The Planck – Kleinert Crystal

Marek Danielewski

Interdisciplinary Centre for Materials Modelling, AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Kraków, Poland

Reprint requests to Prof. M. D.; E-mail: daniel@agh.edu.pl

Z. Naturforsch. 62a, 564 – 568 (2007); received May 10, 2007

The Planck – Kleinert Crystal hypothesis is analyzed for an ideal cubic fcc crystal formed by Planck particles. In this type of a quasi-continuum the energy, momentum, and mass transport are described by the classical balance equations. The transverse wave is the electromagnetic wave, and its velocity equals the velocity of light. The quasi-stationary collective movement of mass in the crystal is equivalent to the particle (body), and such an approach enables derivation of the Schrödinger equation. The diffusing interstitial Planck particles create a gravity field, and the computed value of $G$ is within the accuracy of experimental data. The model predicts four different force fields and a vast amount of the “dark matter and dark energy” in the crystal lattice. It allows for a self-consistent interpretation of multiscale phenomena.

Key words: Planck Scale Physics; Gravity; Dark Matter; Maxwell Equations.

1. Introduction

Physics Today was opened in 1999 by F. Wilczek’s article “The persistence of ether”, concerned with speculations on the physical vacuum and continuum, named ether [1]. Arguments supporting the continuum concept are provided by statistical mechanics [2]. When the ratio between macro- and microscopic scales becomes very large, collective properties emerge that are not noticeable in the behaviour of individual particles. To some degree, quantum space can be considered as analogue to a crystal which, through its atomic structure, changes the propagation of light relative to that through a vacuum [3]. Already in 1987 Kleinert proposed a simple three-dimensional Euclidean World Crystal model [4]. He successfully recovered Einstein’s gravity from a defect model of a crystal in which some elastic terms vanish. Recently, Kleinert and Zaanen explained the absence of torsion in space-time [5].

In the present work the World Crystal is a three-dimensional quasi-continuum, $x \in \mathbb{R}^3$, that will be called the Planck – Kleinert Crystal (PKC). The PKC is a cubic fcc crystal showing the Frenkel disorder. The proposed model is based on: (i) the solid solution assumption, (ii) the ability of defects to diffuse, (iii) the classical conservation laws connecting the mechanic and thermodynamic laws, (iv) the postulate that gravity is the result of diffusing defects (diffusing Planck particles), (v) the deformation and velocity vanish when defining the collective Planck mass movement, and (vi) the assumption that the electric field strength equals the negative of the rotation of the displacement vector. The properties of the crystal are shown in Table 1.

2. The Planck – Kleinert Crystal

The building blocks of a PKC are Planck particles, $P$, that obey the laws of mass, momentum and energy conservation. Each particle exerts a short range force at the Planck length, $l_P$. The crystal is an ideal (central forces act between particles) cubic fcc crystal showing the Frenkel disorder. The PKC is mechanically simple, i.e., its mechanical properties do not depend on strain, Table 1.

2.1. Defects in the PKC

Defects are treated as an ideal solid solution. The process one can consider is the formation of intrinsic Frenkel defects: $P_P \rightleftharpoons V_P + P_i$, where $P_P$, $V_P$ and $P_i$ denote a particle in lattice position, vacancy and interstitial particle, respectively. The densities of lattice sites ($\rho = |P_P| + |V_P|$) and defects ($\rho_d = |P_i| > |V_P|$) differ by orders of magnitude, $\rho \approx 4 l_P^{-3} = \text{const} \gg \rho_d$, where $\rho_d = N_{P} \rho$ and $N_{P}$ is the concentration ratio of...
interstitial Planck particles to lattice sites. The self-diffusivities of the Planck mass, \( D^* \), and of interstitial defects, \( D^* \), are related by \( D^* = (p_D/p) D^*_p \). From the Nernst-Einstein relation (\( D^* = kB T \)) [8], the analogous relation holds for the mobilities:

\[
B = (p_D/p) B_p, \tag{1}
\]

where \( B_p \) and \( B \) denote the mobilities of the defects and Planck particles, respectively.

The collective movement (“the particle-like” behaviour) is common in fluids (e.g., vortex, soliton) and solids (e.g., complex defects and standing waves [9]). The particle (i.e., body) is equivalent to such a wave and is characterized by its energy, \( M c^2 = 4\pi m p^{-3} \int_{-\infty}^{\infty} \rho \sigma \epsilon \), where \( M \) is the mass that is attributed to the particle and \( \epsilon \) denotes the total energy density in the PKC. If the particle is treated as complex, then by analogy, its mobility, \( B_M \), equals

\[
B_M = (m_p/M) B_p. \tag{2}
\]

The drift velocity concept is used to define fluxes [10, 11]. The local (\( u_i \)), diffusional (\( u_i^d \)) and common drift velocities (drift due to deformation, \( u_i^{\text{drift}} = \sigma \)) in PKC are related by

\[
u_i := u_i(t,x) = u_i^d(t,x) + \sigma(t,x), \tag{3}
\]

where \( \sigma \) is the displacement vector. The diffusion fluxes, \( J_i^d(t,x) \), are given by the Nernst-Planck formula [12]

\[
J_i^d = \rho_i u_i^d = \rho_i B_i F = \rho_i B_i \text{grad } \mu^m, \tag{4}
\]

where \( i \) denotes the Planck particle (\( P_i \)) or complex (\( M \)), \( F \) and \( \mu^m \) are the force and mechanical potential.

The calorimetric equation of state implies that \( U = U(S,V,m) \). When a fixed volume is considered: \( U/V = \epsilon = \epsilon(\rho_m, \rho_m) \) or \( \rho_m \epsilon = \epsilon(\rho_m s, \rho_m) \), where \( \epsilon \) and \( s \) denote the internal energy and entropy expressed as the energy per unit volume, and the Gibbs equation becomes [13]

\[
d(\rho_m \epsilon) = T d(\rho_m s) + \mu^m d\rho_m
\]

or

\[
d(\rho e) = T d(\rho s) + \mu^m d\rho,
\]

where \( \mu^m = \partial(\rho m \epsilon)/\partial \rho_m \) denotes the mechanical potential.

Equation (5) is used to compute the rate of entropy production as a result of defect formation and diffusion [14]. The formation of defects and their diffusion does not affect the energy density. Both processes decrease the thermal (\( Ts \rho \)) and increase the mechanical (\( \mu^m \rho \)) energy and are represented by the last terms in (11) and (12).

When the deformation is small, the mechanical potential depends on the total energy density:

\[
\mu^m = \mu_0^m + \mu_l^m \ln(e/\mu_0^m)
\]

or

\[
e = \mu_0^m \exp\left([\mu^m - \mu_0^m]/\mu_0^m\right)
= \frac{\mu_0^m}{2} \exp(2(\mu^m - \mu_0^m)/\mu_0^m),
\]

where \( \mu_0^m \) and \( \mu_0^m \) are the Planck and standard mechanical potentials, see Table 1. To distinguish the energy of the local deformation field, \( \mu_l^m \), and the energy of the “internal process”, \( \mu_l^m \), the following notation is used:

\[
\mu^m = \mu_0^m + \mu_l^m - i \mu_0^m.
\]

Combining (6) and (7) and the relation between the potentials (Table 1), the energy density is expressed by

\[
e = \frac{\mu_0^m}{2} \exp(2(\mu_l^m - i \mu_0^m)/\mu_0^m).
\]
2.2. Mass, Momentum and Energy Balances

These are described by classical formulae with rigorous use of the calorific equation of state. To simplify and shorten the problem, the assumptions of a flat temperature field in the crystal (negligible heat flux, $J_q = 0$) and negligible viscosity are introduced. Consequently, the conservation of mass (particles), momentum and energy are given by

$$\frac{\partial \rho}{\partial t} = - \text{div} (\rho \mathbf{u}) = - \text{div} (\rho \mathbf{u}^\sigma + \rho \mathbf{u}^d), \quad (9)$$

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{\partial \mathbf{u}}{\partial t} = \rho \mathbf{u} \text{ grad } \mathbf{v} + \text{ div } \mathbf{\sigma}, \quad (10)$$

$$\frac{\partial \langle \mu \rangle \rho}{\partial t} = - \text{div} (\langle \mu \rangle \rho \mathbf{u}) + \mathbf{\sigma} \mathbf{:} \text{ Grad } \mathbf{v} + \rho \mathbf{u} \text{ grad } \mu + \gamma \rho e, \quad (11)$$

$$\frac{\partial (T s \rho)}{\partial t} = - \text{div} (T s \rho \mathbf{u}) - \rho \mathbf{u} \text{ grad } \mu + \gamma \rho e, \quad (12)$$

$$\frac{\partial (\rho e)}{\partial t} = \text{ div } (\mathbf{\sigma} \mathbf{\circledast} \mathbf{u} - \rho \mathbf{e} \mathbf{v}), \quad (13)$$

where $e = e + 1/2 \mathbf{u}^2 = T s + \mu^m + 1/2 \mathbf{u}^2$ is the energy density (sum of the internal and kinetic energy) and $\mathbf{\sigma} = \mathbf{\sigma} / m_\rho$. In the thermal, (12), and mechanical parts, (11), of the total energy balance, (13), the terms describing the Frenkel defect formation, $\pm \gamma \rho e$, and diffusion, $\pm \rho \mathbf{u} \text{ grad } \mu^m$, were introduced [14].

2.3. Mechanical Properties

The stress tensor of an ideal regular crystal is given by $\mathbf{\sigma} = \mathbf{\sigma} / m_\rho = (\lambda / m_\rho) \mathbf{D} + 2\mu \mathbf{B} D \mathbf{D} \mathbf{\mu},$ where $\mathbf{D}$ denotes the deformation tensor (the symmetrical part of the strain tensor) and $\lambda = \mu_\ell$ are the Lamé coefficients of an ideal regular crystal.

It was shown by Cauchy and Saint Venant that, if the particles composing a regular crystal interact pairwise through central forces, then there is an additional symmetry requiring $C_{44} = C_{12}$ [15]. This implies the Poisson ratio 0.25 and $\mu_\ell = \lambda$ [15]. Using the identity: grad div $\mathbf{u}$ = div grad $\mathbf{u}$ + rot rot $\mathbf{u}$, we have

$$\text{div } \mathbf{\sigma} = 2\lambda / m_\rho \text{ grad } \mathbf{u} + \lambda / m_\rho \text{ grad } \mathbf{u}$$

$$\equiv 3\lambda / m_\rho \text{ grad } \mathbf{u} + \lambda / m_\rho \text{ rot } \mathbf{rot } \mathbf{u}. \quad (14)$$

The Young modulus and the ratio of the longitudinal to the transverse wave velocities equal $(c_L / c_T)^2 = 3$ and $E = 2.5\lambda$ [15].

Only the isothermal processes are analyzed, consequently $f_T$, $T_P$, $B_P$, $\lambda$, $\gamma$ = const. Upon introducing the mass diffusion flux, (3) and (4), and stress tensor, (14), formulae (9)–(13) become

$$\frac{\partial \rho}{\partial t} = \text{ div } (\rho N_B \mathbf{B} \text{ grad } \mu^m) - \text{ div } (\rho \mathbf{u}^\sigma), \quad (15)$$

$$\rho \mathbf{D} \mathbf{u} / \mathbf{D} t = \frac{\lambda}{m_\rho} \text{ grad } \mathbf{u} + \frac{\lambda}{m_\rho} \text{ rot rot } \mathbf{u}, \quad (16)$$

$$\frac{\partial (\mu^m \rho)}{\partial t} = - \text{div} (\mu^m \rho \mathbf{u}) + \mathbf{\sigma} \mathbf{:} \text{ Grad } \mathbf{v} + \rho \mathbf{u} \text{ grad } \mu^m + \gamma \rho e, \quad (17)$$

$$\frac{\partial (\rho e)}{\partial t} = \text{ div } (\mathbf{\sigma} \mathbf{\circledast} \mathbf{u} - \rho \mathbf{e} \mathbf{v}). \quad (18)$$

The unknowns are $\rho$, $\mathbf{u}$, $\mu^m$ and $e$. Below, only uncoupled, elementary processes will be analyzed.

2.4. Newtonian Constant of Gravitation

The mechanical energy conservation, (17), allows us to estimate the gravity field generated by the immobile body, mass $M$, in the PKC. Gravity is created by diffusing Planck particles. The immobile body implies a quasi-stationary situation and the deformation, mass and energy in a space occupied by the body are “fixed”: $\mu^m = \mu^m(x)$, $\rho = \rho(x)$ and $\mathbf{v}^0 = \partial \mathbf{u} / \partial t \equiv 0$. One can assume $\mu^m \text{ grad } \mathbf{v} \equiv 0$. Thus, combining (3) and (17):

$$\mu^m \text{ div } (\rho \mathbf{u}^d) = \gamma \rho e. \quad (19)$$

The diffusion velocity depends on the real part of the mechanical potential $\mu^m \equiv \mu^m_{\text{Re}}$, and becomes

$$\mathbf{v}^d = -B_P \text{ grad } \mu^m_{\text{Re}}(t, x). \quad (20)$$

The mass-energy equivalence in the PKC, $m_\rho e = c^2 \rho M (E = Mc^2)$, implies:

$$e = c^2 \rho_M / (m_\rho \rho). \quad (21)$$

Assuming that the defect concentration does not vary markedly, $\rho_T \cong \text{ const.}$, and combining (19)–(21):

$$\text{ div } \text{ grad } \mu^m_{\text{Re}} \cong - \rho_M \gamma \rho \mathbf{B} \rho \mathbf{u} \mathbf{B} \rho \mathbf{p}. \quad (22)$$

When the local energy density due to the mass $M$ does not change markedly, the average value of the mechanical potential in the PKC can be used: $\mu^m \cong \tilde{\mu}^m = 2 \mu_0^m / \pi = \mu_0 / \pi$. The density of defects and the frequency of their formation are given by $\rho_T \cong \alpha \tau e^{-3 \exp(-E_f / kT_P)}$ and $\gamma \cong f_T \alpha e^{-3 \exp(-E_f / kT_P)}$, where $\alpha$ and $f_T$ are the geometrical factor and the
Planck frequency [16]. Upon substituting the above relations and data from Tables 1 and 2, (22) becomes

\[ \text{div grad } \mu_{\text{Re}}^m = -4\pi \rho_M (\mu_0^m f^2 c^2) / (m_B \mu_0 B_p) \]
\[ = -4\pi \rho_M (\mu_0^m f^2 c^2) / m_p \]
\[ = -4\pi G \rho_M, \]

which is the equation discovered by the French mathematician Siméon-Denis Poisson, and \( G = 6.674189 \cdot 10^{-11} \) is the Newtonian constant of gravitation (NIST data: \( G = 6.6742(10) \cdot 10^{-11} \) [7]). It was shown that the diffusing interstitial Planck particles are equivalent to weakly interactive massive particles and create the gravitational interaction between matter. One can consider the analogies between the “dark matter” and diffusing Planck particles as well as between the “dark energy” and energy of the diffusing Planck particles.

### 2.5. The Time-Dependent Schrödinger Equation

The deformation and its velocity as well as diffusion of defects are now assumed to be negligible, \( v^\sigma = v^\phi = 0 \) and \( \text{div } (\sigma \otimes v) \cong 0 \). The process that governs the de Broglie waves is the fast internal process. We analyze the case when the driving force of the transport (the collective Planck mass movement, i.e., the movement of a complex of particles showing an energy \( E \) and mass \( M = E c^{-2} \)) is controlled by the imaginary part of the mechanical potential, \( \mu^m = \mu^m_{\text{Im}} \). The flux given by (4) equals \( \mathcal{F}^m = -\rho \sigma \mathcal{B} \text{grad } \mu^m_{\text{Im}} = -\rho \sigma \mathcal{B} \text{grad } \mu^m_{\text{Im}} \), and (18) becomes

\[ \partial(\rho e)/\partial t = B_M \text{div } (\rho e \text{ grad } \mu^m_{\text{Im}}). \]  

By combining (2), (8) and (24) one gets

\[ \partial \exp(2(\mu_{\text{Re}}^m - i\mu_{\text{Im}}^m) / c^2) / \partial t = \]
\[ (m_p B_p) / M \text{ div } \exp(2(\mu_{\text{Re}}^m - i\mu_{\text{Im}}^m) / c^2) \text{ grad } \mu_{\text{Im}}^m. \]

By denoting \( \psi = \exp((\mu_{\text{Re}}^m - i\mu_{\text{Im}}^m) / c^2) \), \( E = m_p/ (2M c^2) B_p (\text{grad } \mu_{\text{Im}}^m)^2 + c^2 \text{grad } \mu_{\text{Re}}^m (\text{grad } \mu_{\text{Re}}^m)^2 \) and using the identity \( \psi^{-1} \Delta \psi = \Lambda (\mu_{\text{Re}}^m - i\mu_{\text{Im}}^m) / c^2 + (\text{grad } \mu_{\text{Re}}^m - i\mu_{\text{Im}}^m) / c^2 \), (25) becomes the standard Schrödinger equation:

\[ i \partial \psi / \partial t = -(m_p B_p e^2) / (2M) \text{ div } \text{ grad } \psi + E \psi \]
\[ = -h / (4\pi M) \text{ div } \text{ grad } \psi + E \psi. \]

Using data shown in Table 1 one can compute the value of Planck’s constant, Table 2.

#### 2.6. Transverse Waves in Planck – Kleinert Crystal

The wave propagation can be analyzed in two limiting situations, the longitudinal wave: \( \text{rot } \mathbf{u} = 0 \), and the transverse one: \( \text{div } \mathbf{u} = 0 \). When mass transport occurs solely due to deformation (\( \mathbf{v} = v^\sigma \)), the deformation is low (\( \rho \cong \text{const.} \)) and the longitudinal wave is excluded, (16) becomes the equation of a transverse wave with the velocity given by \( c = \sqrt{\mu_0^m / 4\pi m_p} \):

\[ \partial v^\sigma / \partial t = \partial^2 u / \partial t^2 = \lambda_\lambda / (4\pi m_p) \text{rot } \mathbf{u}. \]  

#### 2.7. Maxwell Equations in Vacuum

Suppose that \( \mathbf{E} = -\text{rot } \mathbf{u}, \mathbf{B} = v^\sigma \) and the permeability equals \( \mu_0^m = 4\pi m_p / 4\pi = \rho_m / \lambda \). Then, (27) becomes

\[ \mu_0^m \partial \mathbf{B} / \partial t = -\mathbf{E}. \]
The second equation follows straightforwardly. Differentiation of the electric field strength, $E = -\text{rot} \, u$, yields $\frac{\partial E}{\partial t} = -\frac{\partial \text{rot} \, u}{\partial t} = -\frac{\partial \text{rot} \, u}{\partial t}$, which by substituting $B = \nu = \frac{\partial u}{\partial t}$ results in

$$\varepsilon_0 \frac{\partial E}{\partial t} = -\text{rot} \, B. \tag{29}$$

The permittivity, $\varepsilon_0$, equals one, and $\varepsilon_0 \mu_0 = c^{-2}$. The other relations that form the Maxwell system are the consequence of an “empty lattice” and low deformation. From (15), the usual formula for noncompressible flow follows: $\text{div} \, u = 0$. Thus, from the assumption of low deformation:

$$\text{div} \, B = \text{div} \, \nu = 0. \tag{30}$$

The charged particles are not included in the present analysis. Upon differentiating the constitutive equation for $E$ one gets: $\text{div} \, E = -\text{div} \, \text{rot} \, u$. Consequently, from the identity $\text{div} \, \text{rot} \, u = 0$, it follows that

$$\text{div} \, E = 0. \tag{31}$$

Equations (28) – (31) form the Maxwell system in vacuum.

3. Summary

The results support the physical reality at the Planck scale and allow for the interpretation of gravity, quantum mechanics and electromagnetic phenomena. Faster than light velocities of the longitudinal wave might be considered as unphysical, and this requires “fine-tuning” or higher symmetry to make all “sound speeds” equal. Here, the different velocities are related to specific force fields and were analyzed as real quantities that mark different time scales. The diffusing interstitial Planck particles (defects) create the gravitational interaction between matter. The collective behaviour of the particles forming the Planck-Kleinert Crystal is equivalent to the particle (body). Such an approach enables the Schrödinger equation to be derived. The transverse wave in the PKC is equivalent to an electromagnetic wave in vacuum. The PKC model allows the derivation of the classical Newton’s law of gravity (the Poisson equation). The consequence of the equation of internal energy conservation is the existence of waves involving temperature, but not the mechanical potential variations. They are analogous to “the second sound” described by Landau and Lifschitz [14].

Acknowledgements

This work was supported by the Ministry of Science and Higher Education, project COST/247/2006. The author thanks B. Wierzba for help in mathematics and Prof. S. Mrowec for helpful comments.