FIELD THEORY OF COLLECTIVE EXCITATIONS
IV CONDENSATION OF THREE- AND FOUR-PARTICLE CLUSTERS IN BOSE SYSTEMS

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Cluster effects in bosonic many-body systems are governed by the higher effective action $\Gamma[\Phi, G, \alpha_3, \alpha_4]$ whose extrema determine physical configurations of field expectation Φ, interacting Green's function G, and three- and four-particle vertex functions α_3, α_4. The expansion of Γ in powers of α_3 and α_4 leads to a hierarchy of non-perturbative effects. In particular, each of the variables can be spontaneously generated via a gap-type equation.

Four-particle clusters are known to play an important role in the physics of larger nuclei, very similar to Cooper pairs in superconductors and superfluid ^{3}He. Recently [1], we have investigated the effective action whose extrema account for condensation processes of such clusters in arbitrary many-fermion systems. It is the purpose of this note to extend this discussion to bosons. Here the situation is somewhat more involved since the products of an odd number of fields can have non-vanishing expectation values. We shall consider the general action of a Bose field:

$$\mathcal{A}[\varphi] = \tfrac{1}{2}\varphi\, \mathrm{i} G_0^{-1}\varphi - (1/3!)V_3\varphi\varphi\varphi - (1/4!)V_4\varphi\varphi\varphi\varphi\,, \tag{1}$$

where all internal and space–time indices have been suppressed. The matrix G_0 denotes the free-field propagator $G_0 \equiv \overline{\varphi\varphi}$. The form (1) incorporates the usual non-relativistic many-body problem (such as ^{4}He) if we understand φ to be a doubled field,

$$\varphi \equiv \begin{pmatrix} a \\ a^+ \end{pmatrix} = C\varphi^+ \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\varphi\,, \tag{2}$$

which combines the usual creation and annihilation operators in a single symbol, if we choose G_0 as the matrix

$$G_0 = \begin{pmatrix} 0 & -\mathrm{i}\partial_t - \epsilon \\ \mathrm{i}\partial_t - \epsilon & 0 \end{pmatrix}, \tag{3}$$

and if we set $V_3 = 0$. But also more "collective" field theories of the Landau–De Gennes type for liquid crystals [2], crystallization processes [3], or pion condensation [4] are contained in the same formalism which will account for a wide variety of fluctuation phenomena.

Starting point is the well-known generating functional of all connected Green's functions

$$W[j,K] = -\mathrm{i}\log Z[j,K] = -\mathrm{i}\log \int \mathcal{D}\varphi \exp(\mathrm{i}\mathcal{A}[\varphi] + \mathrm{i}j\varphi + \tfrac{1}{2}\mathrm{i}\varphi K\varphi)\,, \tag{4}$$

for which the equations of motion lead to the differential equation [‡1]

[‡1] Subscripts denote functional differentiation, i.e. $W_j = \delta W/\delta j$. The equation follows directly from the identity $\int \mathcal{D}\varphi\,(\delta/\delta\varphi)\exp(\mathrm{i}\mathcal{A} + \mathrm{i}j\varphi + \tfrac{1}{2}\mathrm{i}\varphi K\varphi) = 0$.

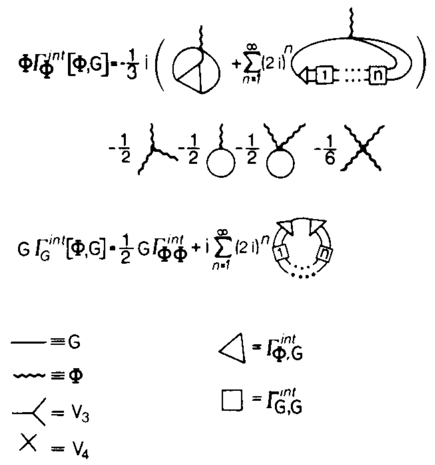

Fig. 1. The integral equation for the effective potential $\Gamma[\Phi, G]$. The symbols are explained in the figure.

$$(\mathrm{i}G_0^{-1} + K)W_j + (V_3/2!)(\mathrm{i}W_{jj} - W_j^2) + (V_4/3!)(W_{jjj} + 3\mathrm{i}W_{jj}W_j - W_j^3) + j = 0\,. \tag{5}$$

An intermediate effective action $\Gamma[\Phi, G]$ may be introduced as the Legendre transform of $W[j, K]$,

$$\Gamma[\Phi, G] = W[j, K] - W_j j - W_K K\,, \tag{6}$$

by choosing the independent variables Φ and G to be the expectation value of the field and the fully interacting connected Green's function, respectively, i.e.

$$\Phi = \langle\varphi\rangle = W_j[j, K] = \Phi[j, K]\,, \qquad G \equiv \langle T\varphi\varphi\rangle_c = 2W_K[j, K] - \Phi\Phi\,. \tag{7}$$

We can then decompose $\Gamma[\Phi, G]$ into a free and an interacting part

$$\Gamma[\Phi, G] = \tfrac{1}{2}\Phi\,\mathrm{i}G_0^{-1}\Phi + \tfrac{1}{2}\mathrm{tr}\,(\mathrm{i}G_0^{-1}G) + \tfrac{1}{2}\mathrm{i}\,\mathrm{tr}\log G^{-1} + \Gamma^{\mathrm{int}}[\Phi, G]\,, \tag{8}$$

and find from eq. (5) the following equation for the interacting part:

$$\Phi\Gamma_\Phi^{\mathrm{int}}[\Phi, G] = -(V_3/2!)(G + \Phi^2)\Phi - (V_4/3!)\,[2\mathrm{i}G^3\Phi\Gamma_{\Phi G}^{\mathrm{int}}(1 - 2\mathrm{i}G^2\Gamma_{GG}^{\mathrm{int}})^{-1} + 3G\Phi^2 + \Phi^4]\,, \tag{9}$$

whose index contractions are pictured graphically in fig. 1. Since W_j and W_K are not independent but related by

$$2\mathrm{i}W_K = W_{jj} + \mathrm{i}W_j^2\,, \tag{10}$$

there is also a constraint on the Φ, G dependence of $\Gamma^{\mathrm{int}}[\Phi, G]$:

$$0 = G\,[\Gamma_{\Phi\Phi}^{\mathrm{int}} - 2\Gamma_G^{\mathrm{int}} - 2\mathrm{i}\Gamma_{\Phi G}^{\mathrm{int}}GG(1 - 2\mathrm{i}G^2\Gamma_{GG}^{\mathrm{int}})^{-1}\Gamma_{G\Phi}^{\mathrm{int}}]\,, \tag{11}$$

which is again pictured in fig. 1. The two equations can be solved iteratively with the lowest-order result [5,6]

$$\Gamma^{\mathrm{int}}[\Phi, G] = -(1/3!)V_3\Phi^3 - (1/4!)V_4\Phi^4 - \tfrac{1}{2}V_3G\Phi - \tfrac{1}{4}V_4G\Phi^2 - \tfrac{1}{8}V_4G^2 + \tfrac{1}{48}\mathrm{i}V_4G^4V_4$$
$$+ \tfrac{1}{12}\mathrm{i}(V_3 + V_4\Phi)^2G^3 + \tfrac{1}{48}V_4^3G^6 + \tfrac{1}{8}V_4(V_3 + V_4\Phi)^2G^5 - \tfrac{1}{24}\mathrm{i}(V_3 + V_4\Phi)^4G^6 + \dots\,. \tag{12}$$

The expansion can easily be continued by observing that what emerges is precisely the sum of all two-particle irreducible vacuum graphs in which lines stand for the full Green's function G and vertices with three or four legs for $(1/3!)(V_3 + V_4\Phi), (1/4!)V_4$, respectively. The significance of this effective action is that by definition (6) its derivatives satisfy

$$\Gamma_\Phi = -j - K\Phi\,, \quad \Gamma_G = -\tfrac{1}{2}K\,, \tag{13}$$

and amount to $\Gamma[\Phi, G]$ being extremal for vanishing external sources, thereby leading to non-perturbative equations of motion for Φ and G. These are known to account for the condensation of particles as well as of Cooper pairs.

It is natural to suppose that by continuing the technique of Legendre transforms to multi-particle sources, also condensation processes of higher clusters will become accessible to simple extremality principles. Therefore we Legendre transform $\Gamma[\Phi, G]$ further with respect to V_3 and V_4:

$$\Gamma[\Phi, G, \alpha_3, \alpha_4] = \Gamma[\Phi, G] - \Gamma_{V_3}[\Phi, G]\, V_3 - \Gamma_{V_4}[\Phi, G]\, V_4 \,, \tag{14}$$

and choose to define the new variables α_3, α_4 as the three- and four-particle vertex functions which are obtained from $\Gamma_{V_3}, \Gamma_{V_4}$ by removing disconnected parts and amputating external legs:

$$W_{V_3}[j, K] \equiv -(1/3!)(-\mathrm{i}\alpha_3 G^3 + 3G\Phi + \Phi^3) = \Gamma_{V_3}[\Phi, G]\,, \tag{15}$$

$$W_{V_4}[j, K] = -(1/4!)(-\mathrm{i}\alpha_4 G^4 - 3\alpha_3^2 G^5 - 4\mathrm{i}\alpha_3 G^3\Phi + 2G^2 + 6G\Phi\Phi + \Phi^4) = \Gamma_{V_4}[\Phi, G] \,.$$

Obviously, only the interacting part of $\Gamma[\Phi, G]$ is affected by (14) and we find for it the simple Legendre transform

$$\Gamma^{\mathrm{int}}[\Phi, G, \alpha_3, \alpha_4] = -\tfrac{1}{12}\,\mathrm{i}\alpha_3^2 G^3 - \tfrac{1}{48}\,\mathrm{i}\alpha_4^2 G^4 + \tfrac{1}{48}\alpha_4^3 G^6 - \tfrac{1}{24}\,\mathrm{i}\alpha_3^4 G^6 + \ldots \,. \tag{16}$$

By construction, the new effective action $\Gamma[\Phi, G, \alpha_3, \alpha_4]$ satisfies the "extremality conditions"

$$\Gamma_\Phi = -j - K\Phi + \tfrac{1}{2}\Phi^2 V_3 + \tfrac{1}{6}\Phi^3 V_4 + \tfrac{1}{2}G(V_3 + V_4\Phi) - \tfrac{1}{6}\,\mathrm{i}\alpha_3 G^3 V_4 \,, \tag{17}$$

$$\Gamma_G = -\tfrac{1}{2}K + \tfrac{1}{2}\Phi V_3 + \tfrac{1}{4}\Phi^2 V_4 + \tfrac{1}{4}G V_4 - \tfrac{1}{2}\,\mathrm{i}\alpha_3 G^2(V_3 + V_4\Phi) - \tfrac{1}{6}\,\mathrm{i}\alpha_4 G^3 V_4 - \tfrac{5}{8}\alpha_3^2 G^4 V_4 \,, \tag{18}$$

$$\Gamma_{\alpha_3} = -\tfrac{1}{6}\,\mathrm{i}G^3(V_3 + V_4\Phi) - \tfrac{1}{4}\alpha_3 G^5 V_4 \,, \quad \Gamma_{\alpha_4} = -\tfrac{1}{24}\,\mathrm{i}G^4 V_4 \,. \tag{19, 20}$$

The physical situation is given by $j = 0$, $K = K^{\mathrm{cp}}$ = chemical potential, V_3, V_4 = physical couplings ‡2.

Explicitly, (17)–(20) lead to the following equations of motion:

$$V_4 = \alpha_4 + \tfrac{3}{2}\,\mathrm{i}\alpha_4\, G^2\alpha_4 + \ldots \,, \tag{21}$$

$$V_3 + V_4\Phi - \tfrac{3}{2}\,\mathrm{i}\alpha_3 G^2 V_4 = \alpha_3 + \mathrm{i}\alpha_3^3 G^3 + \ldots \,, \tag{22}$$

$$\Sigma \equiv \mathrm{i}G_0^{-1} + K^{\mathrm{cp}} - G^{-1} = \Phi(V_3 + \tfrac{1}{2}V_4\Phi) + \tfrac{1}{2}GV_4 - \tfrac{1}{2}\,\mathrm{i}\alpha_3 G^2(V_3 + V_4\Phi) - \tfrac{1}{6}\,\mathrm{i}\alpha_4 G^3 V_4 - \tfrac{1}{2}\alpha_3^2 G^4 V_4 \,, \tag{23}$$

$$(\mathrm{i}G_0^{-1} + K^{\mathrm{cp}})\Phi = \tfrac{1}{2}\Phi^2 V_3 + \tfrac{1}{6}\Phi^3 V_4 + \tfrac{1}{2}G(V_3 + V_4\Phi) - \tfrac{1}{6}\,\mathrm{i}V_4 G^3\alpha_3 \,. \tag{24}$$

This result is displayed graphically in fig. 2.

Notice that the equation for Φ has only a finite number of terms due to Γ^{int} being independent of Φ [see (16)]. The equation for Σ initially involves the whole infinite series for $-2\Gamma_G^{\mathrm{int}}$, which, however, can be resummed completely by using (19) and (20) due to the homogeneity equation

$$G\Gamma_G^{\mathrm{int}}[G, \alpha_3, \alpha_4] = \tfrac{3}{2}\alpha_3\Gamma_{\alpha_3}^{\mathrm{int}} + 2\alpha_4\Gamma_{\alpha_4}^{\mathrm{int}} \,, \tag{25}$$

which follows directly from counting vertices and lines in the vacuum graphs.

Certainly, eq. (23) is precisely the bosonic version of the time-dependent Hartree–Fock–Bogoliubov self-consistency condition

$$G = \mathrm{i}(\mathrm{i}G_0^{-1} + K^{\mathrm{cp}} - \Sigma[\Phi, G, \alpha_3, \alpha_4])^{-1} \,. \tag{26}$$

Eqs. (21)–(24) can be used to investigate large-amplitude collective excitations, and tunneling phenomena as

‡2 Certainly, one could use (15) and add $-\tfrac{1}{2}\Phi K^{\mathrm{cp}}\Phi + \Gamma_{V_3}V_3 + \Gamma_{V_4}V_4$ to $\Gamma[\Phi, G, \alpha_3, \alpha_4]$. The new functional $\Gamma^{\mathrm{new}}[\Phi, G, \alpha_3, \alpha_4]$ would then be truely extremal in all variables, i.e. $\Gamma_\Phi^{\mathrm{new}} = \Gamma_G^{\mathrm{new}} = \Gamma_{\alpha_2}^{\mathrm{new}} = \Gamma_{\alpha_4}^{\mathrm{new}} = 0$.

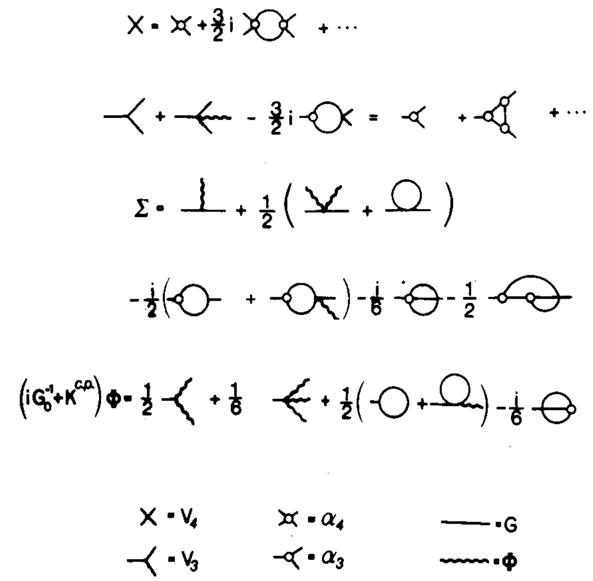

Fig. 2. The four equations of motion for the field expectation, the self-energy $\Sigma \equiv \mathrm{i}G_0^{-1} + K^{\mathrm{cp}} - \mathrm{i}G^{-1}$, and the three- and four-particle vertex functions α_3 and α_4, respectively.

described for Fermi systems in ref. [1], thereby generalizing earlier [6] approaches which were unable to cope with exchange forces.

The important new feature arising in these equations is the possibility of generating spontaneously vertex functions even if the action is free, i.e. eqs. (19) and (20) can have non-trivial solutions for $V_3 = V_4 = 0$. This is similar to the spontaneous generation of mass which is found to lowest approximation $\alpha_3 = V_3$, $\alpha_4 = V_4$, in which case (26) becomes the standard gap equation. Eqs. (19) and (20) may be seen as gap-type equations for the vertex functions α_3 and α_4.

More details [7] as well as the interesting physical consequences will be discussed elsewhere.

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