FIELD THEORY OF COLLECTIVE EXCITATIONS; I. A SOLUBLE MODEL★

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Collective fields are introduced into fermion Lagrangians and graphical rules for a joint treatment of both types of excitations are derived via Feynman path integrals. For the soluble degenerate-shell model with pairing forces used in nuclear and many-body physics the exact collective action is exhibited.

The interplay of collective versus single-particle modes has recently become the subject of detailed investigations in the framework of soluble models [1]. Graphical rules for dealing with both types of excitations simultaneously were invented reproducing the exactly known results. Further it was clarified in which way the interactions succeed in maintaining Pauli's principle in the face of the initial overcompleteness of the basis.

The purpose of this note is to point out a model independent access to the same rules via path integral methods [2] which can easily be generalized to more complicated systems [3]. Moreover, for simple models, such methods lead to an explicit specification of all interactions.

The Lagrangian under consideration consists of Ω fermions $a_m^\dagger$, $b_m^\dagger$ ($m = 1, ..., \Omega$) with a pairing interaction:

$$\mathcal{L}(t) = \sum_m |a_m^\dagger(t)\,(\mathrm{i}\partial_t - \epsilon)a_m(t) + b_m^\dagger(t)\,(\mathrm{i}\partial_t - \epsilon)b_m(t)|$$

$$+ \sum_{m,m'} a_m^\dagger(t) b_m^\dagger(t) b_{m'}(t) a_{m'}(t)\,. \qquad (1)$$

All fermion Green's functions can be obtained from the functional derivatives with respect to the external sources $\eta(t)$, $\lambda(t)$ of

$$Z[\eta^\dagger, \eta, \lambda^\dagger, \lambda] = \int \mathrm{D}a^\dagger \mathrm{D}a \mathrm{D}b^\dagger \mathrm{D}b$$

$$\times \exp\{\mathrm{i}\int \mathrm{d}t[\mathcal{L}(t) + (\mu^\dagger a + \lambda^\dagger b + \mathrm{h.c.})]\}. \qquad (2)$$

Our goal is the replacement of the action by another one in which pairing modes described by the composite field $\Sigma_m\, a_m^\dagger(t) b_m^\dagger(t)$ occur as *independent* variables.

For this purpose we first introduce a *dependent* field $S(t)$ by rewriting the Lagrangian in the form

$$\widetilde{\mathcal{L}}(t) = \sum_m \{a_m^\dagger(\mathrm{i}\partial_t - \epsilon)a_m + b_m^\dagger(\mathrm{i}\partial_t - \epsilon)b_m$$

$$+ V(a_m^\dagger b_m^\dagger S(t) + S^\dagger(t) b_m a_m)\} - V S^\dagger(t) S(t). \qquad (3)$$

Variation with respect to $S(t)$ displays the desired dependence

$$S^\dagger(t) = \sum_m a_m^\dagger(t) b_m^\dagger(t) \qquad (4)$$

which upon reinsertion into eq. (3) proves the action to be unaltered. As a consequence, the generating functional Z of eq. (2) can be calculated with $\widetilde{\mathcal{L}}(t)$ instead of $\mathcal{L}(t)$ if an additional integration over $\mathrm{D}S^\dagger(t)\mathrm{D}S(t)$ is performed. Introducing two-component notation for the fermions $f_m^\dagger \equiv (a_m^\dagger, b_m)$, their sources $j_m^\dagger \equiv (\eta_m^\dagger, \lambda_m)$, and a matrix

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$$iG^{-1}_{S^\dagger,S}(t,t') \equiv \begin{pmatrix} i\partial_t - \epsilon & VS(t) \\ VS^\dagger(t) & i\partial_t + \epsilon \end{pmatrix} \delta(t,t') \quad (5)$$

this integral takes the form

$$Z[j^\dagger, j] = \int Df^\dagger Df DS^\dagger DS \exp\{i\mathcal{A} - i\int VS^\dagger(t)S(t)dt\} \quad (6)$$

with the action:

$$\mathcal{A} = \int dt dt' [f^\dagger(t) iG^{-1}(t,t') f(t') + (j^\dagger(t) f(t) + \text{h.c.})\, \delta(t,t')] . \quad (7)$$

The integral in the fermion fields $\int Df^\dagger Df$ is Gaussian and can be performed after quadratic completion‡ yielding

$$Z[j^\dagger, j] = \int dS^\dagger dS \exp\{i\mathcal{A}[S^\dagger, S] - \int dt dt' j^\dagger(t)\, G_{S^\dagger S}(t,t')\, j(t')\} \quad (8)$$

where $\mathcal{A}[S^\dagger, S]$ denotes the resulting *collective* action:

$$\mathcal{A}[S^\dagger, S] = \int dt\{-VS^\dagger(t)S(t) - i\,(\mathrm{tr}\log iG^{-1}_{S^\dagger S})(t,t)\}. \quad (9)$$

Varying $\mathcal{A}[S^\dagger, S]$ gives the equation of motion for the collective excitations:

$$S(t) = -\frac{\Omega}{2}\,\mathrm{tr}(\sigma^- G_{S^\dagger S}(t,t'))|_{t'=t-0}; \quad \sigma^- \equiv \sigma^1 - i\sigma^2. \quad (10)$$

By definition (5), $G_{S^\dagger S}$ is the propagator of the fermions in the external field $S(t)$. Expanding in powers of $S(t)$

$$G_{S^\dagger S}(t,t') = G_0(t,t') + iV\int dt'' G_0(t,t'') \begin{pmatrix} 0 & S(t'') \\ S^\dagger(t'') & 0 \end{pmatrix} G_0(t'',t) + \ldots \quad (11)$$

brings the second term in (9) to the form‡‡

$$-i\int dt\,(\mathrm{tr}\log iG^{-1}_{S^\dagger S})(t,t) = -i\int dt\,(\mathrm{tr}\log iG_0^{-1})(t,t) + i\sum_{n=1}^{\infty} V^{2n}\,\mathrm{tr}\,\frac{(-1)^n}{2n}\left[G_0\begin{pmatrix} 0 & S \\ S^\dagger & 0 \end{pmatrix} G_0\begin{pmatrix} 0 & S \\ S^\dagger & 0 \end{pmatrix}\right]^n \quad (12)$$

revealing the $(S^\dagger S)^n$ interaction to be due to one-fermion loops absorbing n collective lines S from the past and emitting the same number into the future in alternative order#. Notice that fermions appear only as external sources. By inserting the expansion (11) also in the source term of (8) we see that an external fermion may emit and absorb successively an arbitrary number of collective modes S, $S^\dagger$ before leaving the interaction zone#. There are *no* fermion loops, all being contained implicitly in the collective action $\mathcal{A}[S^\dagger, S]$.

The graphical rules stated are in complete agreement with those found empirically in ref. [1]. It is obvious that the derivation presented here does not really depend on the specific form of the pairing force but can be generalized to arbitrary long-range forces‡.

Let us now describe the explicit form of $\mathcal{A}[S^\dagger, S]$. First we note that the propagator can be written as

$$G_{S^\dagger S}(t,t') = U(t)\begin{pmatrix} \theta(t-t') & 0 \\ 0 & -\theta(t'-t) \end{pmatrix} U(t')^{-1} \quad (13)$$

if $U(t)$ is a unitary matrix satisfying

$$[i\partial_t U(t)]\, U^{-1}(t) = \epsilon\sigma_3 - V(S_1\sigma^1 - S_2\sigma^2). \quad (14)$$

Parametrizing $U(t)$ in terms of Euler angles $\exp(i\alpha\sigma^3/2)$ $\exp(i\beta\sigma^2/2)$ $\exp(i\gamma\sigma^3/2)$ brings (14) to three complex differential equations*

$$\begin{aligned} -\dot\beta \sin\alpha + \dot\gamma \sin\beta\cos\alpha &= -2VS_1(t) \\ \dot\beta\cos\alpha + \dot\gamma\sin\beta\sin\alpha &= -2VS_2(t) \\ \dot\alpha + \dot\gamma\cos\beta &= -2\epsilon. \end{aligned} \quad (15)$$

Eliminating the third equation and introducing a field

‡ And dropping an irrelevant constant energy $-\epsilon\Omega$.

‡‡ Multiplication of the matrices $G_0(t,t')$ is understood also in the functional sense, i.e. intermediate times have to be integrated over.

The coupling strength is V.

‡ See ref. [3] for such generalizations in other contexts.

* The solution of eq. (15) corresponds to the kinematic problem of finding the positions of a rigid body given its angular velocities.

$$\phi(t) \equiv \sin\beta e^{i\alpha} \tag{16}$$

eq. (15) becomes:

$$(\dot\phi + 2i\epsilon\phi)/\sqrt{1-\phi^\dagger\phi} = 2Vi S(t). \tag{17}$$

Suppose a solution $\phi_S(t)$ were known then the equation of motion (10) would read

$$S(t) = \frac{\Omega}{2}\phi_S(t). \tag{18}$$

Differentiating these and inserting (17) yields

$$\dot S(t) = -2i\epsilon S + V\Omega i\sqrt{1 - \frac{4}{\Omega^2}S^\dagger S}\, S. \tag{19}$$

The action $\mathscr{A}[S, S^\dagger]$ is highly non-local in S. This can be illustrated by solving eq. (17) for small S:

$$\phi_S(t) = 2Vi\int_{-\infty}^{t} e^{-2i\epsilon(t-t')}S(t')dt' + ... \tag{20}$$

such that the lowest order part of the action $\mathscr{A}_2[S, S^\dagger]$, quadratic in the fields $S^\dagger$, S, becomes:

$$\mathscr{A}_2[SS^\dagger] = \int dt dt' \{-VS^\dagger(t)S(t)\delta(t-t')$$
$$+ iV^2\Omega S^\dagger(t)S(t')e^{-2i\epsilon(t-t')}\theta(t-t')\}.$$

Functional integration of this part in Z renders the bare propagator for a perturbation theory in the collective fields S:

$$\langle 0|T(S(t)S^\dagger(t'))|0\rangle$$

$$= -\int\frac{dE}{2\pi} i\frac{1}{V + V^2\Omega(E-2\epsilon)^{-1}}e^{-iE(t-t')}. \tag{21}$$

This might be split as

$$= -\frac{i}{V}\delta(t-t') + \Omega\int\frac{dE}{2\pi}\frac{i}{E-2\epsilon+\Omega V}e^{-iE(t-t')}. \tag{22}$$

Thus, in addition to a standard propagator of energy $2\epsilon - \Omega V$, there appears a contact term. This agrees exactly with the empirical results of ref. [1].

While the full action is non-local in $S^\dagger$, S it has a very simple form as a functional of $\phi_S(t)$. Using $\delta S(t)/\delta\phi(t')$ and $\delta S(t)/\delta\phi^\dagger(t')$ from (17) and reversing the functional differentiation leading to (18) one obtains the explicit result:

$$\mathscr{A}[\phi^\dagger, \phi] = \int dt\left\{-\frac{1}{4V}\frac{|\dot\phi + 2i\epsilon\phi|^2}{1-\phi^\dagger\phi}\right.$$
$$\left. - i\frac{\Omega}{4}[(\dot\phi + 2i\epsilon\phi)\phi^\dagger - \text{h.c.}]\frac{\sqrt{1-\phi^\dagger\phi}^{\,-1} - 1}{\phi^\dagger\phi}\right\}. \tag{23}$$

The action $\mathscr{A}[S, S^\dagger]$ can now be obtained by solving eq. (17) successively for higher powers in S, $S^\dagger$ and inserting the result in eq. (23).

Observe that the static field configuration, $\dot\phi(t) \equiv 0$, proves to be stable only if $v \equiv V\Omega/2\epsilon < 1$, i.e. for sufficiently small pairing force. Only then does the expansion in powers of S converge justifying the graphical rules stated above. For stronger pairing force the static action density

$$-\Omega\epsilon\left(\frac{1}{v}\frac{|\phi|^2}{2(1-|\phi|^2)} - \sqrt{1-|\phi|^2}^{\,-1}\right) - \Omega\epsilon \tag{24}$$

develops a new minimum at $|\phi_0|^2 = 1 - v^{-2}$. Close to this minimum, the bare quanta consist of zero-energy azimuth motions ("would-be Goldstone bosons") and radial oscillations of energy* $2\epsilon(v^2-1)$. The corresponding rules can be obtained by expanding $G_{S^\dagger,S}$, in eq. (11), or directly the final action (23) around this minimum. Certainly, the vacuum expectation value $|S_0| = \frac{1}{2}\Omega|\phi_0| = \frac{1}{2}\Omega\sqrt{1-v^{-2}}$ can also be obtained by solving (10) for constant S_0 giving $S_0 = \frac{1}{2}S_0 V\Omega/\sqrt{\epsilon^2 + V^2|S_0|^2}$ which is recognized as the standard gap equation.

A further discussion of the collective Lagrangian will be presented elsewhere.

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* This can also be read off eq. (19).

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