Fractional quantum statistics

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Thanks to:
Anders, Eddy, Emil, Jainendra, Jon-Magne, Juha, Maria, Susanne, ...

Outline:
- **What is fractional statistics?**
- **Where does the quantum Hall effect enter?**
- **What is non-Abelian fractional statistics?**
- **Anyons and Topological Field Theory**

- **Why does Microsoft care - topologically protected quantum computing.**
- **What are the experiments?**

Wave function for identical particles:

$$P_{ij}\Psi(\vec{x}_1 \ldots \vec{x}_i \ldots \vec{x}_j \ldots \vec{x}_N) = e^{i\theta_{ij}}\Psi(\vec{x}_1 \ldots \vec{x}_j \ldots \vec{x}_i \ldots \vec{x}_N)$$

The particles are **identical** since an overall phase is not observable.

But changing back amounts to nothing!

$$P_{ij}^2\Psi(\vec{x}_1 \ldots \vec{x}_i \ldots \vec{x}_j \ldots \vec{x}_N) = \Psi(\vec{x}_1 \ldots \vec{x}_i \ldots \vec{x}_j \ldots \vec{x}_N)$$

$$P_{ij}^2 = 1 \quad \Rightarrow \quad e^{i\theta} = \pm 1$$

So there are only two alternatives:

$$P_{ij} = 1 \quad ; \quad \theta_{ij} = 0$$ Bosons

$$P_{ij} = -1 \quad ; \quad \theta_{ij} = \pi$$ Fermions

BUT!!

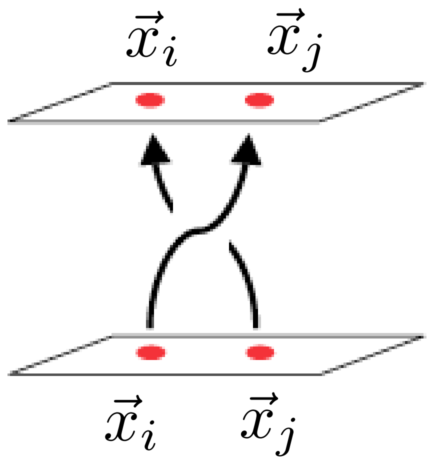
Permutations and braidings:

The basic idea:

The permutation : $\vec{x}_i \rightleftharpoons \vec{x}_j \quad , \quad sgn(P_{ij}) = \pm 1$

can also be viewed as a

the exchange :



$$\Rightarrow \quad e^{i\theta} \; , \quad \theta = 0, \pi$$

For general θ, the particles are ANYONS!

Leinaas & Myrheim, 77

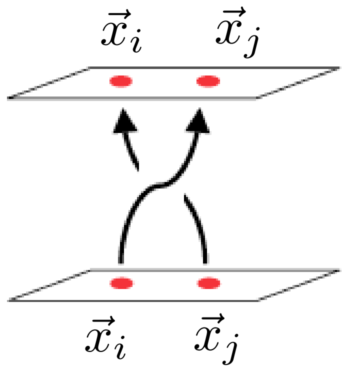
But what is meant by “exchange”?

In a path integral description, different classical exchange paths corresponds to different braids.

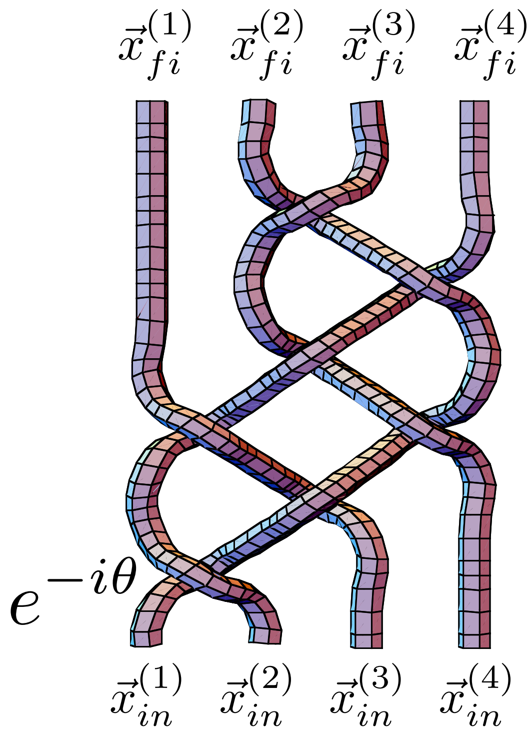
All paths that are described by the same braid, are assigned the same exchange phase!

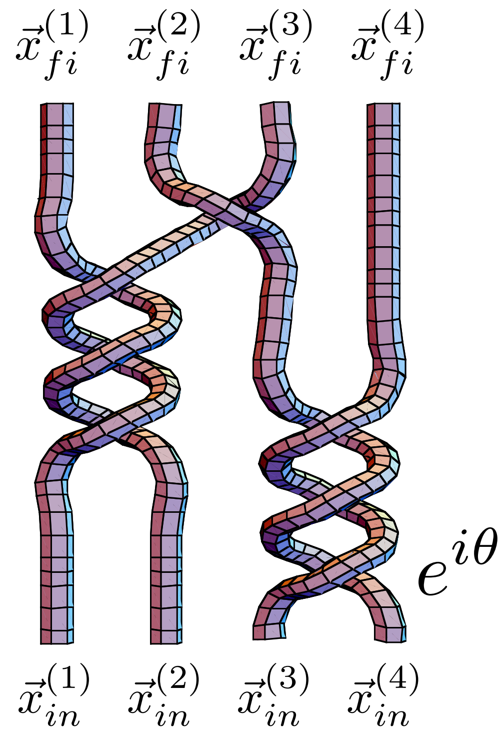
In a Hamiltonian description, we can imagine to “pin down” and move the particles around. In such a real exchange process, the wave function picks up a Berry phase!

Pathintegral for identical particles



$e^{i\theta}$

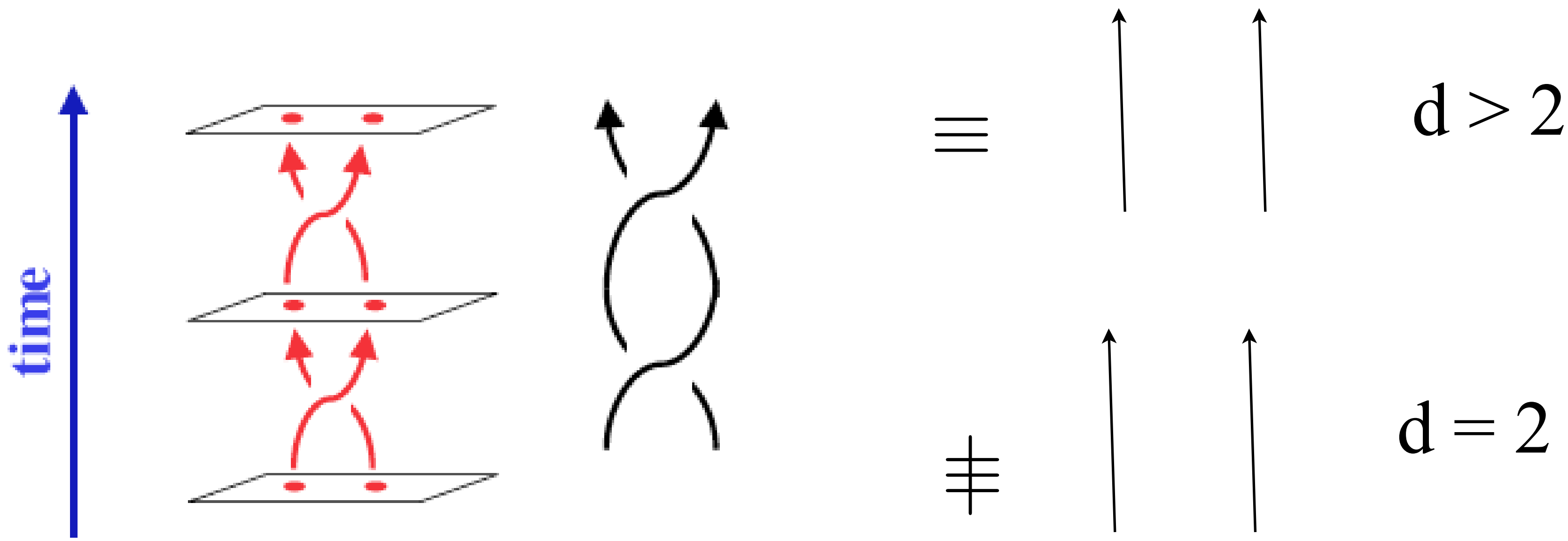




$\theta_{br} = (3-4)\theta = \theta$ $\qquad$ $\theta_{br} = (6-1)\theta = 5\,\theta$

$$A(\vec{x}_{in}^{(\alpha)} \to \vec{x}_{fi}^{(\alpha)}) = \sum_{braids} e^{i\theta_{br}} \int_{\vec{x}^{(\alpha)}=\vec{x}_{in}^{(\alpha)}}^{\vec{x}^{(\alpha)}=\vec{x}_{fi}^{(\alpha)}} \mathcal{D}[\vec{x}^{(\alpha)}] e^{\frac{i}{\hbar}S}$$

Why are two dimensions special ?



So there are no braids for d>2, and we are back to just having the permutation symmetry, and thus only bosons and fermions!

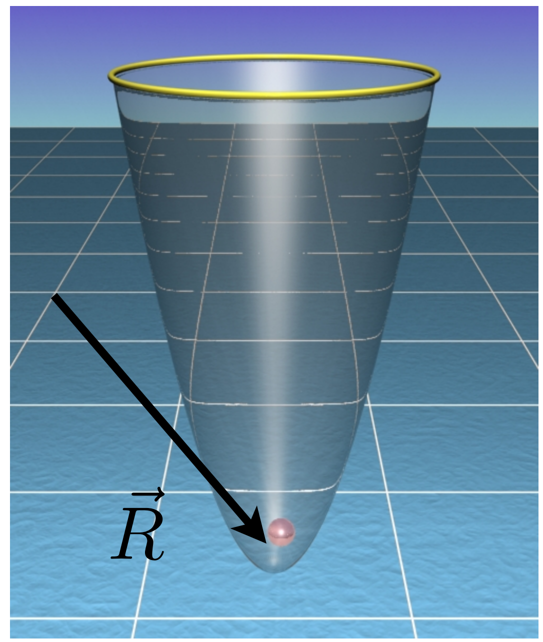
Wave functions for identical particles

- "Pin down" the particles
- Move them around
- Calculate the statistical phase

We can pin down a particle by putting it in a box:

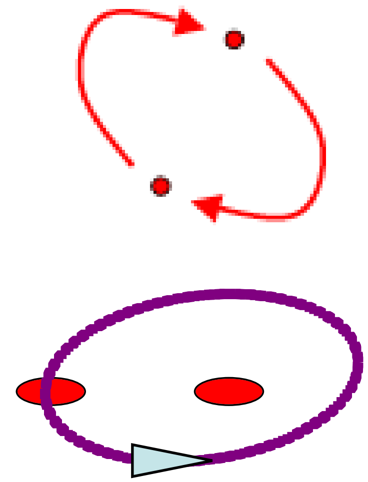
$$H = \frac{p^2}{2m} - V_{\mathrm{box}}(\vec{x} - \vec{R})$$

The particle can now be moved by changing $\vec{R}$.



Question:

What happens when two particles are *exchanged*, or alternatively, one *encirles* the other:



?

Answer:

The w.f. "picks up" a phase factor:

$$\Psi(\vec{x}_i, \vec{x}_j) \rightarrow e^{i\theta}\Psi(\vec{x}_j, \vec{x}_i)$$ **Abelian**

$$\Psi_\alpha(\vec{x}_i, \vec{x}_j) \rightarrow e^{i\theta_a T^a_{\alpha\beta}}\Psi_\beta(\vec{x}_j, \vec{x}_i)$$ **Non-Abelian**

But how do we calculate?

Michael Berry's phase:

Assume: $H(R_i)\psi_n(\vec{r};R_i) = E_n(R_i)\psi_n(\vec{r};R_i)$

How will $\psi_n(\vec{r};R_i)$ evolve under *adiabatic* time evolution if the parameters become timedependent: $R_i \to R_i(t)$?

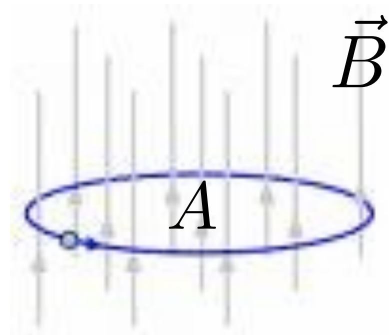
Not **but** $\psi_n(\vec{r},T;R_i) = e^{i\gamma_B}\, e^{-\frac{i}{\hbar}\int_0^T dt E_n(R(t))}\psi_n(\vec{r},0;R_i)$

where the Berry phase is given by, $\gamma_B = i\int_0^T dt\,\dot{R}^a\langle\psi_n|\frac{\partial}{\partial R_a}|\psi_n\rangle$

can be calculated from the wave function!

For a closed path, in parameter space Berry's phase is a geometric property - *i.e.* it does not depend on any arbitrary choices of phase of $|\psi_n(R_i)\rangle$.

Example: Particle in a magnetic field



$$\gamma_B = \frac{\Phi}{\phi_0} = \frac{BA}{\phi_0}$$

where $\phi_0 = 2\pi\frac{\hbar}{e} = \frac{h}{e}$ is the elementary flux quantum.

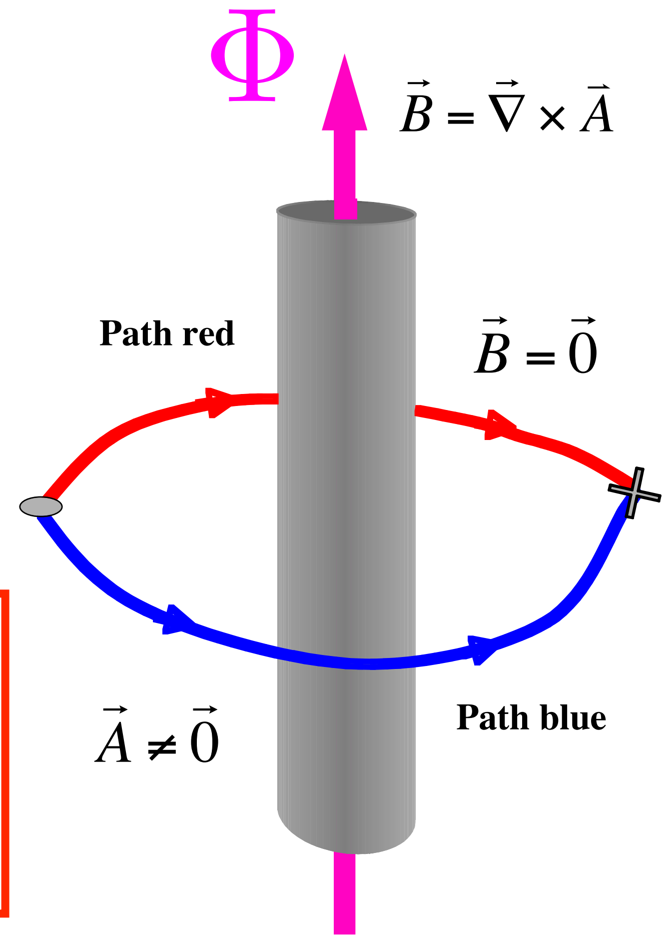
This result is independent of gauge choice!!

In certain situations the Berry's phase is a topological property - *i.e.* it does not depend on the geometry of the paths $R_i(t)$ involved.

Example:

The Aharonov-Bohm effect:

$$\gamma_B = \frac{\Phi}{\phi_0}$$



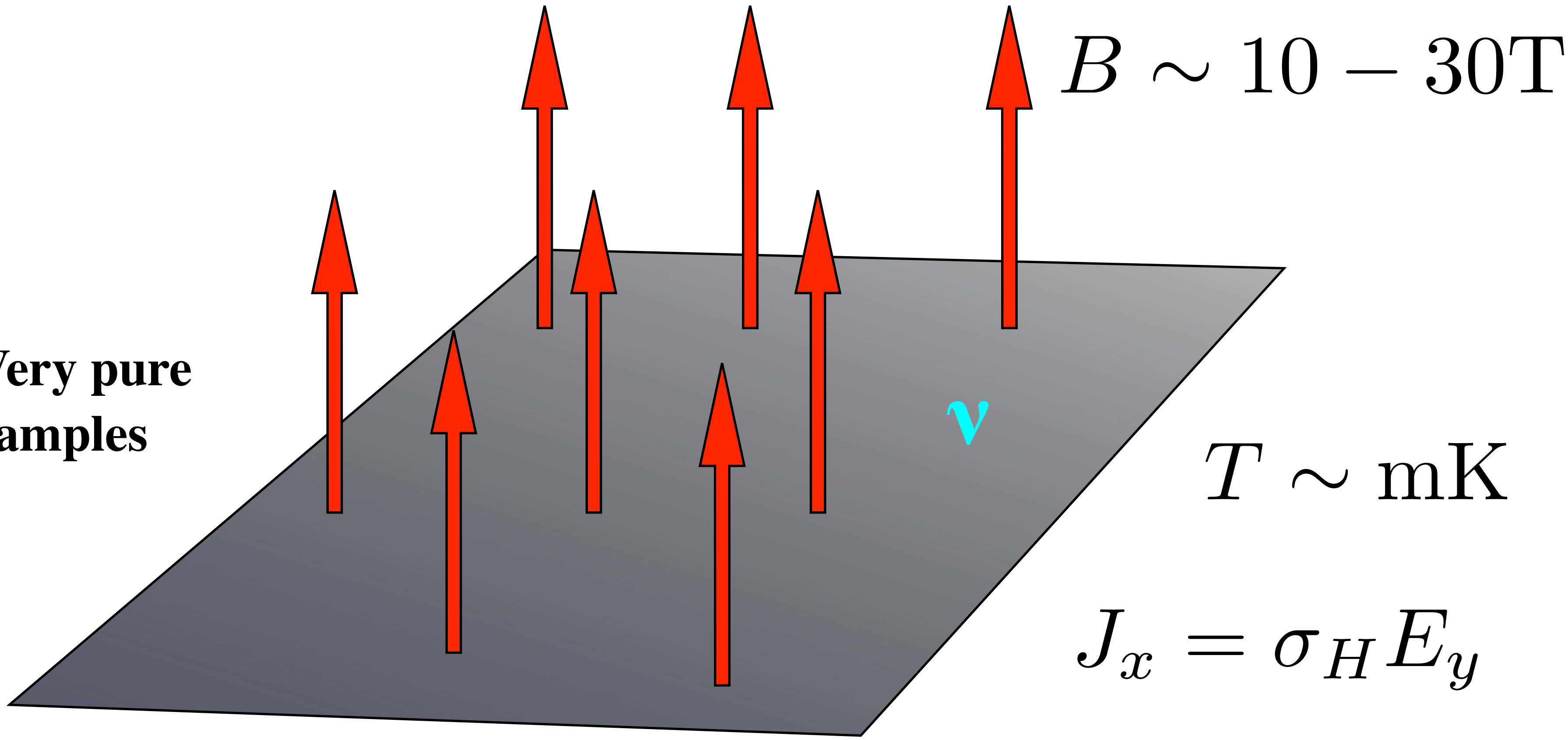
The phase depends only on the flux and if the path is clock wise or not!
No adiabatic assumption needed!!

First main message of this talk:

The statistics of identical particles can be calculated as the topological Berry phase corresponding to a path where two particles are exchanged!

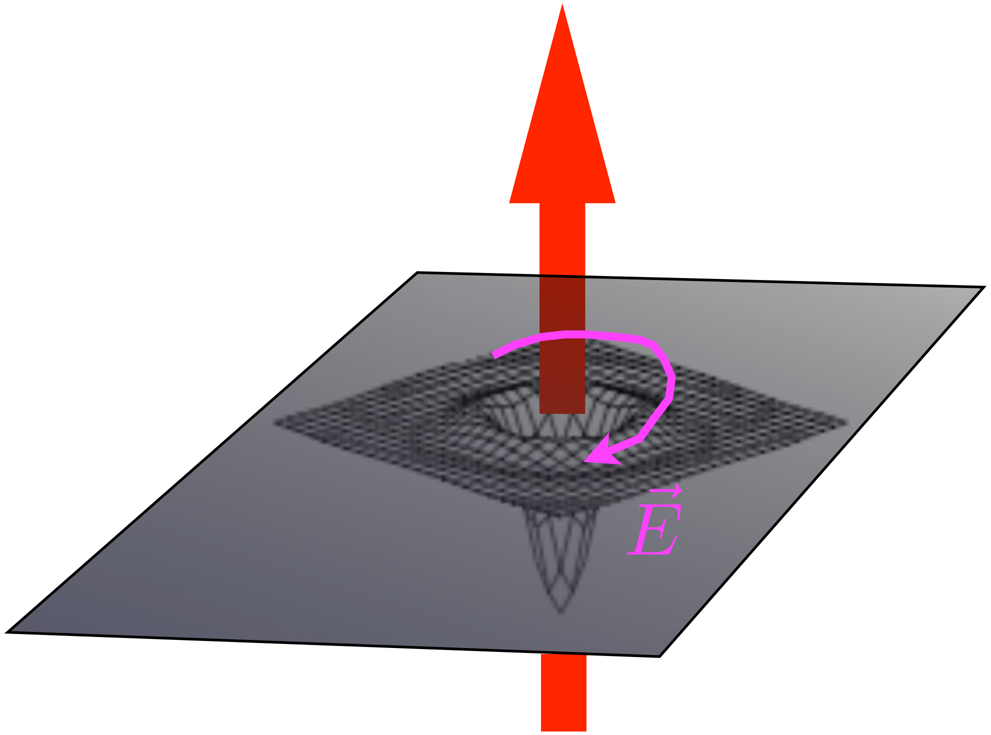
- Since the particles are identical, an exchange path is closed in parameter space.
- The statistical phase factors can be thought of as AB phases due to a statistical gauge field.
- The particles carry thin flux tubes of this gauge field.

The Quantum Hall liquids



Incompressible electron liquids with conductance quantized to an extreme precision at rational values of the "filling fraction" ν !

Laughlin's gauge argument, ν=1/m



A unit of flux through an empty hole can always be removed.......

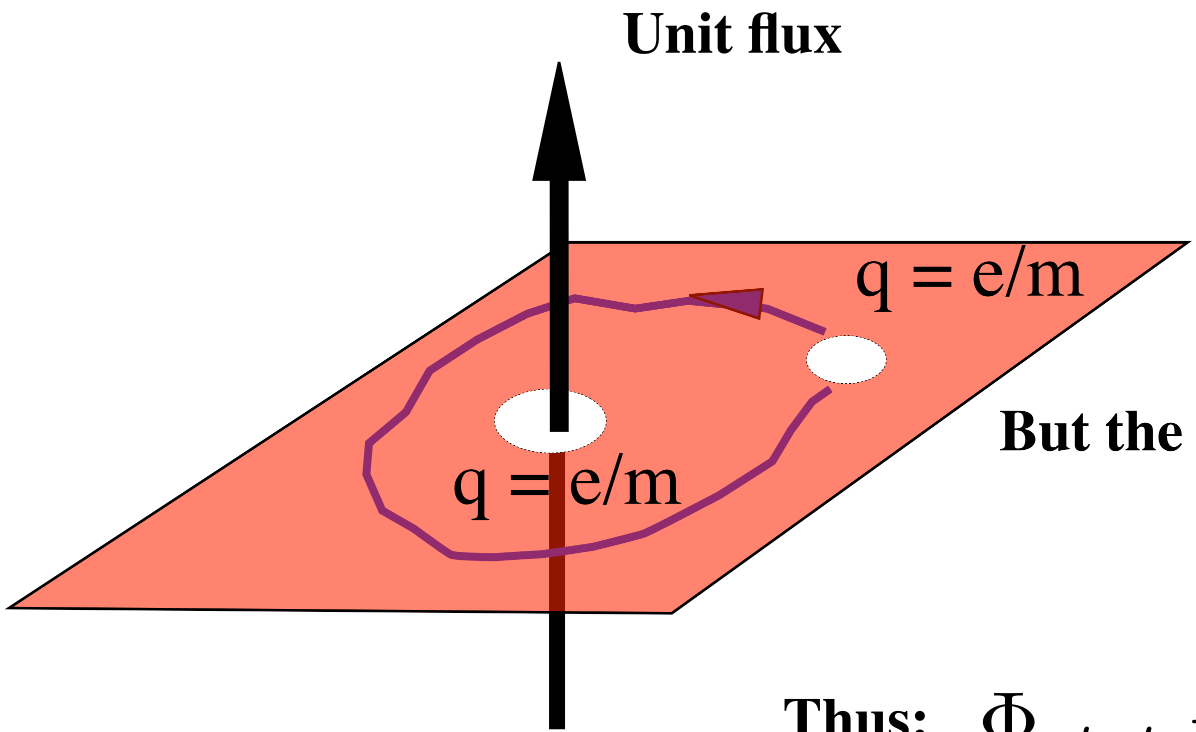
Turn on magnetic flux to push out the liquid and form a hole

Leaving us an excited eigenstate of the original Hamiltonian, and

It is not hard to prove that the sharp charge is e/m !

Fractional charge & fractional statistics

The Kivelson - Roček argument



Aharonov-Bohm phasefactor:

$$\Phi_{AB} = e^{i\frac{2\pi}{m}}$$

But the system consists of electrons!

$$\Phi_{AB}\Phi_{stat} = 1$$

Thus: $$\Phi_{stat} = e^{i2\pi\frac{m-1}{m}}$$ **!!!**

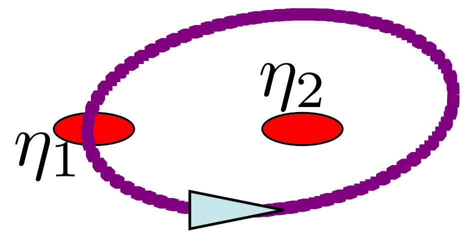
- **Connects the local charge to the non-local fractional statistics.**

This is why we believe that the QH particles are anyons although there is no clear experiment

Archetypical example: Two Laughlin holes

$$\Psi_{2\,qh}(\eta_1, \eta_2; z_i \dots z_N) = N(\eta_i, \bar{\eta}_i) \prod_i (z_i - \eta_1)(z_i - \eta_2) \Psi_L(z_i \dots z_n)$$

One hole encircles the other:



Arovas et.al. derived: $\gamma_B = \frac{\Phi}{\phi_0} - \frac{2\pi}{3}$

AB-phase
geometrical

Statistical phase
topological

Statistical angle: $\theta = -\frac{1}{2}\gamma_B^{top} = \frac{\pi}{3}$ $i.e.$ anyons!

But more strange things *could* happen:

There is an observed FQHS at ν=5/2, where the candidate wave function has the following properties:

- The wave function is paired - very similar to that of a $p_x + p_y$ superconductor.
- The quasiparticles have charge 1/4 rather than 1/2 as expected from Laughlin's argument - this is due to pair-breaking.
- States with 2n quasiholes are degenerate for fixed positions of the quasiholes; degeneracy = 2^{n-1}.
- The quasiholes obey non-Abelian fractional statistics.
- There is a well developed mathematical machinery for calculating the braiding matrices.

Berry phases in presence of degeneracy:

Simplest example - two fold degeneracy:

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} \psi_n^{(1)} \\ \psi_n^{(2)} \end{pmatrix} = E_n \begin{pmatrix} \psi_n^{(1)} \\ \psi_n^{(2)} \end{pmatrix}$$

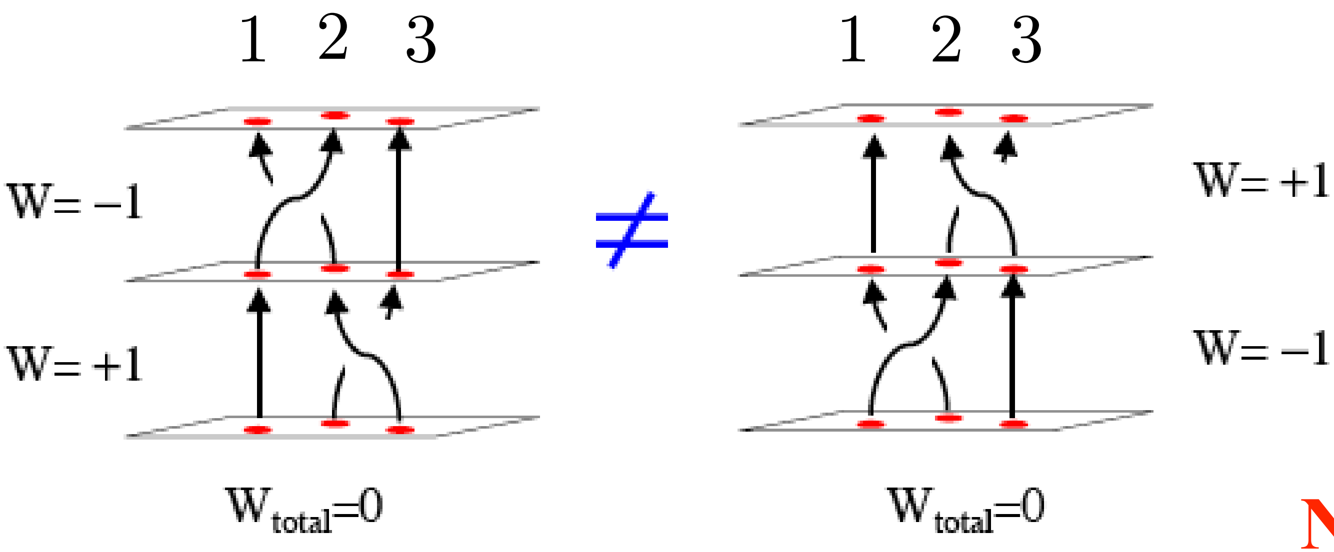
After a cyclic change of parameters, the vectors in the degenerate subspace change by a unitary transformation:

$$\begin{pmatrix} \psi_n^{(1)} \\ \psi_n^{(2)} \end{pmatrix} \to \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} \psi_n^{(1)} \\ \psi_n^{(2)} \end{pmatrix}$$

The matrix U, generalizes the Berry phase factor:

$$e^{i\gamma_B} \longrightarrow \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \equiv e^{i\mathbf{T}_B}$$

In the case of non-Abelian statistics, exchange paths corresponding to different braids can be non equivalent even though the total winding might be the same:



$$U_{21}U_{32}^{\dagger} \neq U_{32}^{\dagger}U_{21}$$

Non-Abelian representation of the braid group!

$$\Psi_{\alpha}(\vec{x}_i, \vec{x}_j) \rightarrow e^{i\theta_a T^a_{\alpha\beta}}\Psi_{\beta}(\vec{x}_j, \vec{x}_i)$$

Second main message of this talk:

- There are candidate physical systems where the n - quasiparticle states are degenerate for fixed positions.
- Braiding the particles amounts to a rotation among the degenerate states.
- Since these rotations are non-commuting, the particles are said to obey non-Abelian fractional statistics

- **There is no generally accepted experimental evidence for non-abelian statistics.**

The mathematics of anyons

Questions:

1. **What is the mechanism for attaching flux to charge?**
2. **What is a Topological Field Theory (TFT)?**

Answer 1:

Abelian Anyons = ordinary fermions or bosons coupled to a gauge field with a Chern-Simons action

$$\mathcal{L} = \mathcal{L}_{\rm CS} - a_\mu j^\mu = \frac{1}{4\theta}\epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho - a_\mu j^\mu$$

$$a_0 \quad \text{e.o.m.}: \quad \epsilon^{ij}\partial_i a_j = 2\theta j^0 \qquad \Leftrightarrow \qquad b(\vec{r}) = 2\theta\rho(\vec{r})$$

Integrate to get: $\mathbf{\Phi = 2\theta\, Q}$

The CS-theory a simple example of an topological field theory, which does not depend on the metric.
As a consequence, correlation functions of gauge invariant operators, *i.e.* Wilson loops,

$$W[L_i] = e^{i \oint_{L_i} dx^\mu a_\mu}$$

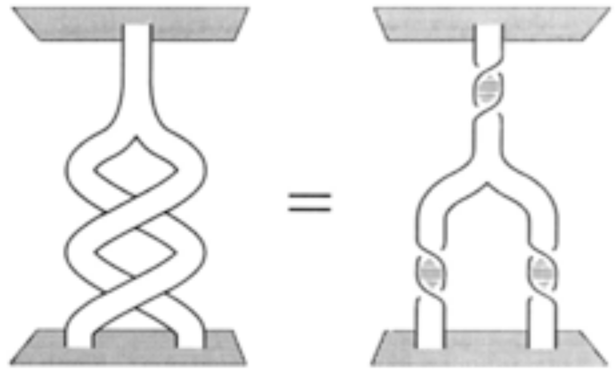
only depend on the topology of the braiding, and knotting of the loops.

The effective low-energy theories for the abelian QH liquids are multi-component CS theories, that encodes:

- **The QH conductance**
- **The fractional charges of quasiparticles**
- **Their fractional statistics**
- **The g.s. degeneracy on higher genus surfaces**

The effective low-energy theories for the non-abelian QH liquids, and other non-abelian phase are topological field theories characterized by:

- **Fusion rules** $a \times b = \sum_c N_c^{ab} c$
- **Spin factors** $\theta_a = e^{2\pi i h_a}$



- **Monodromies, or statistical phases** $e^{2\pi i (h_c - h_a - h_b)}$
- **Modular transformations**
  - **The corresponding mathematical structures are modular ribbon categories**
  - **The topological field theories are closely related to Conformal Field Theory, and the effective field theories of non-abelian QH liquids are CFTs with nontrivial fusion**

Third main message of this talk:

- Flux-charge attachment can be obtained by coupling charged particles to a gauge field with a Chern-Simons term
- General topological field theories, defined by topological spin, fusion, and braiding, is the natural mathematical language to describe non-abelian anyons.
- The topological field theories are closely related to the conformal field theories used in string theory

Thank You for Listening!

Basics:

J.M. Leinaas and J. Myrheim, Nuovo Cimento B 37, 1 (1977)

F. Wilczek, Phys. Rev. Lett. 49, 957 (1982).

F Wilczek and A. Zee, Phys. Rev. Lett., 1984

CS theory for QHE:

S.C. Zhang, Int. Jour. of Mod. Phys. B, 1992

X.-G. Wen, Int. Journ. of Mod. Phys. B, 6, 1711 (1992).

TFT and CFT

E. Witten, Comm. Math. Phys. 121, 351 (1989).

G. Moore and N. Read, Nucl. Phys. B 360, 362 (1991).

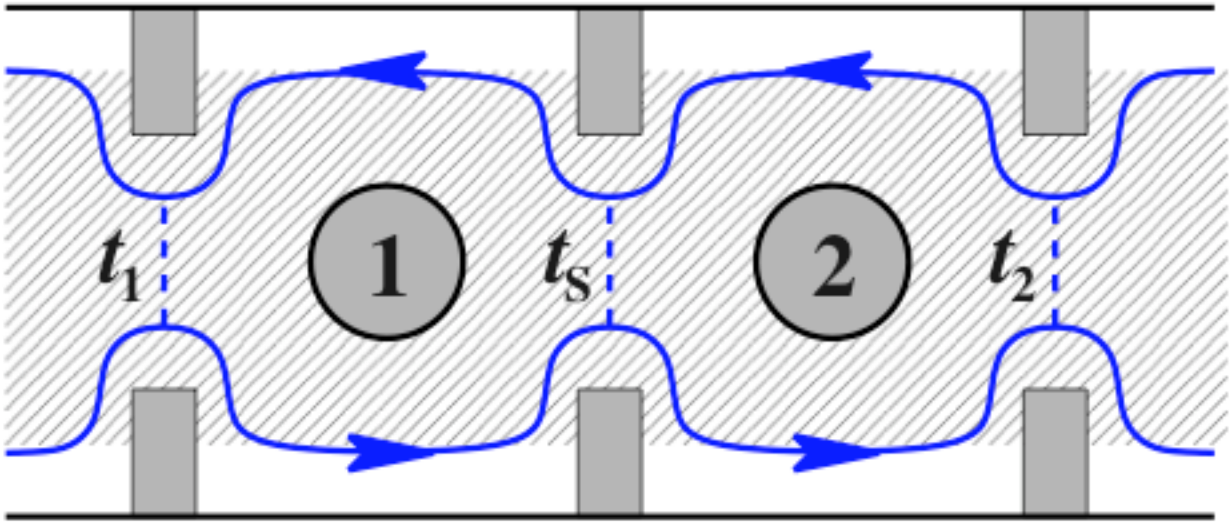
F. A. Bais, and J. K. Slingerland, Phys. Rev. B 79, 045316 (2009)

What can be measured, and how?

- **Fractional charge**
  - ✦ Shot noise in point contact
  - ✦ AB-interference in quantum dot ??
- **Fractional statistics**
  - ✦ Fabry-Pérot interferometer
  - ✦ Mach-Zehnder interferometer

- **Fractional charge is generally considered to be confirmed.**
- **The fractional statistics experiments are very much under debate and there is no consensus about their status.**

How to measure non-Abelian statistics



Das Sarma, Freedman & Nayak, 2005

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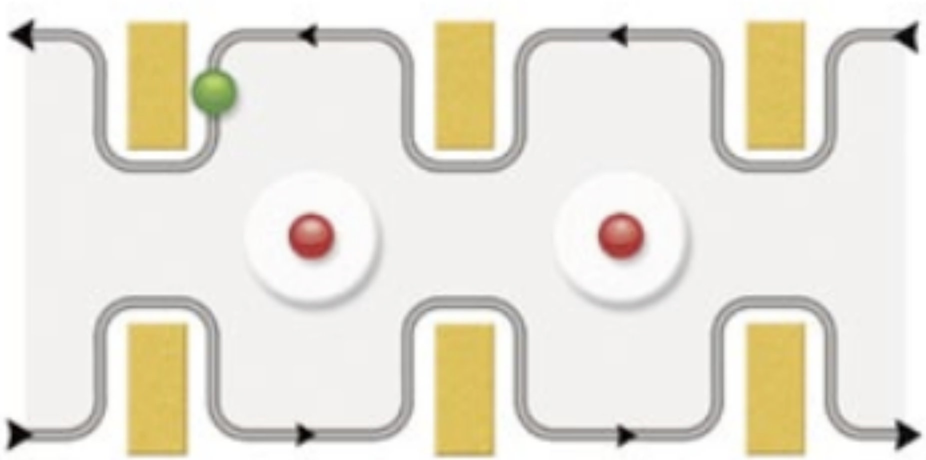
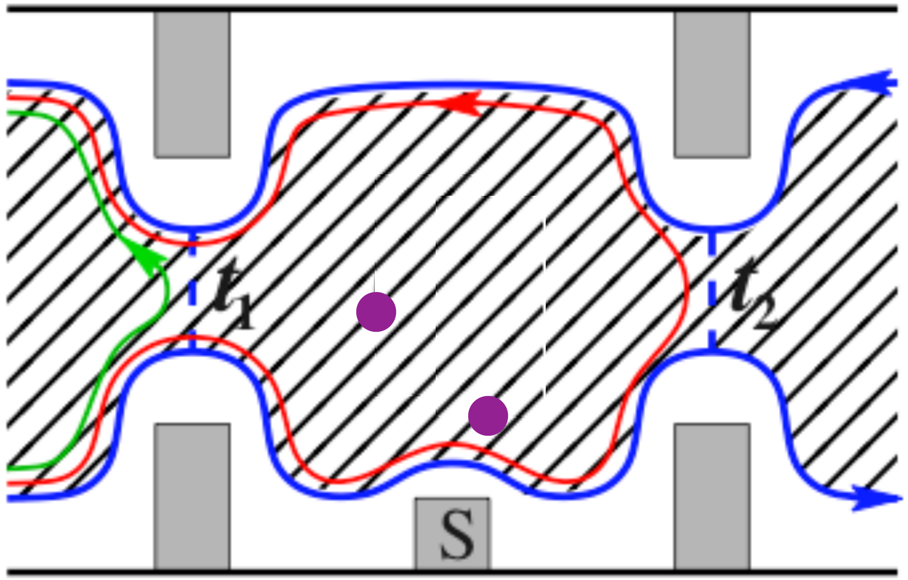


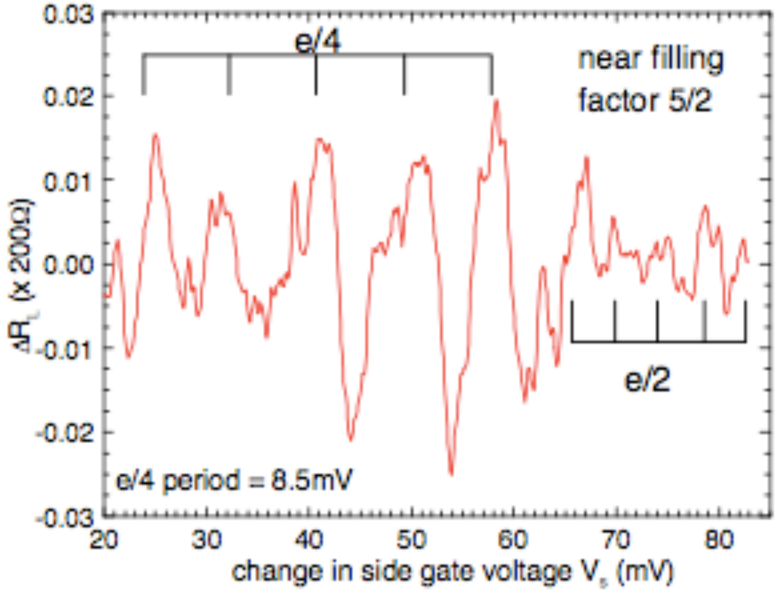
Hard to do, so instead..

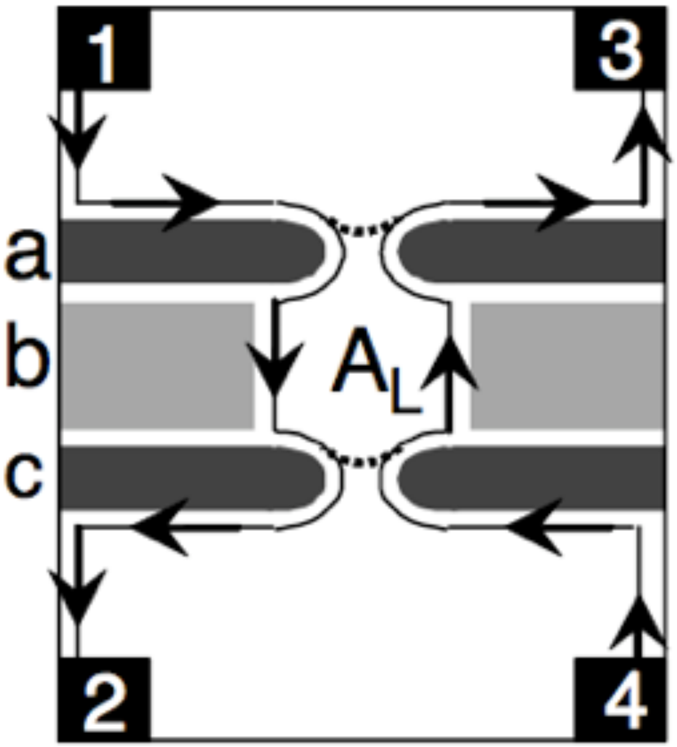


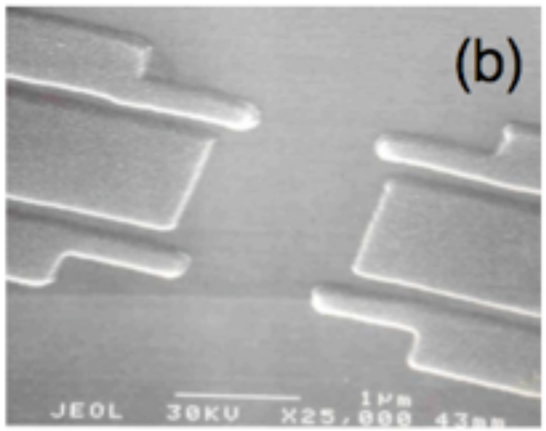
- Use side gate to push the edge across pinned non-Abelian quasiparticles
- Random change between periods in the interference pattern is a sign of non-Abelian quasiparticle tunneling

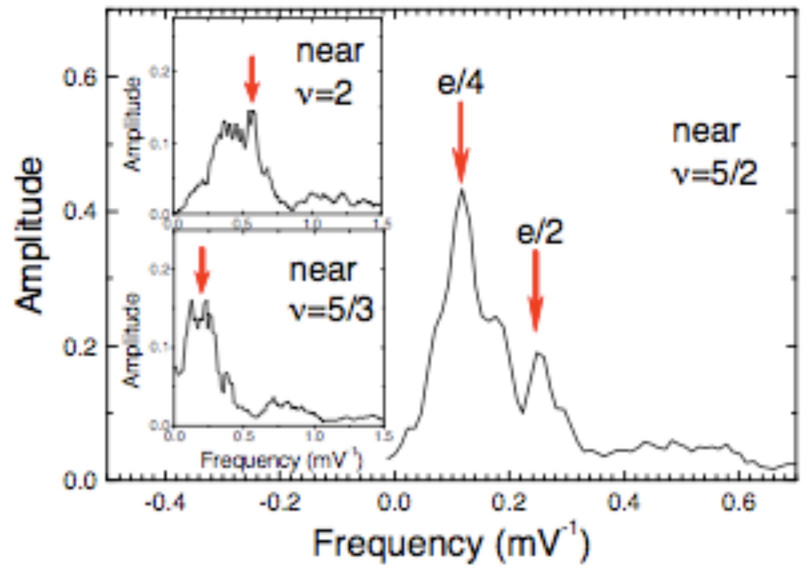
The experiment

R.L. Willett*, L.N. Pfeiffer, K.W. West, 2009









Suggestive, but not conclusive.

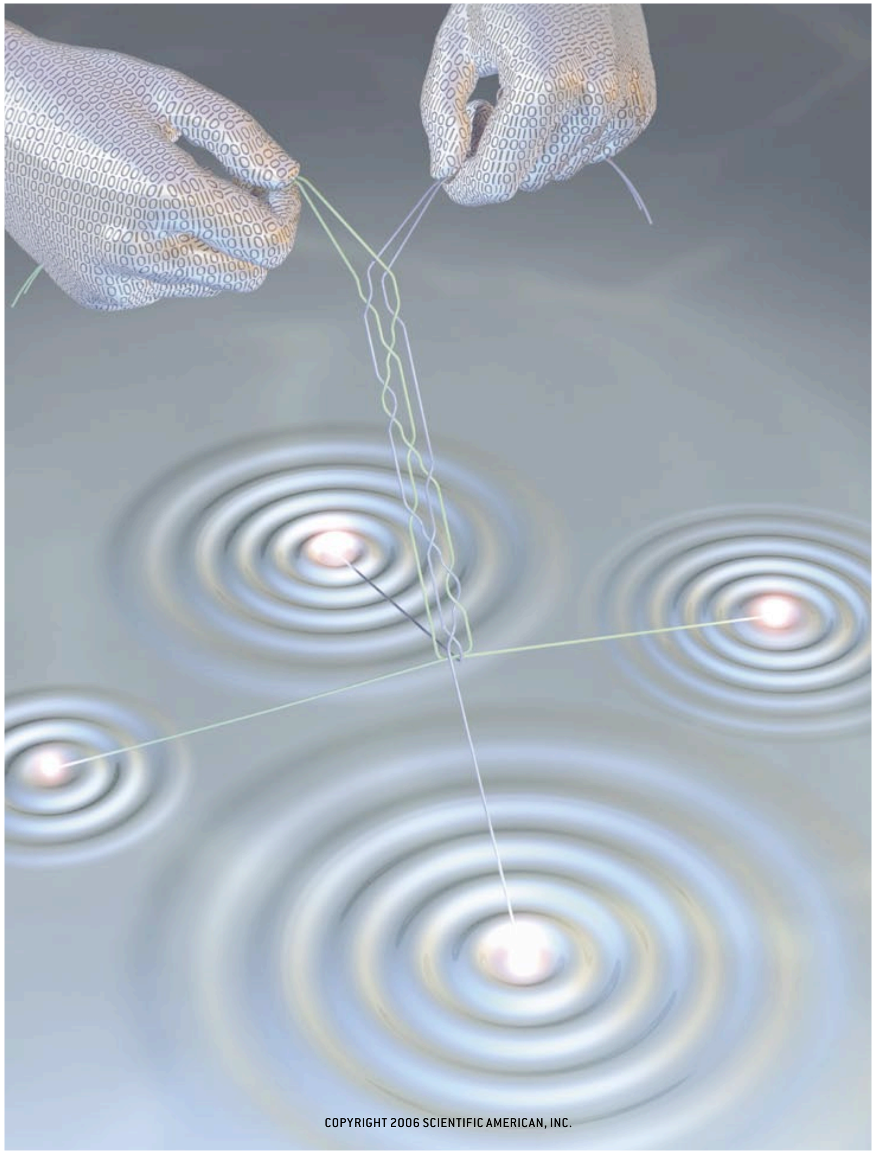
Fourth main message of this talk:

- There are serious attempts to measure the non-Abelian quantum statistics in the $\nu=5/2$ quantum Hall state.
- The results are still controversial
- There are scant experimental evidence for Abelian fractional statistics.

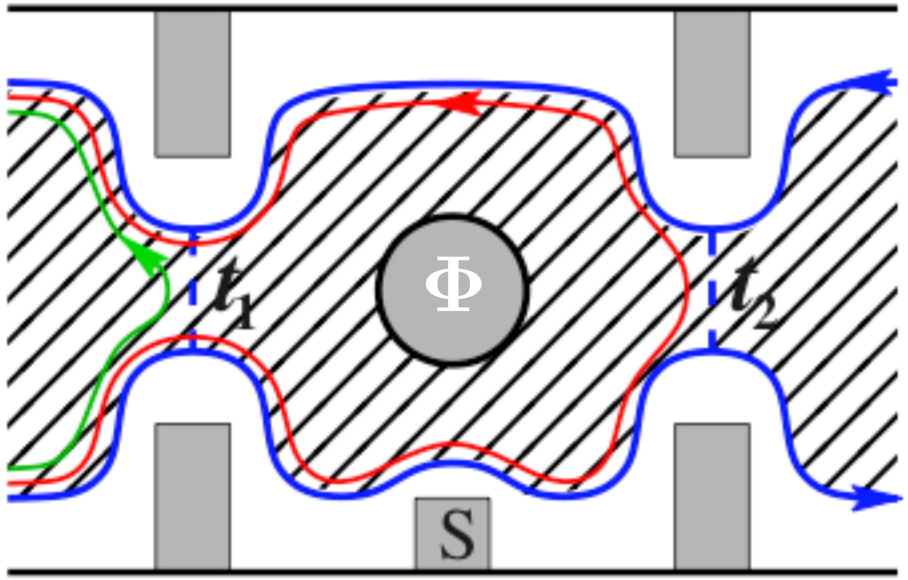
Topologically protected quantum computing

The main idea:

- The qubits are coded in the degenerate subspace.
- When the qusiparticles are far separated no local dirt can dephase the qubit.
- The quntum gates are the braids.
- Braiding can be done by moving the quasiparticles.



The Fabry-Pérot interferometer



C. de C. Chamon *et al.*, Phys. Rev. B **55**, 2331 (1997).

- **Vary the flux Φ between the point contacts.**
- **Vary the charge on the "island" by a back gate.**
- **Vary source-drain voltage.**
- **Measure interference patterns in transmitted current!**

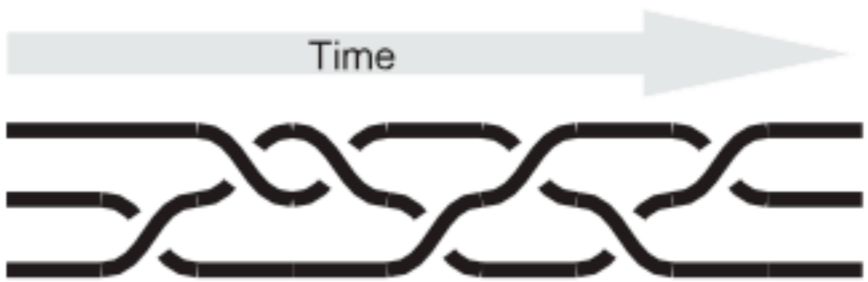
Unitary operators as braids - examples

Fibonacci anyons

$$\sigma_1 = \left(\begin{array}{cc|c} e^{-i4\pi/5} & 0 & 0 \\ 0 & -e^{-i2\pi/5} & 0 \\ \hline 0 & 0 & -e^{-i2\pi/5} \end{array}\right)$$

$$\sigma_2 = \left(\begin{array}{cc|c} -\tau\, e^{-i\pi/5} & -i\sqrt{\tau}\, e^{-i\pi/10} & 0 \\ -i\sqrt{\tau}\, e^{-i\pi/10} & -\tau & 0 \\ \hline 0 & 0 & -e^{-i2\pi/5} \end{array}\right)$$

$|\psi_i\rangle$



$|\psi_f\rangle = M^{-1}|\psi_i\rangle$

$$M = \sigma_1^{-1}\sigma_2\ \sigma_2\ \sigma_1^{-1}\sigma_2^{-1}\sigma_1\ \sigma_2^{-1}$$

Fifth main message of this talk:

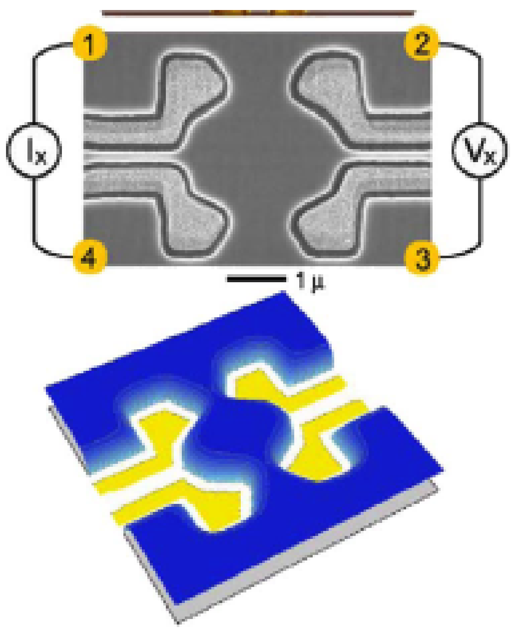
- Non-Abelian anyons could be used to build topologically protected qubits.
- Braiding these particles in a controlled manner would amount to having a protected quantum gate.

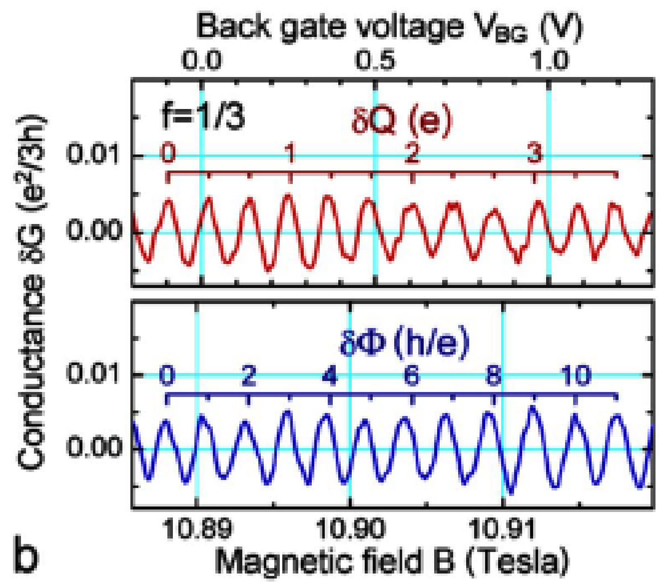
The Stony-Brook experiments

PRL 98, 076805 (2007) PHYSICAL REVIEW LETTERS week ending 16 FEBRUARY 2007

$e/3$ Laughlin Quasiparticle Primary-Filling $\nu = 1/3$ Interferometer

F. E. Camino, Wei Zhou, and V. J. Goldman

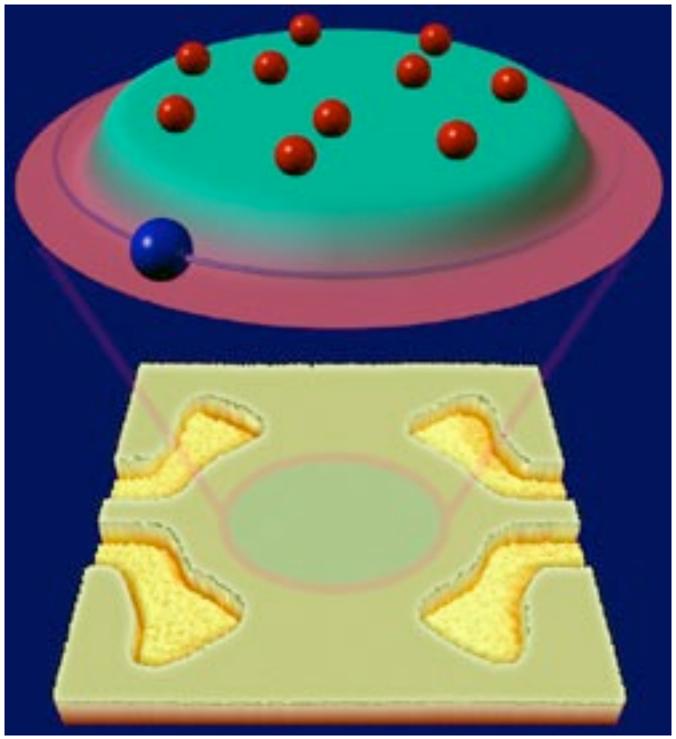




- **Coherent (?) oscillations observed!**
- **The periods are consistent with fractional statistics.**

More complicated systems:

2/5 island in a 1/3 background:



Similar interferometer to the one used for the pure 1/3 case

- Claim of a superperiod 5
- Consistent with fractional statistics, but
- No simple clean interpretation
- Alternative explanations have been proposed

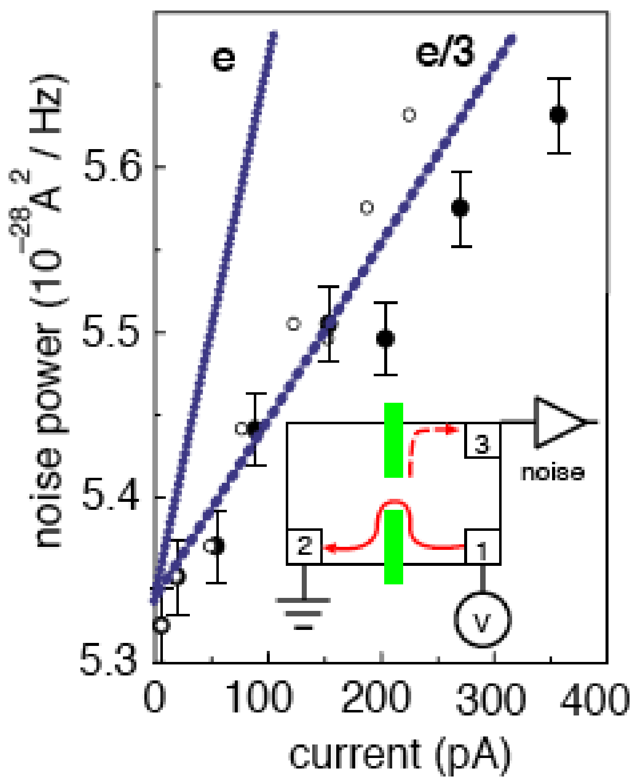
Fractional charge (shot noise):

VOLUME 79, NUMBER 13 PHYSICAL REVIEW LETTERS 29 SEPTEMBER 1997

Observation of the $e/3$ Fractionally Charged Laughlin Quasiparticle

L. Saminadayar and D. C. Glattli
Service de Physique de l'État Condensé, CEA/Saclay, F-91191 Gif-sur-Yvette Cedex, France

Y. Jin and B. Etienne
Laboratoire de Microstructures et Microélectronique, CNRS, B.P. 107, F-92225 Bagneux Cedex, France
(Received 30 June 1997)



Assuming the tunneling to be a Poisson process, and measuring the noise power spectrum, S, the Fano factor,

$$F = \frac{S}{2\langle I \rangle}$$

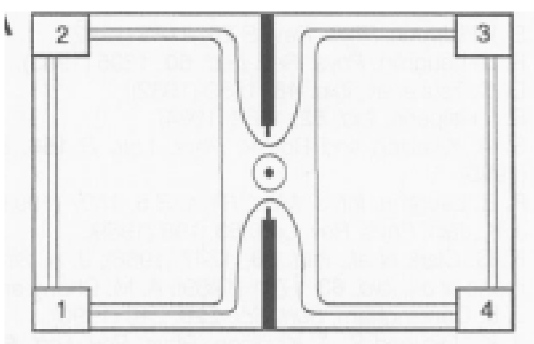
directly gives the charge of the carrier!

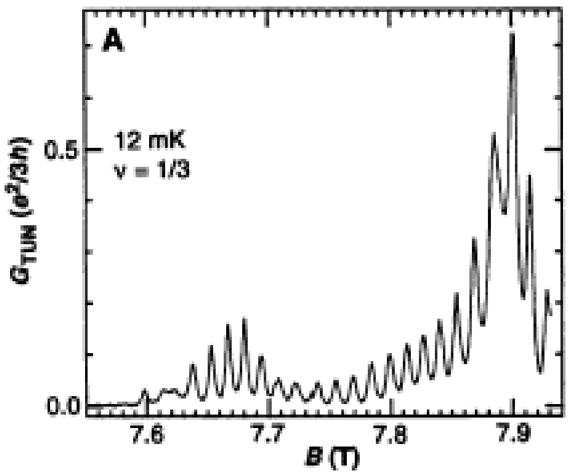
Fractional charge (AB-interference):

Resonant Tunneling in the Quantum Hall Regime: Measurement of Fractional Charge

V. J. Goldman* and B. Su

SCIENCE • VOL. 267 • 17 FEBRUARY 1995





The Aharonov-Bohm effect gives fluctuating current across the quantum dot - the period determines the charge to e/3 !