Comment on Path Integral for General Time-Dependent Solvable Schrödinger Equation.

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Summary. – We explicitly perform the infinitely many integrations in the path integral of an arbitrary time-dependent quantum-mechanical fluctuation problem whose Schrödinger equation is solvable.

Some time ago, an explicit solution has been found for the path integral of the Coulomb problem (1) and this has led to a renewed interest in quantum-mechanical path integrals (2-6). In this note we show, as a pedagogical exercise, how to perform the path integral associated with any solvable Schrödinger equation

$$H(-i\partial, \boldsymbol{x}, t)\,\psi(\boldsymbol{x}, t) = i\partial_t \psi(\boldsymbol{x}, t)\,, \tag{1}$$

where the Hamiltonian may be time dependent.

The quantum-mechanical transition amplitude to be calculated is

$$(\boldsymbol{x}_b, t_b|\boldsymbol{x}_a, t_a) = \int \mathscr{D}^3 x(t)\, \frac{\mathscr{D}^3 p(t)}{(2\pi)^3} \exp\left[i\int\limits_{t_a}^{t_b} \mathrm{d}t[\boldsymbol{p}\dot{\boldsymbol{x}} - H(\boldsymbol{p}, \boldsymbol{x}, t)]\right]. \tag{2}$$

This formal expression is defined, as usual, by grating the time axis into an infinitesimally narrow lattice $t_n = n\varepsilon + t_a$, $n = 0, 1, \ldots, N+1$ with $t_0 = t_a$, $t_{N+1} = t_b$, setting

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(2) G. A. RINGWOOD and T. D. DEVREESE: *J. Math. Phys.* (*N. Y.*), **21**, 139 (1980).
(3) PH. BLANCHARD and M. SIRUGUE: *J. Math. Phys.*, **22**, 1372 (1981).
(4) R. HO and A. INOMATA: SUNYA preprint (1981).
(5) S. ALBEVERIO: Lecture presented at the *1981 International Conference on Mathematical Physics, Berlin*.
(6) C. GERRY: *Phys. Lett. A* (in press).

$\boldsymbol{x}(t_n) \equiv \boldsymbol{x}_n$, $\boldsymbol{p}(t_n) = \boldsymbol{p}_n$, $\boldsymbol{x}_0 = \boldsymbol{x}_a$, $\boldsymbol{x}_{N+1} = \boldsymbol{x}_b$, and performing the product of integrals

$$\mathscr{D}^3 x(t) \frac{\mathscr{D}^3 p(t)}{(2\pi)^3} = \prod_{n=1}^{N} \left(\int \mathrm{d}^3 x_n \int \frac{\mathrm{d}^3 p_n}{(2\pi)^3} \right) \int \frac{\mathrm{d}p_{N+1}^3}{(2\pi)^3}, \tag{3}$$

which become infinitely many in the limit $\varepsilon \to 0$, $N = (t_b - t_a)/\varepsilon - 1 \to \infty$.

The corresponding grated version of the action may require ordering prescriptions between p_n and x_n's, for example, px^2 can be $p_n x_n^2$, $p_n x_{n-2}^2$, $p_n x_{n-1} x_n$, or a combination of these.

It will be convenient to consider time and energy as fluctuating variables. The corresponding paths may be parametrized as $\boldsymbol{x}(s), \boldsymbol{p}(s), t(s), E(s)$. Then we can rewrite (2) as

$$(\boldsymbol{x}_b, t_b | \boldsymbol{x}_a t_a) = \int_{s_a}^{\infty} \mathrm{d}s_b f(\boldsymbol{x}(s_b)) (\boldsymbol{x}_b t_b, s_b | \boldsymbol{x}_a t_a, s_a), \tag{4}$$

where

$$(\boldsymbol{x}_b t_b, s_b | \boldsymbol{x}_a t_a s_a) = \tag{5}$$
$$= \int \mathscr{D}^4 x(s) \frac{\mathscr{D}^4 p(s)}{(2\pi)^4} \exp\left[i \int_{s_a}^{s_b} \mathrm{d}s \{ \boldsymbol{p}(s) \boldsymbol{x}'(s) - E(s) t'(s) - f(\boldsymbol{x}(s)) [H(\boldsymbol{p}(s) \boldsymbol{x}(s), t(s)) - E(s)] \} \right]$$

is the amplitude that a particle moves from $\boldsymbol{x}_a t_a$ to $\boldsymbol{x}_b t_b$ in the parameter interval $s_b - s_a$. The path integral $\mathscr{D}^4 x(\mathscr{D}^4 p/(2\pi)^4)$ is defined as $\mathscr{D}^3 x(\mathscr{D}^3 p/(2\pi)^3) \mathscr{D}t(\mathscr{D}E/2\pi)$ with a grated s-axis, in complete analogy with (3).

The function $f(\boldsymbol{x})$ is completely arbitrary (*) and may be chosen later to simplify the problem.

Equation (4) is verified as follows: First one integrates out $\mathscr{D}E(s)$ which leads to a δ-functional in $t' - f(\boldsymbol{x})$ such that $\mathscr{D}t(s)$ can be performed trivially. By watching out for the grated variables, eq. (5) becomes

$$(\boldsymbol{x}_b t_b, s_b | \boldsymbol{x}_a t_a, s_a) = \tag{6}$$
$$= \delta\left(t_b - t_a - \int_{s_a}^{s_b} f(\boldsymbol{x}(s)) \, \mathrm{d}s \right) \int \mathscr{D}^3 x \frac{\mathscr{D}^3 p}{(2\pi)^3} \exp\left[i \int_{s_a}^{s_b} \mathrm{d}s \left[\boldsymbol{p}\boldsymbol{x}' - \frac{\mathrm{d}t(s)}{\mathrm{d}s} H(\boldsymbol{p}(s) \boldsymbol{x}(s), t(s)) \right] \right].$$

Integrating this over $\int_{s_a}^{\infty} \mathrm{d}s_b$ with $f(\boldsymbol{x}(s_b))$ as a factor removes the δ-function such that the action is the same as in (2). In addition, this leads to the correct path-dependent relations between t_b, t_a and $s_b - s_a$. Explicitly, eq. (5) amounts to the following infinite product of integrals:

$$(\boldsymbol{x}_b t_b, s_b | \boldsymbol{x}_a t_a, s_a) \underset{N \to \infty}{\equiv} \tag{7}$$
$$\underset{N \to \infty}{\equiv} \prod_{n=1}^{N} \left(\int \mathrm{d}^3 k_n \int \mathrm{d}t_n \right) (\boldsymbol{x}_b t_b, s_b | \boldsymbol{x}_N t_N, s_N)(\boldsymbol{x}_N t_N, s_N | \boldsymbol{x}_{N-1} t_{N-1}, s_{N-1}) \dots (\boldsymbol{x}_1 t_1, s_1 | \boldsymbol{x}_a t_a, s_a)$$

with

$$(\boldsymbol{x}_n t_n, s_n | \boldsymbol{x}_{n-1} t_{n-1} s_{n-1}) \equiv \tag{8}$$
$$\equiv \int \frac{\mathrm{d}^3 p_n}{(2\pi)^3} \int \frac{\mathrm{d}E_n}{2\pi} \exp[i \{ \boldsymbol{p}_n(\boldsymbol{x}_n - \boldsymbol{x}_{n-1}) - E_n(t_n - t_{n-1}) - \varepsilon f(\boldsymbol{x}_n) [H(p_n, \boldsymbol{x}_n, t_n) - E_n] \}],$$

(*) A possible p-dependence has been omitted, for brevity's sake.

where the end points $\boldsymbol{x}_{N+1}=\boldsymbol{x}_b$, $t_{N+1}=t_b$, $\boldsymbol{x}_0=\boldsymbol{x}_a$, $t_0=t_a$ are kept fixed. The exponential factor can be removed from each factor and (8) takes the form

$$(9)\qquad (\boldsymbol{x}_n t_n, s_n|\boldsymbol{x}_{n-1}t_{n-1}s_{n-1})\approx$$
$$\approx\left\{1-\varepsilon f(\boldsymbol{x}_a)\left[H\left(\frac{1}{i}\frac{\partial}{\partial\boldsymbol{x}_n},\,\boldsymbol{x}_n,\,t_n\right)-i\frac{\partial}{\partial t_n}\right]\right\}\int\frac{\mathrm{d}^3p_n}{(2\pi)^3}\frac{\mathrm{d}E_n}{2\pi}\exp\left[i\{\boldsymbol{p}_n(\boldsymbol{x}_n-\boldsymbol{x}_{n-1})-E_n(t_n-t_{n-1})\}\right]$$

in which the $\mathrm{d}^3p_n\,\mathrm{d}E_n$ integrals simply give δ-functions

$$(10)\qquad \delta^{(3)}(\boldsymbol{x}_n-\boldsymbol{x}_{n-1})\,\delta(t_n-t_{n-1})\,,$$

such that

$$(11)\qquad (\boldsymbol{x}_b t_b, s_b|\boldsymbol{x}_a t_a s_a)\approx\prod_{n=1}^{N}\left(\int\mathrm{d}^3x_n\int\mathrm{d}t_n\right)\cdot$$
$$\cdot\prod_{n=1}^{N+1}\left\{1-i\varepsilon f(\boldsymbol{x}_n)\left[H\left(\frac{1}{i}\frac{\partial}{\partial\boldsymbol{x}_n},\,\boldsymbol{x}_n,\,t_n\right)-i\frac{\partial}{\partial t_n}\right]\right\}\delta^{(3)}(\boldsymbol{x}_n-\boldsymbol{x}_{n-1})\,\delta(t_n-t_{n-1})\,.$$

In order to integrate out the remaining infinitely many variables $\boldsymbol{x}_n$ and t_n, it is useful to expand the δ-functions in a factorized form in terms of a complete set of orthonormal functions

$$(12)\qquad \delta^{(3)}(\boldsymbol{x}_n-\boldsymbol{x}_{n-1})\,\delta(t_n-t_{n-1})=\sum_{\alpha_n}\psi_{\alpha_n}(\boldsymbol{x}_n t_n)\,\psi^*_{\alpha_n}(\boldsymbol{x}_{n-1}t_{n-1})\,.$$

Then each integral involves expressions

$$(13)\qquad \sum_{\dots\alpha_{n+1},\alpha_n\dots}\dots\int\mathrm{d}^3x_n\int\mathrm{d}t_n\,\dots\,\psi^*_{\alpha_{n+1}}(\boldsymbol{x}_n,\,t_{n+1})\,\psi_{\alpha_n}(\boldsymbol{x}_n t_n)\,\dots\,,$$

which simply reduce to $\sum_{\alpha_n}$, independent of the specific choice of the set $\psi_\alpha(\boldsymbol{x}\,t)$.

Of particular advantage is a choice which diagonalizes the differential operator

$$(14)\qquad f(\boldsymbol{x})\left[H\left(\frac{1}{i}\frac{\partial}{\partial\boldsymbol{x}},\,\boldsymbol{x},\,t\right)-i\frac{\partial}{\partial t}\right]\psi_\alpha(\boldsymbol{x}t)=\varepsilon_\alpha\,\psi_\alpha(\boldsymbol{x}t)\,.$$

Then (11) becomes

$$(15)\qquad (\boldsymbol{x}_b t_b, s_b|\boldsymbol{x}_a t_a, s_a)\prod_{n=1}^{N}\left(\int\mathrm{d}^3x_n\int\mathrm{d}t_n\right)\prod_{n=1}^{N+1}\sum_{\alpha_n}[1-i\varepsilon\varepsilon_{\alpha_n}]\,\psi_{\alpha_n}(\boldsymbol{x}_n t_n)\,\psi^*_{\alpha_n}(\boldsymbol{x}_{n-1}t_{n-1})\,.$$

Using the property (13) all intermediate integrals can be trivially done and we arrive at

$$(16)\qquad \sum_{\alpha}(1-i\varepsilon\varepsilon_\alpha)^{N+1}\,\psi_\alpha(\boldsymbol{x}_b t_b)\,\psi^*_\alpha(\boldsymbol{x}_a t_a)\,.$$

In the continuum limit this reduces to

$$(17)\qquad (\boldsymbol{x}_b t_b, s_b|\boldsymbol{x}_a t_a, s_a)=\sum_{\alpha}\exp\left[-i\varepsilon_\alpha(s_b-s_a)\right]\psi_\alpha(\boldsymbol{x}_b t_b)\,\psi^*_\alpha(\boldsymbol{x}_a t_a)\,.$$

Now we can perform the integral $\int_{s_a}^{\infty} ds_b$ in (4) and find the desired propagator

$$(x_b, t_b|x_a, t_a) = \sum_\alpha -\frac{i}{\varepsilon_\alpha} \psi_\alpha(x_b t_b) \psi_\alpha(x_a t_a) . \tag{18}$$

In the frequent case that H corresponds to a solvable time-independent Schrödinger equation, $f(x)$ can be chosen as $f(x) \equiv 1$ and equation (14) separates such that it is solved by a factorized ansatz $\psi_\nu(x) \exp[-iEt]$ with $\psi_\nu(x)$ obeying

$$H\left(\frac{1}{i}\frac{\partial}{\partial x}, x\right) \psi_\nu(x) = E_\nu \psi_\nu(x) . \tag{19}$$

Then $\varepsilon_\alpha = E_\nu - E$ and (18) reduces to the well-known form

$$(x_b, t_b|x_a, t_a) = \sum_\nu \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{i}{E - E_\nu + i\varepsilon} \exp[-iE(t_b - t_a)] \psi_\nu(x_b) \psi_\nu^*(x_a) = \tag{20}$$

$$= \theta(t_b - t_a) \sum_\nu \exp[-iE_\nu(t_b - t_a)] \psi_\nu(x_a) \psi_\nu^*(x_a) .$$

In the case of the H atom, the choice $f(x) = r$ is the best and leads to the four-dimensional oscillator wave functions $\psi_\alpha(x, t)$.

Of course, what we have done is nothing but give another proof of the equivalence of path integrals and Schrödinger equations for the time-dependent case in a way which explicitly performs all the intermediate integration.

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