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SEMICLASSICAL APPROACH TO LARGE-AMPLITUDE COLLECTIVE NUCLEAR EXCITATIONS

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Abstract: We propose a semiclassical functional treatment for large-amplitude collective oscillations and illustrate the method by an application to the degenerate shell model with pairing forces.

1. Introduction

In recent years, an increasing effort has been devoted to investigating large-amplitude collective excitations in nuclei, either via the generator coordinate method [1]) or the adiabatic time-dependent Hartree-Fock (TDHF) equations [2]). Model studies reveal the necessity of including non-adiabatic effects [4]). But then the problem of quantizing TDHF equations becomes quite cumbersome. This problem can be circumvented by employing the recently proposed [5]) path integral techniques for the study of collective nuclear phenomena. They have proven successful in the description of other involved many-body systems such as plasma [6]), superconductors [refs. [6,7])], super-liquid ^{3}He [ref. [8])], and strongly interacting particles [9]). In nuclear physics, they have led to a proper foundation of the rules of the so-called nuclear field theory [10]).

Certainly, to the practical nuclear physicist, the conceptual advantages would not give sufficient reason to warrant the introduction of an unfamiliar theoretical frame work unless there is a range of phenomena in which previous tools fail while the new methods render a description of superiour simplicity.

It is the purpose of this paper to point out that path integrals are ideally suited for a theory of large-amplitude nuclear excitations. The main reason lies in the fact that here, beyond the reach of conventional operator methods, the motion becomes semiclassical and it is precisely in this limit, that path integrals are most powerful.

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The necessary quantization techniques for systems of many degrees of freedom are available [12]. They have been developed for the description of non-linear quantum effects in field theories and have led to the quantum-mechanical understanding of a new type of quasi-particles (topological solitons, soliton-antisoliton bound states) which was inaccessible to previous field theoretic treatments.

Here we would like to take the first step towards the introduction of these techniques in nuclear physics by demonstrating the semi-classical quantization of the collective fields for the degenerate-shell BCS model [11].

Since this model is quite familiar, exactly soluble, and also contains important non-linear effects (even though none are of the most interesting soliton nature) we believe it to be ideally suited for a simple introduction of the new methods.

Although we are dealing with the BCS model, we shall formulate the semiclassical approach in a way which can be easily extended to more complicated systems. The explicit treatment of the general case is based on the recently proposed [14]) path integral formulation of time-dependent self-consistent field theories, and will be given in a subsequent paper.

2. The collective field theory

Consider the degenerate shell model containing 2Ω fermions coupled via a pairing force with a Hamiltonian

$$H = \sum_{i=1}^{\Omega} \varepsilon(a_i^+ a_i - b_i b_i^+) - \tfrac{1}{2}V\{\sum_i a_i^+ b_i^+, \sum_j b_j a_j\}. \tag{1}$$

The model is, of course, soluble by observing that

$$L^+ = \sum_{i=1}^{\Omega} a_i^+ b_i^+, \qquad L^- = \sum_{i=1}^{\Omega} b_i a_i,$$
$$L_3 = \tfrac{1}{2} \sum_{i=1}^{\Omega} (a_i^+ a_i - b_i b_i^+) \tag{2}$$

form an SU(2) quasi-spin algebra and that H can be written as

$$\begin{aligned} H &= 2\varepsilon L_3 - \tfrac{1}{2}V\{L^+, L^-\}, \\ &= 2\varepsilon L_3 - V(L^2 - L_3^2). \end{aligned} \tag{3}$$

The eigenfunctions may be constructed by repreated application of L^+ to states containing the seniority number v of unpaired particles

$$(L^+)^n|v\rangle. \tag{4}$$

Their quasi-spin is

$$L^2 = \tfrac{1}{2}(\Omega - v)(\tfrac{1}{2}(\Omega - v) + 1) \tag{5}$$

with the third component L_3 taking the values

$$L_3 = \tfrac{1}{2}(N-\Omega) = -\tfrac{1}{2}(\Omega-v), \ldots, \tfrac{1}{2}(\Omega-v), \tag{6}$$

where N is the total number of particles. As a consequence, the energy spectrum is

$$E = 2\varepsilon(n-\tfrac{1}{2}(\Omega-v)) - V[\tfrac{1}{2}(\Omega-v)(\tfrac{1}{2}(\Omega-v)+1)-(n-\tfrac{1}{2}(\Omega-v))^2] \tag{7}$$

where we have relabelled the eigenvalues of L_3 as

$$L_3 = n-\tfrac{1}{2}(\Omega-v), \tag{8}$$

such that n is the number of pairs which for seniority v can take the values

$$n = 0, 1, 2, \ldots, \Omega-v.$$

Notice that the more conventional form

$$\begin{aligned} H' &= \sum_{i=1}^{\Omega} \varepsilon'(a_i^+ a_i + b_i^+ b_i) - V \sum_{i,j} a_i^+ b_i^+ b_j a_j, \\ &= 2\varepsilon'(L_3+\tfrac{1}{2}\Omega) - V L^+ L^-, \\ &= (2\varepsilon' - V)L_3 - \tfrac{1}{2}V\{L^+, L^-\} + \varepsilon'\Omega, \end{aligned}$$

can be obtained by renormalizing the single-particle energy as well as the zero point:

$$\varepsilon' = \varepsilon + \tfrac{1}{2}V, \qquad H' = H + \varepsilon'\Omega. \tag{9}$$

Of course, the energy E' is

$$E' = 2\varepsilon' N - \tfrac{1}{4}V[(\Omega-v)(\Omega-v+2)-(N-\Omega)^2]. \tag{10}$$

Consider now the path integral formulation of such a quantum theory. Here one does not work with operators and a Hamiltonian but employs instead, the classical action

$$A[a, b] = \int_{-T/2}^{T/2} \mathrm{d}t \sum_{i=1}^{\Omega} (a_i^+ i\partial_t a_i(t) + b_i i\partial_t b_i(t) - \int_{-T/2}^{T/2} \mathrm{d}t H(t), \tag{11}$$

where a, b are classical (but anticommuting) objects.

The full quantum theory is obtained by noticing that the partition function

$$Z = \mathrm{tr}(\exp[-iHT/\hbar]) \tag{12}$$

can be written in the form [13])

$$Z = \int \mathrm{D}a\mathrm{D}a^+ \mathrm{D}b\mathrm{D}b^+ \, \mathrm{e}^{iA[a,b]/\hbar} \tag{13}$$

The functional integrations run over all classical field configurations $a_i(t)$, $b_i(t)$. Since the attractive force causes pairing between a and b particles there exists a completely equivalent description of the system involving only collective fields for the particle *pairs*. In order to derive this one introduces an auxiliary dependent field S which [19]) classically equals the composite pair $V\sum_i a_i^+ b_i^+$ by adding to the action

$$\frac{1}{V}|S - V\sum_{i=1}^{\Omega} a_i^+ b_i^+|^2. \tag{14}$$

The partition function Z does not change if one also integrates functionally over the fields S, S^+. In this way S, S^+ become quantum variables which fluctuate in a Gaussian fashion around the composite pair field $V\sum_i a_i^+ b_i^+$.

The point is now that the choice (14) eliminates the quartic interaction and leads to a Yukawa type of theory which can simply be written in a matrix form as

$$A[a,b] = \int \mathrm{d}t \left\{ \sum_{i=1}^{\Omega} (a_i^+(t) b_i(t)) \begin{pmatrix} i\partial_t - \varepsilon & S^+ \\ S & i\partial_t + \varepsilon \end{pmatrix} \begin{pmatrix} a_i(t) \\ b_i^+(t) \end{pmatrix} - \frac{1}{V}|S(t)|^2 \right\}. \tag{15}$$

But now the original fermions appear only quadratically such that the Fermi fields can be integrated out of Z leaving a theory described only in terms of the collective field S:

$$Z = \int \mathrm{D}S\mathrm{D}S^+ \, \mathrm{e}^{iA[S]/\hbar} \tag{16}$$

with an action

$$A[S] = -i \operatorname{tr} \log \begin{pmatrix} i\partial_t - \varepsilon & S^+(t) \\ S(t) & i\partial_t + \varepsilon \end{pmatrix} - \frac{1}{V}\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}t |S(t)|^2. \tag{17}$$

The form of this collective action is universal to all pairing theories [5-8]) and does not depend on the particular simplicity of the model. In fact, in superconductors and ^{3}He one only has to take

$$\varepsilon = -\frac{1}{2m}\partial^2 - \mu, \qquad S = S(x,t)$$

and the action (17) describes extremely well a great variety of phenomena.

The simplicity of the model is helpful, however, when it comes to calculating the trace of the logarithm explicitly. This has previously been done in the limit $T \to \infty$. We shall here extend the discussion to arbitrary time intervals T where also excited states contribute to the trace [12]).

First we observe that the equation of motion which follows from eq. (15) by functional differentiation with respect to S reads ($S = S_1 + iS_2$)

$$S_{1,2}(t) = -\tfrac{1}{2} V \operatorname{tr}(\sigma_{1,2} G_S(t,t'))|_{t'=t+\varepsilon}, \tag{18}$$

where the Green function G_s is the solution of the equation

$$\begin{pmatrix} i\partial_t - \varepsilon & S^+(t) \\ S(t) & i\partial_t + \varepsilon \end{pmatrix} G_S(t,t') = i\delta(t,t') \tag{19}$$

and $\sigma_{1,2}$ are the Pauli matrices. This is solved by an ansatz

$$G_S(t,t') = U^+(t) G_0(t,t') U(t'), \tag{20}$$

where U is a unitary 2×2 rotation matrix satisfying

$$i\dot{U}^{\dagger}U = -V(S_1\sigma_1+S_2\sigma_2)+\varepsilon\sigma_3. \tag{21}$$

The left-hand side defines the angular velocities of the rotation U:

$$i\dot{U}^{\dagger}U = \sum_{a=1}^{3}\tfrac{1}{2}\omega_a\sigma_a. \tag{22}$$

When performing the trace one has to be careful with the boundary condition to be imposed on $G_0(t,t')$. This depends on which of the states

$$|0\rangle,\ a_i^+|0\rangle,\ b_i^+|0\rangle,\ a_i^+b_i^+|0\rangle,$$

is considered as a ground state for each of the shell labels i. Since G_0 corresponds to the free propagator

$$G_0(t,t') = \begin{pmatrix}\langle aa^+\rangle & \langle ab\rangle\\ \langle b^+a^+\rangle & \langle b^+b\rangle\end{pmatrix}, \tag{23}$$

it has the form

$$G_0(t,t') = \begin{pmatrix}\theta(t-t') & 0\\ 0 & -\theta(t'-t)\end{pmatrix} \tag{24}$$

if the ground state is empty $|0\rangle$;

$$G_0(t,t') = \begin{pmatrix}-\theta(t-t') & 0\\ 0 & -\theta(t'-t)\end{pmatrix} \tag{25}$$

if there is a particle $a_i^+|0\rangle$;

$$G_0(t,t') = \begin{pmatrix}\theta(t-t') & 0\\ 0 & \theta(t-t')\end{pmatrix} \tag{26}$$

if there is a particle $b_i^+|0\rangle$, and

$$G_0(t,t') = \begin{pmatrix}-\theta(t'-t) & 0\\ 0 & \theta(t-t')\end{pmatrix} \tag{27}$$

if both a_i^+ and b_i^+ are present. Therefore $G_0(t,t')|_{t'=t+\varepsilon}$ becomes

$$\begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & 0\end{pmatrix} \tag{28}$$

in these four cases, respectively, or

$$\tfrac{1}{2}(-1+\sigma_3),\ -1,\ 0,\ \tfrac{1}{2}(-1-\sigma_3). \tag{29}$$

Inserting this into the equation of motion gives contributions

$$-\tfrac{1}{2}V\,\mathrm{tr}\,(\sigma_{1,2}U^{\dagger}(t)MU(t)), \tag{30}$$

with M being any of the four matrices (29). This results in

$$-\tfrac{1}{2}V\begin{cases}\operatorname{tr}(\sigma_{1,2}U^{\dagger}\sigma^{3}U(t))\\ 0\\ 0\\ -\operatorname{tr}(\sigma_{1,2}U^{\dagger}\sigma^{3}U(t)).\end{cases} \tag{31}$$

If we parametrize U in terms of Euler angles

$$U = e^{i\alpha\sigma_3/2}e^{i\beta\sigma_2/2}e^{i\gamma\sigma_3/2} \tag{32}$$

this becomes

$$-\tfrac{1}{2}V\begin{cases}\sin\beta e^{i\gamma}\\ 0\\ 0\\ -\sin\beta e^{i\gamma}\end{cases} \tag{33}$$

The same consideration has to be done for each of the particle states in the shell $i = 1, \ldots, \Omega$.

Consider now an arbitrary state of seniority v. This has $\Omega - v$ empty places while v places are occupied with one unpaired particle a^+ or b^+. Consequently, the trace term in eq. (18) gives

$$-\tfrac{1}{2}V(\Omega - v)\sin\beta e^{i\gamma}, \tag{34}$$

such that on background of v unpaired particles the equation of motion for the collective field S reads

$$S^{(v)}(t) = -V\tfrac{1}{2}(\Omega - v)\sin\beta(t)e^{i\gamma(t)}. \tag{35}$$

Notice that the collective field moves differently for different seniority v and the index v is necessary to keep track of this fact. Certainly, eq. (35) is very implicit since β and γ are still (very complicated) functionals of $S^{(v)}$. Fortunately, it is quite easy to derive explicit equations of motion for each $S^{(v)}$. For this we form the time derivative of (35):

$$\dot{S}^{(v)} = -V\tfrac{1}{2}(\Omega - v)\cos\beta(\dot{\beta} + i\dot{\gamma}\operatorname{tg}\beta). \tag{36}$$

From (21) we find

$$2ViS^{(v)} = (\dot{\beta} - i\dot{\alpha}\sin\beta)e^{i\gamma} = (\dot{\beta} - i(2\varepsilon - \dot{\gamma})\operatorname{tg}\beta)e^{i\gamma}, \tag{37}$$

such that

$$\dot{S}^{(v)}(t) = i[2\varepsilon - V(\Omega - v)\cos\beta(t)]S^{(v)}(t). \tag{38}$$

Inserting once more (35) gives

$$\dot{S}^{(\nu)} = i\left[2\varepsilon - V(\Omega-\nu)\sqrt{1-\frac{4|S^{(\nu)}|^2}{V^2(\Omega-\nu)^2}}\right]S^{(\nu)}. \tag{39}$$

This equation is easily solved. For this one observes that it implies $|S^{(\nu)}|^2$ to remain constant such that $S^{(\nu)}$ describes a pure phase rotation:

$$S^{(\nu)}(t) = S_0 e^{i(\omega t-\theta_0)}, \tag{40}$$

$$-\frac{\omega-2\varepsilon}{2V} = \frac{\Omega-\nu}{2}\sqrt{1-\frac{4|S_0|^2}{V^2(\Omega-\nu)^2}}. \tag{41}$$

The positive square root corresponds to orbits with $\beta\in(0,\frac{1}{2}\pi)$ the negative to those with $\beta\in(\frac{1}{2}\pi,\pi)$. Inverting eq. (41) gives:

$$|S_0|^2 = V^2\left[\frac{(\Omega-\nu)^2}{4} - \left(\frac{\omega-2\varepsilon}{2V}\right)^2\right]. \tag{42}$$

Thus frequency and amplitude S_0 are strongly related; for small amplitudes S_0 with $\beta\in(0,\frac{1}{2}\pi)$ we recover the RPA collective frequences

$$\omega = 2\varepsilon - V(\Omega-\nu). \tag{43}$$

The important improvement of the present work concerns the quantum aspects of the non-linear correction in (41).

Let us now calculate the full classical action (17) for such periodic orbits. For the general fluctuating field S the trace of the logarithm is a very complicated non-local functional of S. In realistic situations like superconductors and superliquid ^{3}He it has been studied only for small long-wavelength oscillations of S either around zero (then one obtains Ginzburg-Landau theories) or around some fixed non-zero gap value $S_0 \neq 0$. Only in a few exactly soluble models the result has been obtained in closed form (for example massless and massive Thirring and Schwinger models). Certainly, the model at hand is of the soluble type and we could resort to this solution. The procedure, however, would have little chance of being applicable also to more realistic situations. We shall therefore at first ignore our extended knowledge of the full fluctuating action and confine ourselves to a calculation of the action for a classical periodic field $S(t)$ only. We shall see that this will be enough to find all energy levels in the semiclassical limit. Only at the end we shall compare our results with the complete information about the collective action which has been obtained in previous works. The only assumption we shall make is that we know about the presence of periodic classical orbits of the form

$$S(t) = S_0 e^{i(\omega t-\theta_0)}. \tag{44}$$

In order to calculate the trace of the logarithm we observe that

$$\exp\left(\operatorname{tr}\log\begin{pmatrix} i\partial_t-\varepsilon & S^+(t) \\ S(t) & i\partial_t+\varepsilon \end{pmatrix}\right) = \det\begin{pmatrix} i\partial_t-\varepsilon & S^+(t) \\ S(t) & i\partial_t+\varepsilon \end{pmatrix} \tag{45}$$

is the product of all eigenvalues λ_μ of the system of equations

$$\begin{aligned} (i\partial_t-\varepsilon)\phi^{(1)}(t)+S^+(t)\phi^{(2)}(t) &= \lambda\phi^{(1)}(t), \\ S(t)\phi^{(1)}(t)+(i\partial_t+\varepsilon)\phi^{(2)}(t) &= \lambda\phi^{(2)}(t). \end{aligned} \tag{46}$$

Since the determinant arose from integrating out anticommuting fermion fields, the boundary condition for ϕ is antiperiodicity

$$\phi^{(1,2)}(t) = -\phi^{(1,2)}(t+T). \tag{47}$$

The solutions can all be found by solving the homogeneous system

$$\begin{aligned} (i\partial_t-\varepsilon)g^{(1)}(t)+S^+(t)g^{(2)}(t) &= 0. \\ S(t)g^{(1)}(t)+(i\partial_t+\varepsilon)g^{(2)}(t) &= 0. \end{aligned} \tag{48}$$

Since $S(t)$ is periodic, the solutions $g^{(1)}, g^{(2)}$ have to satisfy Bloch's theorem. According to this they can be written as

$$g^{(1,2)}(t) = e^{+i\kappa t}h^{(1,2)}(t), \tag{49}$$

where κ is some frequency and $h^{(1)}$ has the same periodicity as $S(t)$, i.e.

$$h^{(1,2)}(t) = h^{(1,2)}(t+T). \tag{50}$$

Obviously, κ is defined only module the addition of a "reciprocal lattice vector"

$$\kappa \to \kappa+2\pi m/T, \qquad m = \pm 1, \pm 2, \ldots.$$

One usually fixes the choice by taking κ within the first Brioullin zone

$$-\pi/T < \kappa < \pi/T. \tag{51}$$

Certainly, $\exp(i\kappa T)$ are the representation phases of the translation group introduced by Floquet.

Now, given the solutions $h^{(1,2)}$, it is easy to construct all antiperiodic eigenfunctions of (46). For this we form

$$\phi^{(1,2)}(t) = \exp^{-i(\kappa+2\pi m/T+\pi/T)t}g^{(1,2)}(t). \tag{52}$$

These functions are antiperiodic due to the additional phase $\exp(-i\pi t/T)$. Because of (48) they solve (46) with energies

$$\lambda_m = \kappa+(2m+1)\pi. \tag{53}$$

Thus, if we restrict ourselves to the 2×2 matrix trace of the logarithm we find

$$\exp\left(\operatorname{tr}\log\begin{pmatrix} i\partial_t-\varepsilon & S^+(t) \\ S(t) & i\partial_t+\varepsilon \end{pmatrix}\right) = \prod_\kappa \prod_{m=-\infty}^{\infty} [\kappa+(2m+1)\pi/T] = \text{const} \prod_\kappa 2\cos\tfrac{1}{2}\kappa T \tag{54}$$

where the product has to be taken over all solutions κ. The constant in front can be determined by taking the special case $S(t) = 0$. Then $\kappa = \pm\varepsilon$ and one has

$$\text{const.}\,(2\cos\tfrac{1}{2}\varepsilon T)^2 = \text{const.}(e^{i\varepsilon T/2}+e^{-i\varepsilon T/2})^2, \tag{55}$$

which coincides with the partition function of two free fermions if we fix const $= 1$.

If the trace over all Ω degenerate shell states is included, (54) becomes

$$\exp\left(\text{tr}\log\begin{pmatrix} i\partial_t-\varepsilon & S^+(t) \\ S(t) & i\partial_t+\varepsilon\end{pmatrix}\right) = \prod_\kappa (2\cos\tfrac{1}{2}\kappa T)^{\Omega}. \tag{56}$$

In general, κ will be a functional of the periodic orbit $S(t)$.

Consider now our simple problem (46). Inserting the second equation into the first gives the second-order differential equation

$$\left[-\partial_t^2-\varepsilon^2+i\frac{\dot{S}^+S}{|S|^2}(i\partial_t-\varepsilon)-|S|^2\right]g^{(1)}(t) = 0. \tag{57}$$

According to our periodic ansatz (44) this becomes simply

$$[-\partial_t^2-\varepsilon^2+\omega(i\partial_t-\varepsilon)-|S_0|^2]g^{(1)}(t) = 0. \tag{58}$$

This equation is trivially solved by

$$g^{(1)}(t) = e^{-i\kappa t}h^{(1)} \tag{59}$$

with a constant $h^{(1)}$ and

$$\kappa_{1,2} = \tfrac{1}{2}\omega \pm \sqrt{\tfrac{1}{4}\omega^2+\varepsilon^2-\varepsilon\omega+|S_0|^2}.$$

Since we may subtract ω from κ, the second solution can also be taken as $\kappa_2-\omega$, i.e.

$$\kappa_{1,2} = \pm\kappa = \pm\tfrac{1}{2}[\omega+\sqrt{(\omega-2\varepsilon)^2+4|S_0|^2}. \tag{60}$$

Therefore, the expression (56) becomes

$$\exp\left(\text{tr}\log\begin{pmatrix} i\partial_t-\varepsilon & S^+(t) \\ S(t) & i\partial_t+\varepsilon\end{pmatrix}\right) = (2\cos\tfrac{1}{2}\kappa T)^{2\Omega} = \sum_{\nu=0}\binom{2\Omega}{\nu}e^{i(\Omega-\nu)\kappa T}. \tag{61}$$

Inserting this into (16) and (17), the partition function for a collective periodic motion of frequency ω in $S(t)$ reads

$$Z(\omega) = \sum_{\nu=0}^{2\Omega}\binom{2\Omega}{\nu}\exp^{i[(\Omega-\nu)\kappa-|S_0|^2/V]T}. \tag{62}$$

At this place, the amplitude S_0 is still undetermined. For a classical orbit each of the exponents has to be extremal. By differentiating with respect to $|S_0|^2$ we find the "gap" equation

$$\frac{1}{V} = \frac{\Omega-\nu}{2}\frac{1}{\sqrt{(\frac{1}{2}(\omega-2\varepsilon))^2+|S_0|^2}} \tag{63}$$

or

$$|S_0|^2 = V^2\left[\left(\frac{\Omega-\nu}{2}\right)^2 - \left(\frac{\omega-2\varepsilon}{2V}\right)^2\right]. \tag{64}$$

This is exactly the result (42) which was obtained previously from a general not necessarily semiclassical evaluation of the trace of the logarithm (whose feasibility depend on the solubility of the model). Therefore we can verify that the index ν of the expansion (62) happens to coincide with the seniority of the fermion wave functions. Notice, however, that in the purely classical calculation of the collective action, this connection would escape our knowledge. Inserting (64) into (60) we obtain a simple expression for κ:

$$\kappa = \tfrac{1}{2}[\omega+(\Omega-\nu)V]. \tag{65}$$

Inserting (65) into (61), (17) we find the classical partition function for orbits of frequency ω:

$$Z_{\rm cl}(\omega) = \sum_{\nu=0}^{2\Omega} Z_{\rm cl}^{(\nu)}(\omega) = \sum_{\nu=0}^{2\Omega} \binom{2\Omega}{\nu} \exp\left[iA_{\rm cl}^{(\nu)}(\omega)\right], \tag{66}$$

where $A^{(\nu)}(\omega)$ are the actions of the classical orbits

$$A^{(\nu)}(\omega) \equiv TL^{(\nu)}(\omega), \tag{67}$$

$$= T\left\{\frac{\Omega-\nu}{2}\omega+V\left[\left(\frac{\Omega-\nu}{2}\right)^2+\left(\frac{\omega-2\varepsilon}{2V}\right)^2\right]\right\}. \tag{68}$$

Here we have found it convenient to introduce the curly bracket as $L^{(\nu)}(\omega)$ which is the value of the Lagrangian for the periodic orbit.

This classical partition function can now directly be used for semiclassical quantization.

3. General method for semiclassical quantization

Consider a general theory of a scalar field $\varphi(t)$ with an action $A(\varphi)$. Then the parition function

$$Z = \operatorname{tr} e^{-iHT/\hbar} \tag{69}$$

equals the path integral

$$Z = \int D\varphi(t) e^{iA[\varphi]/\hbar}, \tag{70}$$

where the integration runs over all periodic paths

$$\varphi(0) = \varphi(T). \tag{71}$$

The semiclassical approximation consists in an expansion of the action around classical closed orbits and keeping only the quadratic fluctuations. Let $\varphi_{cl}(t)$ be such an orbit which is periodic in a time T and solves the equation of motion

$$\left.\frac{\delta A(\varphi)}{\delta\varphi(t)}\right|_{\varphi(t)=\varphi_{cl}(t)} = 0. \tag{72}$$

We may then set

$$\varphi(t) = \varphi_{cl}(t)+\delta\varphi(t)$$

and expand up to quadratic order

$$A[\varphi] = A[\varphi_{cl}]+\tfrac{1}{2}\int dt dt' \delta\varphi(t)\left.\left(\frac{\delta^2 A}{\delta\varphi(t)\delta\varphi(t')}\right)\right|_{\varphi=\varphi_{cl}} \delta\varphi(t'), \tag{73}$$

where the linear term is absent due to (72). Now, the classical action does no longer depend on the value of $\varphi(0) = \varphi(T)$ but just on the orbit and therefore on T:

$$A[\varphi_{cl}] = A_{cl}[T]. \tag{74}$$

Therefore we may write for the semiclassical partition function

$$Z_{s.cl.}(T) = \sum_{\substack{\text{periodic}\\ \text{orbits}}} \exp(iA_{cl}[T])\int D\delta\varphi \exp(i\tilde{A}[\delta\varphi]),$$

where $\tilde{A}[\delta\varphi]$ is the quadratic correction in eq. (73). The partition function Z^{cl} coincides exactly with the WKB approximation in quantum mechanics.

The integral over $\delta\varphi$ can be done with the result:

$$Z_{s.cl.} \approx \left(\frac{i}{2\pi\hbar}\right)^{\frac{1}{2}} T\left[\frac{dE_{cl}}{dT}\right]^{\frac{1}{2}} \exp(iA_{cl}[T]/\hbar), \tag{75}$$

where

$$E_{cl} = -\frac{dA_{cl}(T)}{dT} \tag{76}$$

is the classical energy of the periodic orbit.

For the same period T there are also contributions of a fractional fundamental period

$$\tau = T/n, \qquad m = 1,2,3,4,\dots \tag{77}$$

which are traversed n times. Adding all these gives

$$Z_{s.cl.}(T) = \left(\frac{i}{2\pi n}\right)^{\frac{1}{2}} \sum_n \tau\left[\frac{dE_{cl.}(\tau)}{n d\tau}\right]^{\frac{1}{2}} e^{inA_{cl}[\tau]/\hbar}. \tag{78}$$

If the orbit has two turning points there is an additional phase factor $e^{-in\pi}$. The

Fourier transformation of (78) leads to

$$Z_{\mathrm{s.cl.}}(E) = \mathrm{tr}\left(\frac{i}{E-H}\right)\Bigg|_{\mathrm{semicl.}}$$

$$= \left(\frac{i}{2\pi\hbar}\right)^{\frac{1}{2}} \sum_n \int_0^\infty \frac{\mathrm{d}\tau}{\hbar} \sqrt{n\tau} \left|\frac{\mathrm{d}E_{\mathrm{cl}}}{\mathrm{d}\tau}\right|^{\frac{1}{2}} \mathrm{e}^{inA_{\mathrm{cl}}[\tau]/\hbar}. \tag{79}$$

In the stationary phase approximation only such periods τ with

$$\frac{\partial A_{\mathrm{cl}}}{\partial \tau} + E = 0 \tag{80}$$

contribute. Since at that point the exponent is

$$A_{\mathrm{cl}} + E\tau = \frac{1}{2}\frac{\partial^2 A_{\mathrm{cl}}}{\partial \tau^2}(\delta\tau)^2 + \ldots$$

$$= -\frac{1}{2}\frac{\partial E_{\mathrm{cl}}}{\partial \tau}(\delta\tau)^2, \tag{81}$$

one can perform the τ integral to quadratic order and finds

$$Z(E) = \frac{\tau(E)}{\hbar}\sum_{u=1}^{\infty} \mathrm{e}^{inW(E)/\hbar},$$

$$= \frac{\tau(E)}{\hbar}\frac{\mathrm{e}^{iW(E)/\hbar}}{1-\mathrm{e}^{iW(E)/\hbar}}. \tag{82}$$

Here we have introduced the Legendere transform of $A_{\mathrm{cl}}(\tau)$:

$$W(E) = A_{\mathrm{cl}}(\tau(E)) + E\tau(E). \tag{83}$$

The bound state poles are obtained from

$$W(E_n) = 2\pi n\hbar. \tag{84}$$

If the orbit has two turning points the additional phase leads to the alternative quantization rule

$$W(E_n) = 2\pi(n+\tfrac{1}{2})\hbar. \tag{85}$$

In both cases, due to

$$\frac{\partial W(E)}{\partial E} = \tau(E), \tag{86}$$

the pole terms of $G(E)$ are simply

$$G(E) = \sum_n \frac{i}{E-E_n}. \tag{87}$$

In order to illustrate these rules consider the action of a point particle in a one-dimensional potential well such that

$$L = \tfrac{1}{2}m\dot{\varphi}^2 - V(\varphi). \tag{88}$$

Here, function W is simply

$$W(E) = 2\int \mathrm{d}\varphi p(\varphi), \tag{89}$$

where

$$p(\varphi) \equiv \sqrt{2m(E - V(\varphi))} \tag{90}$$

is the classical momentum of the orbit. There are two turning points at which $p = 0$ such that we have to quantize according to (85) which indeed coincides with the well known WKB result of quantum mechanics. Similarly, for a free particle confined to a circle there are no turning points and

$$W = \int p\mathrm{d}x = 2\pi rp \tag{91}$$

is quantized according to (84) such that $rp = n\hbar$, which are the correct quanta of angular momenta.

It is obvious that all these considerations can be extended to a partition function of the form

$$Z = \sum_{\nu} N_{\nu} Z_{\nu} = \sum_{\nu} \int \mathrm{D}\varphi \exp(iA^{(\nu)}/\hbar). \tag{92}$$

In this case one finds periodic orbits for each $A^{(\nu)}$ separately, each giving an independent quantization rule for $W^{(\nu)}(E^{(\nu)})$.

As a mnemonic rule we may remember: Given a classical partition function for orbits of period $\tau = 2\pi/\omega$

$$Z_{\mathrm{cl}}(\tau) = \sum_{\nu} N_{\nu} Z_{\nu} = \sum_{\nu} N_{\nu} \exp[iA^{(\nu)}(\tau)], \tag{93}$$

the semiclassical Fourier transformed partition function is simply

$$Z_{\mathrm{s.cl.}}(E) = \sum_{\nu} N_{\nu} \frac{\tau(E)}{\hbar} \frac{\exp[iW^{(\nu)}(E)/\hbar]}{1 - \exp[iW^{(\nu)}(E)/\hbar]}, \tag{94}$$

which has bound state poles at

$$W^{(\nu)}(E) = 2\pi n\hbar, \tag{95}$$

whith a pole form

$$Z_{\mathrm{s.cl.}}(E) \approx \sum_{\nu} N_{\nu} \frac{i}{E - E_n^{(\nu)}}, \tag{96}$$

such that N_{ν} is the multiplicity of the states of energy $E_n^{(\nu)}$.

4. Semiclassical quantization of the collective action

Consider now our partition function which in the classical limit reads

$$Z_{\mathrm{cl}}(\omega) = \sum_{\nu=0}^{2\Omega} \binom{2\Omega}{\nu} \exp\left[iA^{(\nu)}(\omega)\right] \tag{97}$$

with

$$A^{(\nu)}(\omega) = TL^{(\nu)}(\omega). \tag{98}$$

The classical energies are calculated from

$$E^{(\nu)} = -\frac{\mathrm{d}A^{(\nu)}(\tau)}{\mathrm{d}\tau}. \tag{99}$$

Now, τ and ω are connected via $\tau = 2\pi/\omega$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = -\frac{\omega}{\tau}\frac{\mathrm{d}}{\mathrm{d}\omega}. \tag{100}$$

Therefore

$$E^{(\nu)} = -\frac{1}{\tau}A^{(\nu)}(\omega) + \frac{2\pi}{\tau}\frac{\mathrm{d}}{\mathrm{d}\omega}L^{(\nu)}(\omega). \tag{101}$$

But then we see directly that

$$W^{(\nu)}(E) \equiv E^{(\nu)}\tau + A^{(\nu)} = 2\pi\frac{\mathrm{d}}{\mathrm{d}\omega}L^{(\nu)}(\omega) = 2\pi\left[\frac{\Omega-\nu}{2} + \frac{\omega-2\varepsilon}{2V}\right] \tag{102}$$

and the quantization condition becomes simply

$$\left.\frac{\mathrm{d}L^{(\nu)}(\omega)}{\mathrm{d}\omega}\right|_{\omega=\omega_n} = \frac{\Omega-\nu}{2} + \frac{\omega_n-2\varepsilon}{2V} = n, \qquad n = 0, 1, \ldots, \Omega-\nu. \tag{103}$$

The upper limit on the quantum number n results from eq. (42) according to which the frequencies satisfy

$$\left(\frac{\omega-2\varepsilon}{2V}\right)^2 < \left(\frac{\Omega-\nu}{2}\right)^2. \tag{104}$$

Consider now the energy

$$\begin{aligned} E_n^{(\nu)} &= n\omega_n - L(\omega_n) \\ &= \tfrac{1}{2}(\Omega-\nu)\omega_n + \frac{(\omega_n-2\varepsilon)^2}{2V} + 2\varepsilon\frac{\omega_n-2\varepsilon}{2V} \\ &\quad -\tfrac{1}{2}(\Omega-\nu)\omega_n - V\left[\left(\frac{\Omega-\nu}{2}\right)^2 + \left(\frac{\omega_n-2\varepsilon}{2V}\right)^2\right] \\ &= 2\varepsilon\frac{\omega_n-2\varepsilon}{2V} - V\left[\left(\frac{\Omega-\nu}{2}\right)^2 - \left(\frac{\omega_n-2\varepsilon}{2V}\right)^2\right]. \end{aligned} \tag{105}$$

We see that this spectrum coincides with (7) if we identify the eigenvalues of the third component of quasispin L_3 with

$$L_3 = \frac{\omega_n - 2\varepsilon}{2V} \tag{106}$$

and if we substitute the eigenvalue of

$$L^2 = L(L+1) = \tfrac{1}{2}(\Omega - v)(\tfrac{1}{2}(\Omega - v)+1) \tag{107}$$

by the semiclassical result

$$L^2 = \left(\frac{\Omega - v}{2}\right)^2. \tag{108}$$

Notice that the quantum number n coincides with the number of pairs in the state of seniority v.

5. Comparison with full quantum collective action

Since the model at hand is exactly soluble we may compare the semiclassical results with an exact quantization. First of all, it was observed in ref. [7]) that the collective action of the full fluctuating S field can be expressed in a local form if not S itself but rather the Euler angles of the rotation matrix U are used. From eq. (37) we have

$$2ViS = (\dot{\beta} - i\dot{\alpha}\sin\beta)e^{i\gamma}. \tag{109}$$

For the 2×2 trace of the logarithm one can find [7])

$$-i\,\mathrm{tr}\log\begin{pmatrix} i\partial_t - \varepsilon & S^+(t) \\ S(t) & i\partial_t + \varepsilon \end{pmatrix} = \int_{-T/2}^{T/2} (\dot{\alpha}(t) + \dot{\gamma}(t))\mathrm{d}t, \tag{110}$$

where due to (21), $\dot{\alpha}$ is related to $\dot{\gamma}$ by

$$\dot{\alpha} = -\frac{\dot{\gamma} - 2\varepsilon}{\cos\beta}. \tag{111}$$

Therefore the full collective Lagrangian, now valid for fluctuating fields, reads

$$L^{(v)}(\beta, \gamma, \dot{\beta}, \dot{\gamma}) = -\tfrac{1}{2}(\Omega - v)(\dot{\gamma} - 2\varepsilon)\left(\frac{1}{\cos\beta} - 1\right) + (\Omega - v)\varepsilon - \frac{1}{4V}(\dot{\beta}^2 + (\dot{\gamma} - 2\varepsilon)^2 \,\mathrm{tg}^2\,\beta), \tag{112}$$

where the first two terms are due to the trace of the logarithm and the last term comes from the expression $-|S|^2/V$. The classical equations of motion for the fields β, γ are

$$\begin{aligned} \ddot{\beta} &= (\dot{\gamma} - 2\varepsilon)^2 \frac{\sin\beta}{\cos^3\beta} + (\dot{\gamma} - 2\varepsilon)(\Omega - v)V\frac{\sin\beta}{\cos^2\beta}, \\ &(\dot{\gamma} - 2\varepsilon)\,\mathrm{tg}^2\,\beta + (\Omega - v)V\left(\frac{1}{\cos\beta} - 1\right) = \mathrm{const.} \end{aligned} \tag{113}$$

The classical orbits for S can be reproduced by choosing

$$\beta = \text{const}, \qquad -\frac{\dot{\gamma}-2\varepsilon}{2V} = \tfrac{1}{2}(\Omega-\nu)\cos\beta \tag{114}$$

with $\dot{\gamma} = \omega$. From the equation of motion (35) for S we see

$$\cos\beta = \sqrt{1-\frac{4|S_0|^2}{(\Omega-\nu)^2}}, \tag{115}$$

such that (114) coincides with (41). Inserting this with (114) into the Lagrangian we find

$$L^{(\nu)}(\omega) = \tfrac{1}{2}(\Omega-\nu)((\Omega-\nu)V+\omega)-\frac{|S_0|^2}{V} \tag{116}$$

in agreement with (65). Thus for the classical orbits our general collective Lagrangian does coincide with the purely classical evaluation (67). We complete the picture by giving also the exact quantization of the collective theory. For this we write

$$L^{(\nu)}(\beta,\gamma,\dot{\beta},\dot{\gamma}) = -\frac{1}{4V}(\omega_1^2+\omega_2^2)+\tfrac{1}{2}(\Omega-\nu)(\dot{\alpha}+\dot{\gamma}) \tag{117}$$

with

$$\omega^{\pm} = \omega_1 \pm i\omega_2 = \pm i(\dot{\beta}+\dot{\alpha}\sin\beta)e^{i\gamma}, \tag{118}$$

$$\dot{\alpha} = (2\varepsilon-\dot{\gamma})/\cos\beta. \tag{119}$$

The canonical momenta are

$$\begin{aligned} p_\beta &= -\frac{1}{2V}\dot{\beta} = -i\partial_\beta-\tfrac{1}{2}i\operatorname{ctg}\beta, \\ p_\gamma &= -\frac{1}{2V}(\dot{\gamma}-2\varepsilon)\operatorname{tg}^2\beta+\tfrac{1}{2}(\Omega-\nu)\left(1-\frac{1}{\cos\beta}\right) \\ &= -i\partial_\gamma+\tfrac{1}{2}(\Omega-\nu)\left(1-\frac{1}{\cos\beta}\right) \end{aligned} \tag{120}$$

We now observe [for a derivation see ref. [7])] that

$$\begin{aligned} L^{\pm} &= -\frac{1}{2V}\omega^{\pm} = -\frac{1}{2V}(\pm i\dot{\beta}-(\dot{\gamma}-2\varepsilon)\operatorname{tg}\beta)e^{i\gamma}, \\ L_3 &= -\frac{1}{2V}(\dot{\gamma}-2\varepsilon)\operatorname{tg}^2\beta-\tfrac{1}{2}(\Omega-\nu)\frac{1}{\cos\beta} \end{aligned} \tag{121}$$

satisfy the commutation rules of angular momentum. The Hamiltonian of (117) becomes

$$H^{(\nu)} = p_\beta \dot{\beta} + p_\gamma \dot{\gamma} - L^{(\nu)}$$

$$= 2\varepsilon \left[-\frac{1}{2V}(\dot{\gamma}-2\varepsilon)\,\mathrm{tg}^2\,\beta - \frac{\Omega-\nu}{2}\frac{1}{\cos\beta} \right] - \frac{1}{4V}(\dot{\beta}^2 + (\dot{\gamma}-2\varepsilon)^2\,\mathrm{tg}^2\,\beta), \quad (122)$$

which has precisely the form

$$H = 2\varepsilon L_3 - V(L_1^2 + L_2^2)$$

if we insert

$$L_3 = p_\gamma - \tfrac{1}{2}(\Omega - \nu). \quad (123)$$

The eigenvalues of p_γ coincide with our previous number n of pairs. The wave functions are of course

$$\psi(\beta, \gamma) = \mathrm{e}^{i(n-(\Omega-\nu)/2)\gamma} d^{(\Omega-\nu)/2}_{n-(\Omega+\nu)/2,\,(\Omega-\nu)/2}\,. \quad (124)$$

6. Conclusion

Path integrals can be used to transform the action of a fermion theory to an equivalent action in terms of collective fields. The determination of classical orbits is sufficient to find also the semiclassically quantized energies. There are no problems arising from quantization rules of composite operators as in conventional Hamiltonian approaches.

The special strength of the method presented seems to lie in the quantization of large amplitude oscillations where non-linear effects are important.

It will be interesting to find out whether some of the new, exclusively non-linear effects, found recently in field theories have an analogon in nuclear collective phenomena.

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