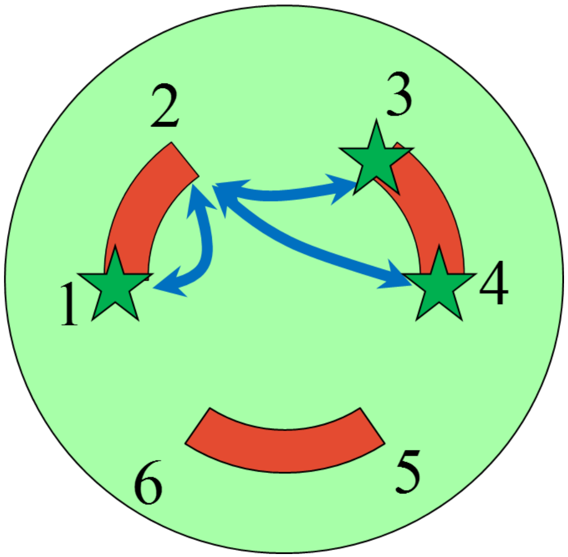


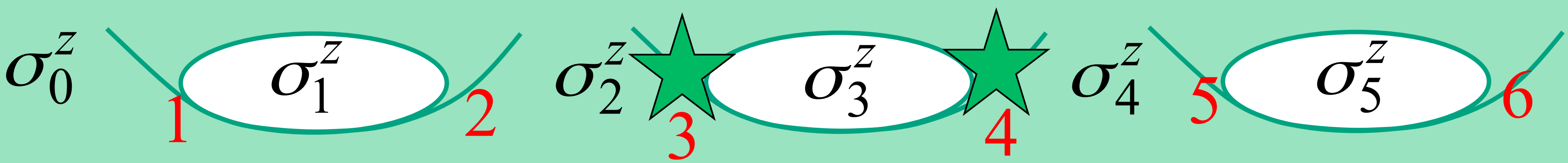
$$W_{mn} = e^{i\pi/8}\left(W_{mk}W_{kn} + h.c\right)$$

Braiding

$$U_{34} = \left(\frac{1+\sigma_3^z}{2}\right)\sigma_2^x\sigma_4^x + \left(\frac{1-\sigma_3^z}{2}\right)e^{i\pi/4}$$

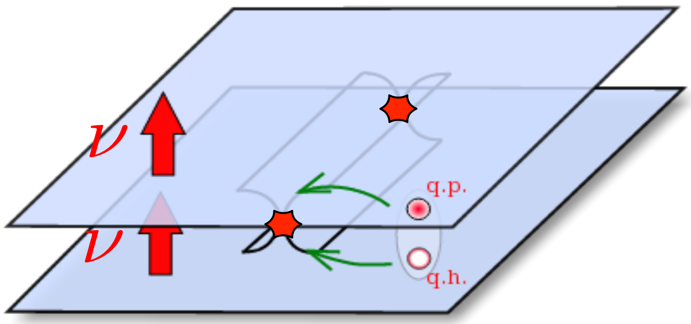
Phase gate needed to make Ising theory universal!

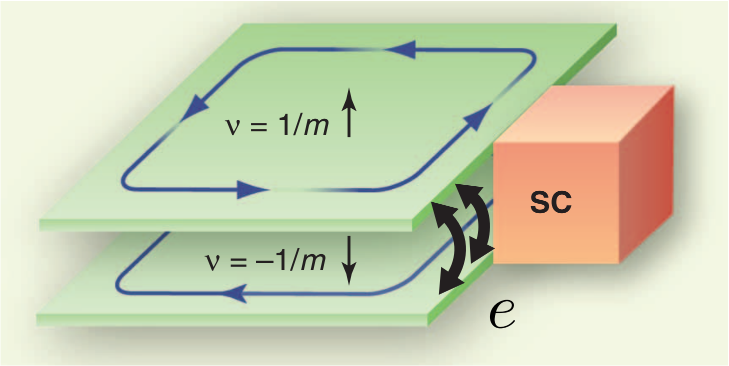




Conclusion

New paradigm for realizing non-abelian anyons: defects on edges of two-dimensional topological phases.





Future directions:

Classification of 1D gapped edge states of 2D topological theories?

Experimental signatures?

Thank you.