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Depairing Critical Current of ^{3}He-B at all Temperatures Including Gap Distortion

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Gap functions and superfluid densities are calculated at all temperatures in the presence of superflow. The critical current undergoes a significant reduction due to gap distortion.

Recently, several laboratories have begun investigating flow properties of superfluid ^{3}He. The Berkeley group,[1] for example, observed at zero pressure a maximal B-phase current proportional to $(1-T/T_c)^{3/2}$ in a tube of 0.02 mm diameter, which is narrow enough to lock the normal component of the superfluid.

Theoretical analyses are available[2] only for $T \approx T_c$. An existing calculation of the critical currents at all temperatures[3] cannot be compared with the data since it neglects the important effect of the current distorting the spherical gap of the B phase.

For $T \approx T_c$, this effect can readily be seen to be important: If the order parameter has the form

$$A_{ai} = \Delta_B \begin{pmatrix} a & & \\ & a & \\ & & a \end{pmatrix} e^{i\gamma} \tag{1}$$

with

$$\Delta_B^2 = \tfrac{1}{6}(1-T/T_c)v_F^2/\xi_0^2 \tag{2}$$

then the free energy becomes

$$f/2f_c = -a^2 + \tfrac{1}{2}\beta a^4 + \xi^2 a^2 (\partial_z \gamma)^2 \tag{3}$$

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where f_c is the weak coupling condensation energy of the B phase,

$$f_c \equiv (1/8m^2\xi_0^2)\rho(1-T/T_c)^2 \tag{4}$$

and

$$\xi = \xi_0\left(1-\frac{T}{T_c}\right)^{-1/2} = \left[\frac{7\zeta(3)}{48\pi^2}\right]^{1/2}\frac{\hbar}{k_{\rm B}}\frac{v_{\rm F}}{T_c}\left(1-\frac{T}{T_c}\right)^{-1/2} \tag{5}$$

is the coherence length (at $p=0$, $\xi_0 \approx 127$ Å). The constant β is the ratio of the Ginzburg–Landau parameter $\beta_{12}+\frac{1}{3}\beta_{345}$ to its weak coupling value. The current can be written as

$$J = jJ_0(1-T/T_c)^{3/2} \tag{6}$$

with the unit current

$$J_0 \equiv (\hbar/\xi_0 m)\rho \equiv v_0\rho[\approx (12.55 \text{ cm/sec})\rho \text{ at } \rho=0] \tag{7}$$

and a dimensionless quantity

$$j = a^2\xi(\partial_z\gamma) \tag{8}$$

The superfluid density is given in terms of a by

$$\rho_s = 2a^2(1-T/T_c)\rho \tag{9}$$

The equilibrium value of a at any current is obtained by minimizing the Legendre transformed energy

$$g = \frac{f - J\,\partial_z\gamma}{2f_c} = \frac{f}{2f_c} - 2j\,\partial_z\gamma = -a^2 + \frac{\beta}{2}a^4 - \frac{j^2}{a^2} \tag{10}$$

which happens at

$$j^2 = a^4(1-\beta a^2) \tag{11}$$

The largest current permitted by this relation is

$$j_c = 2/(3\beta\sqrt{3}) \tag{12}$$

or, at $p=0$,

$$J = (4.83 \text{ cm/sec})\rho(1-T/T_c)^{3/2}/\beta \tag{13}$$

Since at $p=0$ strong coupling corrections *are believed* to be small,[2] we shall neglect the correction factor β in what follows.

The critical current (11) is reduced appreciably if the gap is not artificially forced to stay spherical. With the current running in the z direction one may expect the distortion of the order parameter to be given by

$$A_{ai} = \Delta_B \begin{pmatrix} a & & \\ & a & \\ & & c \end{pmatrix} e^{i\gamma} \tag{14}$$

and finds a free energy

$$g = -\frac{1}{3}(2a^2+c^2) + \frac{1}{15}\left(4a^4+2a^2c^2+\frac{3}{2}c^4\right) - \frac{5}{2a^2+3c^2} j^2 \tag{15}$$

with

$$j = \tfrac{1}{5}(2a^2+3c^2)\xi(\partial_z \gamma) \tag{16}$$

instead of Eq. (8). Now the superfluid density parallel to the flow is given by

$$\rho_s^{\parallel} = \tfrac{2}{5}(2a^2+3c^2)(1-T/T_c)\rho \tag{17}$$

Due to the anisotropy of the gap, a slight additional current orthogonal to the direction of flow is associated with a transverse superfluid density of

$$\rho_s^{\perp} = \tfrac{2}{5}(4a^2+c^2)(1-T/T_c)\rho \tag{18}$$

The equilibrium with respect to variations of a and c now lies at

$$a = 1 \tag{19}$$

with a current

$$j^2 = (1/3 \cdot 25)(1-c^2)(2+3c^2)^2 \tag{20}$$

Thus the gap in the direction orthogonal to the flow ignores the current. Only the longitudinal value c shrinks. The maximal value of j is reached at

$$c = \tfrac{2}{3} \tag{21}$$

with

$$j_c = (20/3)^{1/2}/9 \tag{22}$$

which is smaller than (12) by a factor $\frac{3}{4}(80/81)^{1/2} \approx \frac{3}{4}$.

It is the purpose of this paper to extend this discussion to all temperatures.

The free energy of the B phase is[4]

$$f = -T \sum_{\omega_n, \mathbf{p}} \log[i\omega_n - E(\mathbf{p})] + (E \to -E) + (1/3g)|A_{ai}|^2 \tag{23}$$

In the presence of a current in the z direction, the equilibrium is governed not by f but by $f - \mathbf{vP}$, which is obtained by replacing[5,6]

$i\omega_n \to i\omega_n - \mathbf{vp}$ in (23). The nonspherical gap results in quasiparticle energies

$$E = [\xi^2(\mathbf{p}) + |A_{ai}\hat{p}_i|^2]^{1/2} = [\xi^2(\mathbf{p}) + \Delta_\perp^2(1-r^2z^2)]^{1/2}$$

where*

$$r^2 = 1 - c^2/a^2, \qquad \Delta_\perp^2 = \Delta_B^2 a^2, \qquad \Delta_\parallel^2 = \Delta_\perp^2(1-r^2) \tag{24}$$

The parameter r measures the reduction of the gap along the direction of flow. The last term in (23) is explicitly

$$(1/3g)|A_{ai}|^2 = \frac{1}{g}\Delta_\perp^2(1-\tfrac{1}{3}r^2)$$

By minimizing (22) with respect to $\Delta_\perp^2$ and $\Delta_\perp^2 r^2$ we now have two gap equations:

$$\log\frac{T}{T_c} = \frac{2}{\delta}\,\mathrm{Re}\int_{-1}^{1}\frac{dz}{2}\sum_{n=0}^{\infty}\left\{\frac{1}{[(x_n - i\nu z)^2 + 1 - r^2z^2]^{1/2}} - \frac{1}{x_n}\right\} \tag{25}$$

$$\log\frac{T}{T_c} = \frac{2}{\delta}\frac{3}{2}\,\mathrm{Re}\int_{-1}^{1}\frac{dz}{2}(1-z^2)\sum_{n=0}^{\infty}\left\{\frac{1}{[(x_n - i\nu z)^2 + 1 - r^2z^2]^{1/2}} - \frac{1}{x_n}\right\} \tag{26}$$

The right-hand side depend only on the dimensionless quantities

$$\delta = \Delta_\perp/\pi T, \qquad \nu = v p_F/\Delta_\perp, \qquad x_n = \omega_n/\Delta_\perp = (1/\delta)(2n+1) \tag{27}$$

The superfluid density parallel to the flow is obtained by differentiation,

$$J_s \equiv \rho_s^\parallel v = \partial(f - \mathbf{vP})/\partial v + \rho v \tag{28}$$

with the result

$$\frac{\rho_s^\parallel}{\rho} = 6\int_{-1}^{1}\frac{dz}{2} z \frac{1}{\delta\nu}\,\mathrm{Re}\; i\sum_{n=0}^{\infty}\frac{x_n - i\nu z}{[(x_n - i\nu z)^2 + 1 - r^2z^2]^{1/2}} \tag{29}$$

In order to find the superfluid density orthogonal to the main stream of current one substitutes in the expression for $f - \mathbf{vP}$

$$\nu z \to \nu z + \nu^\perp(1-z^2)^{1/2}\cos\varphi$$

then differentiates with respect to $\nu^\perp$ and averages over the azimuthal angles φ to get

$$\frac{\rho_s^\perp}{\rho} = \frac{3}{\delta}\int_{-1}^{1}\frac{dz}{2}(1-r^2z^2)(1-z^2)\sum_{n=0}^{\infty}\mathrm{Re}\frac{1}{[(x_n - i\nu z)^2 + 1 - r^2z^2]^{1/2}} \tag{30}$$

It is useful to realize that this can be obtained from (26) by substituting $x_n \to x_n/\alpha$, $\nu \to \nu/\alpha$, multiplying by $1/\alpha$, forming the derivative $-2\,d/d\alpha^2$, and setting $\alpha = 1$.

*Here $\Delta_B^2 \equiv \pi^2 T_c^2 \frac{8}{7\zeta(3)}\left(1 - \frac{T}{T_c}\right) \approx (3.06T_c)^2\left(1 - \frac{T}{T_c}\right)$.

Equivalently, one has

$$\frac{\rho_s^\perp}{\rho}=\left[1+2\nu^2\frac{d}{d\nu^2}+\text{“}\left(2x_n^2\frac{d}{dx_n^2}\right)\text{”}\right]\log\frac{T}{T_c}\bigg|_{\text{Eq. (26)}} \tag{31}$$

where the operation in quotation marks has to be performed *within* the Matsubara sum of Eq. (26).

Notice that Eqs. (27)–(30) are exactly the gap equations and superfluid densities for the B and A phases if one inserts the special values $r=0$ and $r=1$, respectively. This is useful for a check of the calculation. Performing the integrals over z gives

$$\log\frac{T}{T_c}=\frac{2}{\delta}\sum_{n=0}^{\infty}\left(\frac{1}{(\nu^2+r^2)^{1/2}}\alpha_n-\frac{1}{x_n}\right) \tag{32}$$

$$\log\frac{T}{T_c}=\frac{2}{\delta}\frac{3}{2}\sum_{n=0}^{\infty}\left\{\left[1-\frac{1}{2(\nu^2+r^2)}+\frac{(\nu^2-r^2/2)x_n^2}{(\nu^2+r^2)^2}\right]\frac{\alpha_n}{(\nu^2+r^2)^{1/2}}\right.$$
$$\left.-\frac{2}{3}\frac{1}{x_n}+\text{Re}\,\frac{\nu^2+r^2-3i\nu x_n}{2(\nu^2+r^2)^2}[(x_n-i\nu)^2+1-r^2]^{1/2}\right\} \tag{33}$$

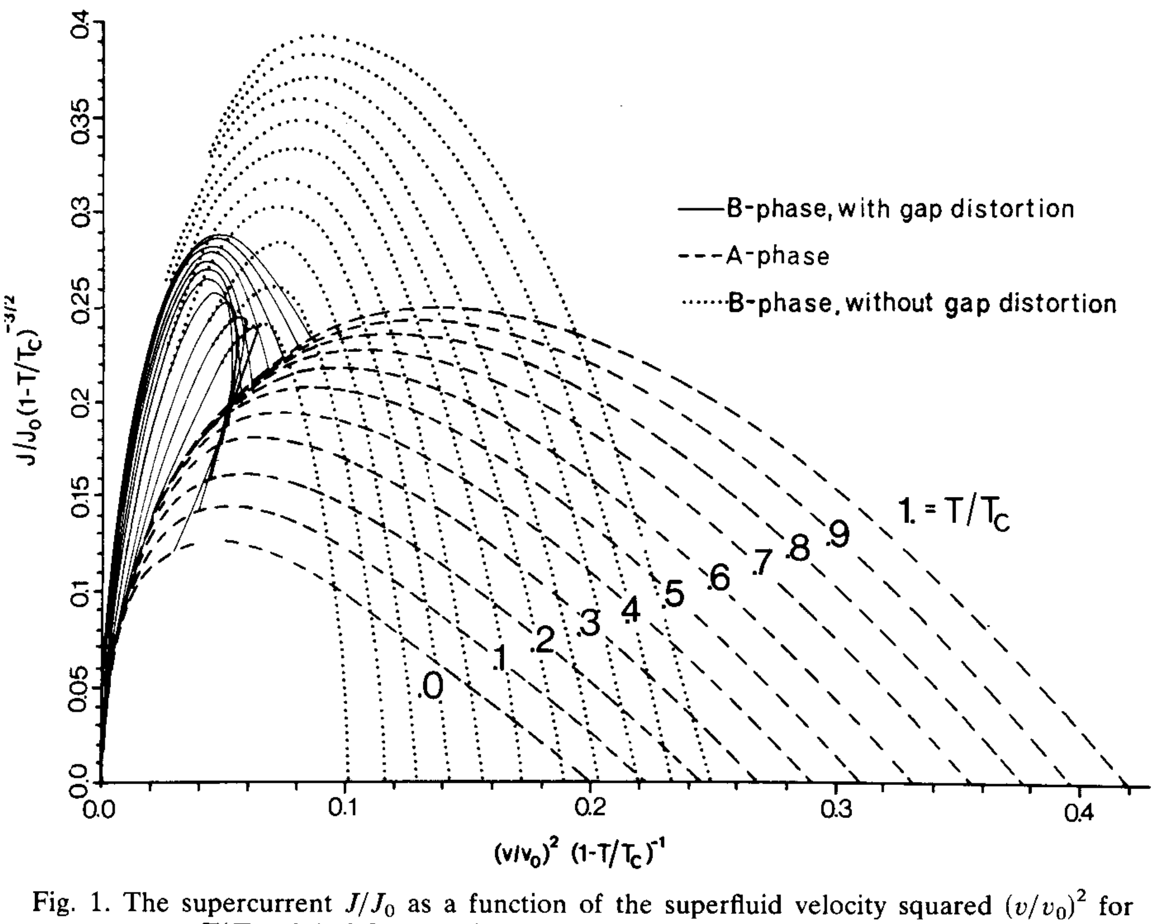

Fig. 1. The supercurrent J/J_0 as a function of the superfluid velocity squared $(v/v_0)^2$ for temperatures $T/T_c=0.1, 0.2, \ldots, 1$ in the B phase including (—) and neglecting (· · · ·) the effects of gap distortion, and in the A phase (- - -). The natural units are $v_0=12.55$ cm/sec and $J_0=\rho v_0$. We have found it convenient to divide out the temperature factors $(1-T/T_c)^{3/2}$ and $(1-T/T_c)$ for J and v^2.

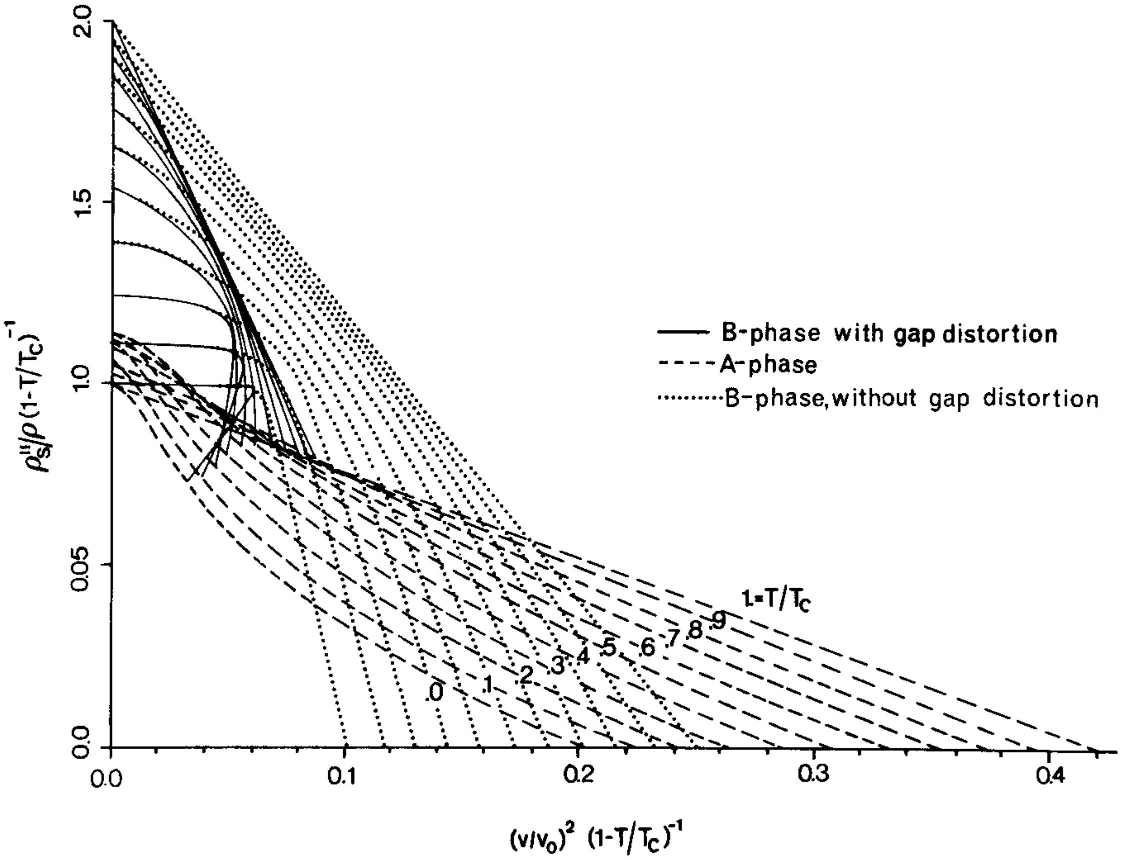

Fig. 2. The superfluid density parallel to the flow $\rho_s^{\parallel}/\rho$, with the same conventions as in Fig. 1. A factor $(1-T/T_c)$ is divided out.

$$\frac{\rho_s^{\parallel}}{\rho}=\frac{3}{\delta}\sum_{n=0}^{\infty}\left\{\frac{1}{(\nu^2+r^2)^2}(3r^2x_n^2+\nu^2+r^2)\frac{\alpha_n}{(\nu^2+r^2)^{1/2}}\right.$$
$$\left.-\mathrm{Re}\,\frac{2}{\nu(\nu^2+r^2)}\left[ix_n+\frac{\nu}{2}\left(1-3i\frac{\nu x_n}{\nu^2+r^2}\right)\right][(x_n-i\nu)^2+1-r^2]^{1/2}\right\} \tag{34}$$

where

$$\alpha_n=\mathrm{Im}\,\mathrm{sh}^{-1}\left(\frac{\nu x+i(\nu^2+r^2)}{(\nu^2+r^2+r^2x^2)^{1/2}}\right) \tag{35}$$

The solutions are found by computer and are shown in Figs. 1–5. In the limit $T\to 0$ the sum over x_n can be replaced by an integral, leading to the gap equations ($\Delta_{\mathrm{BCS}}=\pi e^{-\gamma}T_c\approx 1.76T_c$)

$$\begin{pmatrix}1\\ \frac{2}{3}\end{pmatrix}\log\frac{\Delta^{\perp}}{\Delta_{\mathrm{BCS}}}=-\frac{1}{2}\int_0^{(\nu^2+r^2)^{-1/2}}dz\begin{pmatrix}1\\ 1-z^2\end{pmatrix}\log(1-r^2z^2)$$
$$-\int_{(\nu^2+r^2)^{-1/2}}^{1}dz\begin{pmatrix}1\\ 1-z^2\end{pmatrix}\log\{[(\nu^2+r^2)z^2-1]^{1/2}+\nu z\} \tag{36}$$

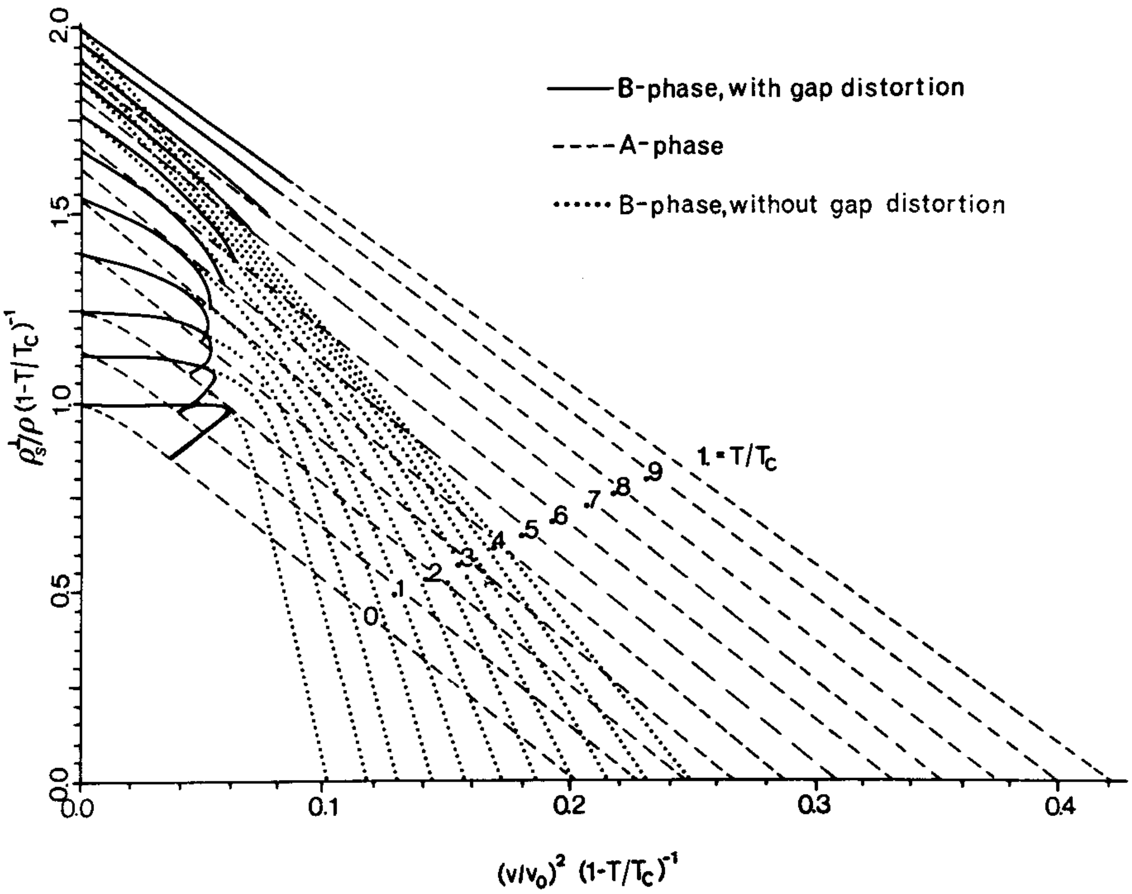

Fig. 3. The superfluid density orthogonal to the flow, with the same conventions as in Fig. 2.

and a superfluid density

$$\frac{\rho_s^{\parallel}}{\rho}=1-\theta(\nu^2+r^2-1)\frac{1}{\nu(\nu^2+r^2)}(\nu^2+r^2-1)^{3/2} \tag{37}$$

The transverse superfluid density is found from the $T=0$ rule corresponding to (31) as

$$\begin{aligned}\frac{\rho_s^{\parallel}}{\rho}&=1+2\nu^2\frac{d}{d\nu^2}\log\frac{\Delta^{\perp}}{\Delta_{\mathrm{BCS}}}\bigg|_{\text{lower Eq. (36)}}\\ &=1-\theta(\nu^2+r^2-1)\frac{\nu}{(\nu^2+r^2)^2}(\nu^2+r^2-1)^{3/2}\end{aligned} \tag{38}$$

As a final check of our equations we take $T\sim T_c$. Now the $x_n\to\infty$ limit of (25), (26), and (29) gives

$$\begin{aligned}1-\frac{T}{T_c}&\approx\frac{2}{\delta}\mathrm{Re}\int_{-1}^{1}\frac{dz}{2}\begin{pmatrix}1\\ \frac{3}{2}(1-z^2)\end{pmatrix}\sum_{n=0}^{\infty}\frac{1+(2\nu^2-r^2)z^2-2i\nu z x_n}{2x_n^3}\\ &=\delta^2\left[1+\begin{pmatrix}\frac{1}{3}\\ \frac{1}{5}\end{pmatrix}(2\nu^2-r^2)\right]\frac{7\zeta(3)}{8}\end{aligned} \tag{39}$$

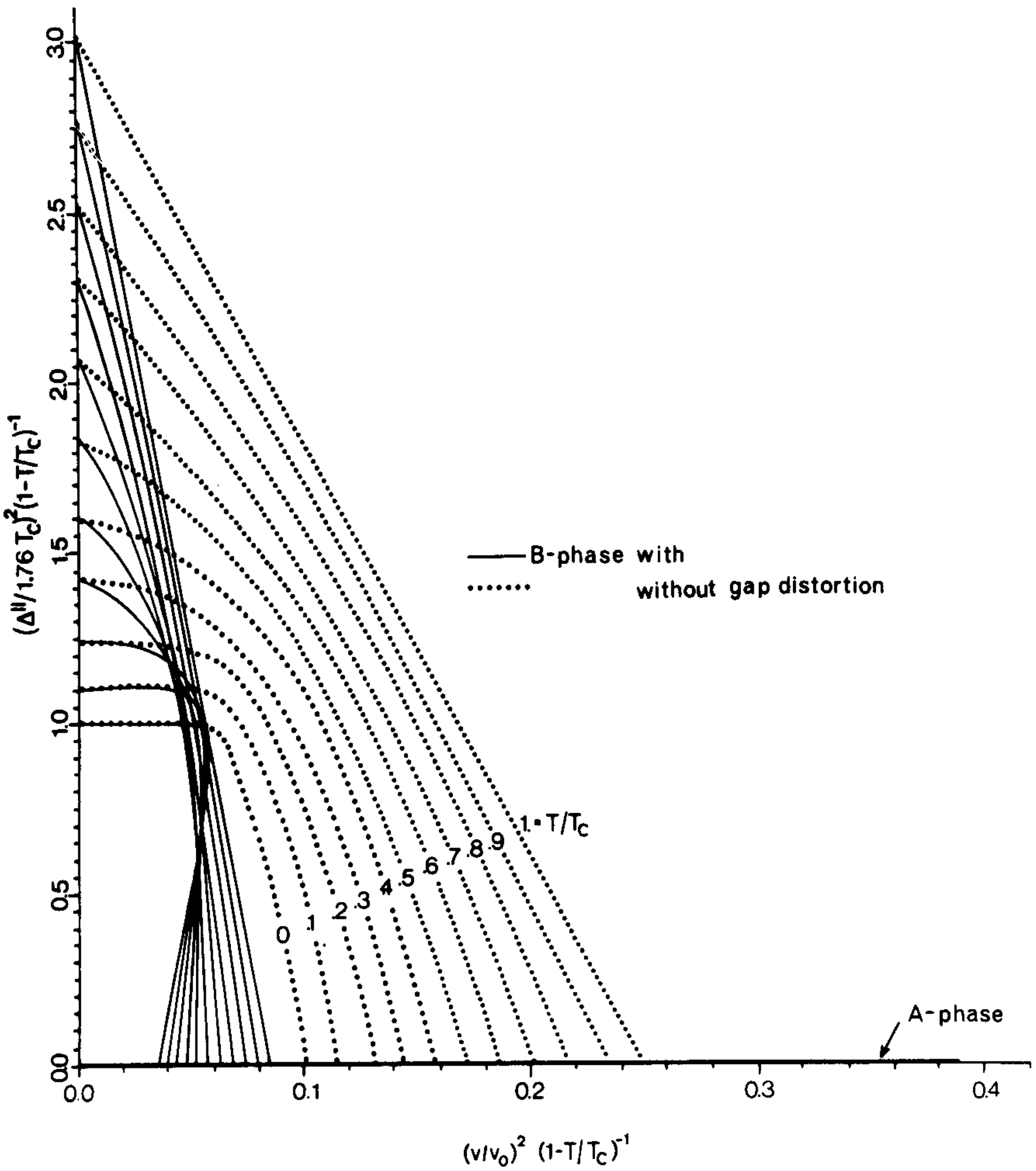

Fig. 4. The gap function $(\Delta^{\|})^2/\Delta_{BCS}^2$ parallel to the flow, in the form $(\Delta^{\|})^2/[\Delta_{BCS}^2(1-T/T_c)]$, where $\Delta_{BCS}\approx 1.76T_c$, with all other conditions as in the previous figures.

$$\frac{\rho_s^{\|}}{\rho}\approx\frac{6}{\delta}\int_{-1}^{1}\frac{dz}{2}z^2(1-r^2z^2)\sum_{n=0}^{\infty}\frac{1}{x_n^3}=6\delta^2\left(\frac{1}{3}-\frac{r^2}{5}\right)\frac{7\zeta(3)}{8}$$

$$\frac{\rho_s^{\perp}}{\rho}\approx\frac{3}{\delta}\int_{-1}^{1}\frac{dz}{2}(1-z^2)(1-r^2z^2)\sum_{n=0}^{\infty}\frac{1}{x_n^3} \qquad (40)$$

$$=2\delta^2\left(1-\frac{r^2}{5}\right)\frac{7\zeta(3)}{8}$$

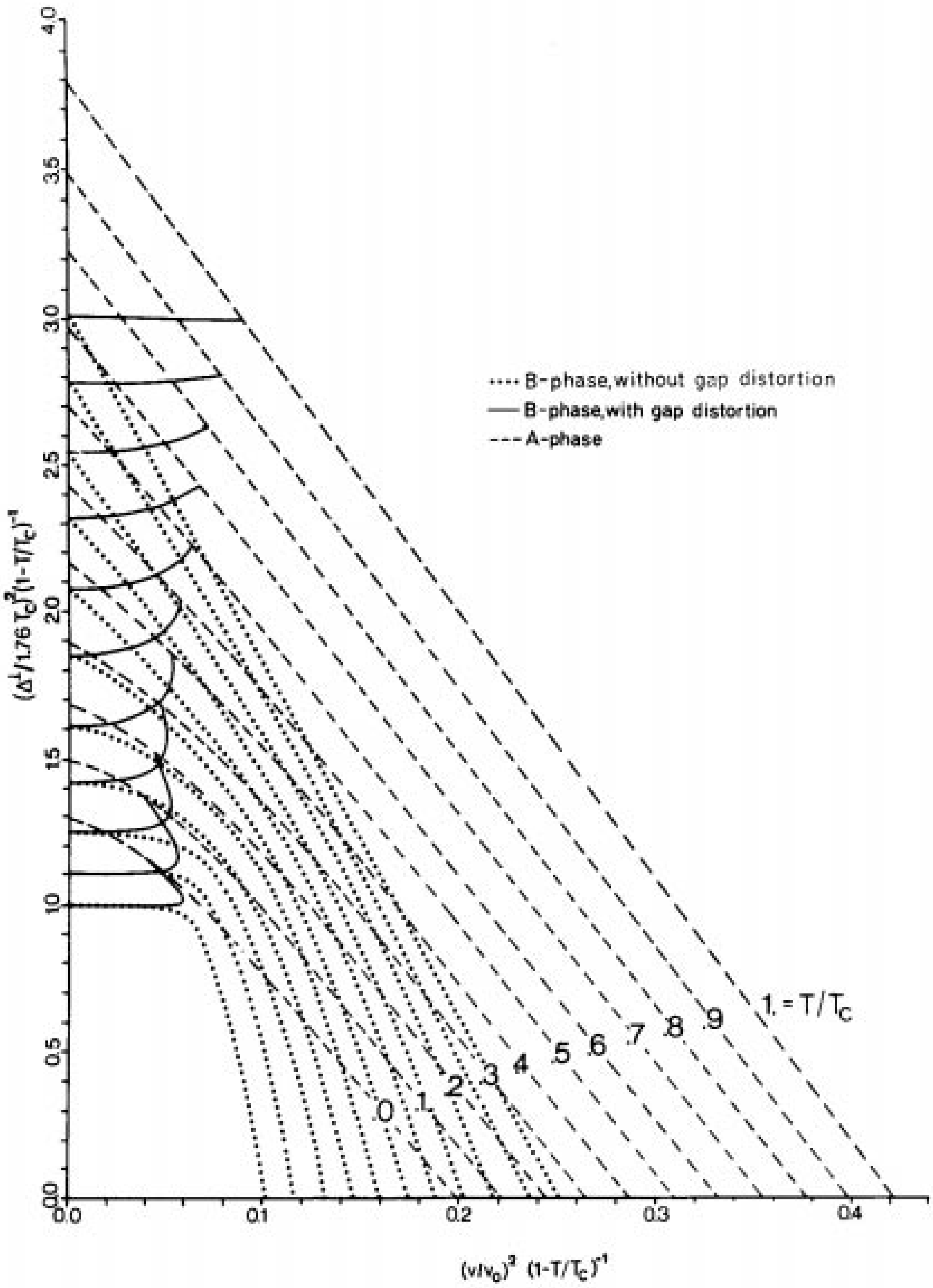

Fig. 5. The gap function orthogonal to the flow in the form $(\Delta^\perp)^2/[\Delta^2_{BCS}(1-T/T_c)]$ similar to Fig. 4.

resulting in

$$\delta^2 \approx (1-T/T_c)8/7\zeta(3) \tag{41}$$

$$r^2 \approx 2\nu^2 \tag{42}$$

such that the supercurrent

$$J_s = \rho_s^{\|} v_s = 6\delta^2\left(\frac{1}{3}-\frac{r^2}{5}\right)\nu\delta\frac{7\zeta(3)}{8}\frac{\pi T_c}{\rho_F} \tag{43}$$

is maximal at

$$\nu^2 = 5/18 \tag{44}$$

which leads to exactly the same value (22) found in the Ginzburg–Landau calculation.*

The inclusion of Fermi liquid corrections has no effect on the critical current.[3] It is taken into account most simply by replacing $v^{\|,\perp}$ in all formulas by the reduced quantity $v^{\|,\perp}/[1+\frac{1}{3}F_1^s\ (1-\rho_s^{\|,\perp}/\rho)]$. This changes the position of the critical current on the velocity axis but leaves the size untouched!†

For a graphical representation of the results it is most convenient to divide out the $(1-T/T_c)$ dependence attached to all quantities in the Ginzburg–Landau domain. Thus we plot

$$\begin{aligned} j &= J/[J_0(1-T/T_c)^{3/2}] \\ \hat{\rho}^{\|} &= \rho^{\|}/[\rho(1-T/T_c)] \\ \hat{\rho}^{\perp} &= \rho^{\perp}/[\rho(1-T/T_c)] \\ (\hat{\Delta}^{\|})^2 &= (\Delta^{\|})^2/(1-T/T_c)(1.76T_c)^{-2} \\ (\hat{\Delta}^{\perp})^2 &= (\Delta^{\perp})^2/(1-T/T_c)(1.76T_c)^{-2} \end{aligned}$$

against $v^2/[v_0^2(1-T/T_c)]$. This is the best way of displaying the deviations from the $T \approx T_c$ limiting forms. We have

$$j \equiv \hat{\rho}^{\|} v = \begin{Bmatrix} 2[1-(36/5)v^2] \\ 2(1-4v^2) \\ [1-(12/5)v^2] \end{Bmatrix} v \quad \text{for} \quad \begin{Bmatrix} \mathrm{B}_{\text{dist. gap}} \\ \mathrm{B}_{\text{undist. gap}} \\ \mathrm{A} \end{Bmatrix} \text{phase} \tag{45}$$

*Note that

$$\frac{\pi T_c}{p_F} = \frac{1}{4\sqrt{3}}[7\zeta(3)]^{1/2} J_0 = \frac{\pi}{1.764}\frac{\Delta_{BCS}}{p_F} = 1.781 \times 0.235 J_0$$

†Notice that $\rho_s^{\|,\perp}$ is reduced by a factor $[1+\frac{1}{3}F_1^s(1-\rho_s^{\|,\perp}/\rho)]^{-1}$ while $v^{\|,\perp}$ receives the inverse correction. Therefore $\rho_s v$ is invariant.

$$\hat{\rho}^{\perp} = 2\begin{Bmatrix} 1-(12/5)v^2 \\ 1-4v^2 \\ 1-(12/5)v^2 \end{Bmatrix} \tag{46}$$

$$(\hat{\Delta}^{\parallel})^2 = 3.015\begin{Bmatrix} 1-12v^2 \\ 1-4v^2 \\ 0 \end{Bmatrix} \tag{47}$$

$$(\hat{\Delta}^{\perp})^2 = 3.015\begin{Bmatrix} 1 \\ 1-4v^2 \\ \frac{5}{4}[1-(12/5)v^2] \end{Bmatrix} \tag{48}$$

In all cases we have also given the corresponding functions for the B phase neglecting gap distortion as well as for the A phase, in order to permit an easy judgment of the differences.

The currents are maximal for

$$v_c = \begin{cases} \frac{1}{6}\sqrt{\frac{5}{3}} \approx 0.215 \\ \dfrac{1}{2\sqrt{3}} \approx 0.288 \\ \sqrt{5}/6 \approx 0.373 \end{cases} \tag{49}$$

with peak values

$$j_c = \begin{cases} \frac{2}{9}\sqrt{\frac{5}{3}} \approx 0.287 \\ \dfrac{2}{3\sqrt{3}} \approx 0.385 \\ \frac{1}{9}\sqrt{5} \approx 0.249 \end{cases} \tag{50}$$

Notice that as v passes beyond v_c for the distorted gap, the current decreases and joins the curve for the A phase at the point where the gap $\Delta^{\parallel}$ vanishes, as it should.

In Fig. 6 we show the behavior of the critical current j_c as a function of $1-T/T_c$ on a double-logarithmic plot. Only for $T<0.15T_c$ are there visible deviations from the Ginzburg–Landau straight line.

For completeness[2] let us mention that an external magnetic field h measured in natural units $H_0 = \{[N(0)/g_z](1-T/T_c)\}^{1/2} = (16.38\ \text{kG})(1-T/T_c)^{1/2}$ would add to (15)* a term c^2h^2 and enter inside the

*The additional magnetic energy density is[2] $f_{mg} = g_z|H_\alpha A_{\alpha i}|^2$, which can be written, together with (1), (2), (4), and $N(0) = \frac{3}{2}\rho/p_F^2$, as $f_{mg} = 2f_cH^2/H_0^2$ or $g_{mg} = c^2h^2$. Then (51) and (52) follow by repeating the steps from (15) to (20) including the term c^2h^2.

curly brackets in (45)–(48) as

$$\begin{Bmatrix} -6h^2 \\ -2h^2 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} 0 \\ -2h^2 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} -6h^2 \\ -h^2 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} \frac{3}{2}h^2 \\ -h^2 \\ 0 \end{Bmatrix} \tag{51}$$

Critical velocities and currents would be reduced to

$$v_c = \begin{Bmatrix} \frac{1}{6}\sqrt{\frac{5}{3}}(1-3h^2)^{1/2} \\ (1/2\sqrt{3})(1-h^2)^{1/2} \\ \sqrt{5}/6 \end{Bmatrix}, \qquad j_c = \begin{Bmatrix} \frac{2}{9}\sqrt{\frac{5}{3}}(1-3h^2)^{3/2} \\ \frac{2}{3}(1/\sqrt{3})(1-h^2)^{3/2} \\ \sqrt{5}/9 \end{Bmatrix}, \tag{52}$$

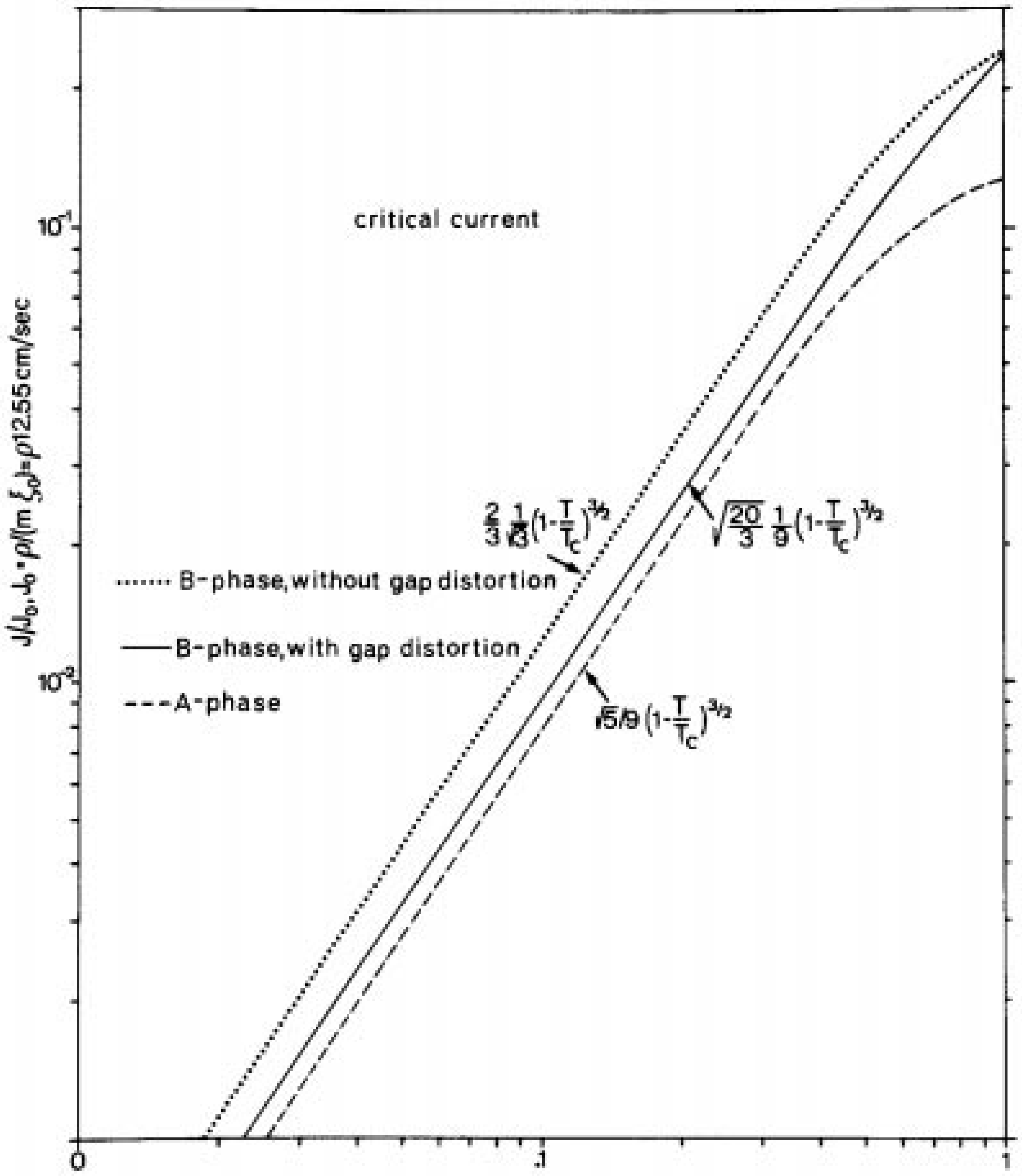

Fig. 6. The critical current J/J_0 as a function of temperature.

Notice that for $h^2 > h_{AB}^2 \equiv \frac{1}{3}[1-(\frac{3}{4})^{1/3}] \approx 0.0305$ the system can support the highest supercurrent in the A phase.

A discussion of the collective excitations in the presence of gap distortion can be found in Ref. 7.

NOTE ADDED IN PROOF

After distributing preprints of this work, I was informed by D. Vollhardt that he and K. Maki were investigating the same problem.

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