QUARK MASSES☆

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In quark gluon theory with very small bare masses, $-\bar{\psi}\mathcal{M}\psi$, spontaneous breakdown of chiral symmetry generates sizable masses M_u, M_d, M_s, ... We find $(M_u + M_d)/2 \approx m_\rho/\sqrt{6} \approx 312$ MeV, and $M_s \approx 432$ MeV. Scalar densities have well determined non-zero vaccum expectations $\langle 0|u^a|0\rangle \equiv \langle 0|\bar{\psi}(x)\,(\lambda^a/2)\psi(x)|0\rangle \approx f_\pi^2 M^a$, i.e. $\langle 0|u^0|0\rangle \approx 8 \times 10^{-3}$ (GeV)^3 at an SU(3) breaking of the vacuum $c' \equiv \langle 0|u^8|0\rangle/\langle 0|u^0|0\rangle \approx -16\%$.

A quark gluon Lagrangian of the form

$$\mathcal{L} = \bar{\psi}\mathrm{i}(\partial\!\!\!/ - \mathrm{i}gG\!\!\!/)\psi - \tfrac{1}{4}G_{\mu\nu}^2 - \bar{\psi}\mathcal{M}\psi \qquad (1)$$

with

$$G_{\mu\nu} \equiv \partial_\mu G_\nu - \partial_\nu G_\mu - \mathrm{i}g[G_\mu G_\nu]$$

and an appropriate choice of internal degrees of freedom is presently the favorite candidate for a possible theory of strong‡1 interactions [1]. The non-abelian colored character of the gluon field makes quark almost free at small and hopefully confined at large distances.

It is the purpose of this note to study the implications of such a quark theory on the low energy properties of the basic mesons π, ρ, σ, A_1, and their SU(3) partners. As far as the interquark potential is concerned, such physics takes place at intermediate distances. Here color aspects should be rather irrelevant. For color supposedly makes sure there exist many resonances high up in mass. But low energy dispersion relations for π, π; π, ρ etc. scattering are known to receive little contribution from such resonances. Otherwise current algebra would not saturate at the observed precocity. In addition meson bound states do not need colored quarks for reasons of statistics‡2.

Thus for our purpose we can approximate the original Lagrangian $\mathcal{L}$ by another one, $\mathcal{L}'$, which neglects color and contains *only* the basic mesons as bound states. This is achieved by using singlet gluons of a very large mass‡3 μ. Calculations become extremely simple by performing a large μ expansion and keeping only the leading term everywhere. In this way, a Lagrangian for low energy meson physics can be *derived* which turns out to be of the standard SU(3) × SU(3) type. The procedure of turning a quark theory into a theory of its hadronic bound states might be called "hadronization".

In the meson Lagrangian obtained via hadronization many previously free parameters are related to quark properties. Moreover, the derivation displays an intimate relation between meson fields and the infinite sea of quark pairs inside hadrons [2].

First we notice that, in analogy to the original Nambu argument, the Lagrangian $\mathcal{L}'$ displays a spontaneous breakdown of chiral symmetry [4, 5]. We rewrite

$$\mathcal{L}' = \bar{\psi}\mathrm{i}\partial\!\!\!/\psi - \bar{\psi}M\psi + \mathcal{L}_\mathrm{i}' \qquad (2)$$

with

$$\mathcal{L}_\mathrm{i}' \equiv g\bar{\psi}G\!\!\!/\psi - \tfrac{1}{4}G_{\mu\nu}^2 + \tfrac{1}{2}\mu^2 G_\mu^2 + \bar{\psi}M\psi - \bar{\psi}\mathcal{M}\psi \qquad (3)$$

and determine a best value of M by making the quark self energy correction due to $\mathcal{L}_\mathrm{i}'$ vanish. The result is a "gap equation"‡4:

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‡1 With large spontaneous breakdown of some of the symmetries it would also describe weak and electromagnetic interactions [1].

‡2 Only the decay $\pi^0 \to \gamma\gamma$ needs color if estimated via *short* distance arguments. The correct result on intermediate distances is obtained by using $g_{\omega\rho\pi}$ and vector meson dominance.

‡3 In the original theory this mass is supposed to be generated spontaneous by gluon self interactions.

‡4 Working to leading order in μ we can define the mass at zero momentum.

$$M - \mathcal{M} = \mathrm{i}g^2 \int \frac{d^4p}{(2\pi)^4} \gamma^\nu \frac{\mathrm{i}}{\not{p} - M} \gamma_\nu \frac{\mathrm{i}}{p^2 - \mu^2} \equiv 4g^2 MQ \tag{4}$$

with a logarithmically divergent factor $Q \approx (4\pi)^{-2} \log \Lambda^2/\mu^2$.

Due to M there will be a splitting among the quark masses, say $M = M_0 + \delta M$, with $M_0 = 4g^2 M_0 Q$ and

$$\mathcal{M} = 8g^2 M_0^2 \delta M(1 + 3\delta M/2M_0) L_0/\mu^2; \qquad L_0 \approx (4\pi)^{-2} \log \mu^2/M_0^2. \tag{5}$$

Notice that, at this point, M_0 is completely arbitrary, since it is related to the irrelevant[‡5] cutoff Λ. Only by studying the properties of bound states, M_0 can be determined in terms of observable quantities (namely m_ρ^2).

The bound state problem can be solved (to leading order in μ) via Bethe Salpeter equations. As a result one finds an almost massless pseudoscalar meson, reflecting the spontaneous breakdown of chiral symmetry, as well as vector, axial-vector and scalar mesons. In the ladder approximation one can also derive the coupling strengths among these mesons. There is a very economic alternative way of deriving and phrasing the same results. It may be imitated from the theory of type II superconductors [7,8,2]. For this one allows in $\mathcal{L}'$ for an x dependent term $\bar{\psi}(x)M(x)\psi(x)$ and minimizes again the quark selfenergy in $\mathcal{L}_\mathrm{i}'$.

The field

$$M(x) \equiv \hat{\sigma}(x) + \hat{\pi}(x)\,\mathrm{i}\gamma_5 + \hat{V}_\mu(x)\gamma^\mu + \hat{A}_\mu(x)\gamma^\mu\gamma_5$$

can be thought of describing coherent gap waves in the superconductive world. In their presence, a quark propagates according to

$$(\mathrm{i}\not{\partial}_x - M(x))S_{M(x)}(x,y) = \mathrm{i}\,\delta^{(4)}(x-y) \tag{6}$$

and eq. (4) turns into a "gap wave equation"[‡6]

$$M(x) - \mathcal{M} = \mathrm{i}g^2 \int \mathrm{d}z\, \gamma^\nu S_{M(x)}(x + z/2, x - z/2)\gamma_\nu D(\mu; z). \tag{7}$$

By expanding $S_{M(x)}$ in powers of $m(x) \equiv M(x) - M_0$ according to

$$S_{M(x)} = S_{M_0} - \mathrm{i}\, S_{M_0} m(x) S_{M_0} + \ldots$$

one finds eqs. of motion for $m(x)$. They are summarized in the hadron Lagrangian [6]

$$\begin{aligned}\mathcal{L}_\mathrm{h} = \mathrm{Tr}\{&[(D_\mu\sigma)^2 + (D_\mu\pi)^2] + 2M_0^2(\sigma^2 + \pi^2) \\ &- \tfrac{2}{3}\gamma^2[(\sigma - \mathrm{i}\pi)(\sigma + \mathrm{i}\pi)]^2 - \tfrac{1}{2}(F_{\mu\nu}^\mathrm{V})^2 - \tfrac{1}{2}(F_{\mu\nu}^\mathrm{A})^2 \\ &+ m_0^2(V_\mu^2 + A_\mu^2) - \sqrt{\tfrac{3}{2}}\frac{\mu^2}{g^2 L_0 \gamma}\mathcal{M}\sigma\}.\end{aligned} \tag{8}$$

It can be shown that in quantized form, the Lagrangian $\mathcal{L}_\mathrm{h}$ is the *exact* "hadronization" of the original quark theory. For this one has to introduce a dependent field $M(x)$ in the generating functional of the theory and integrate out the quark fields.

The discussion of this Lagrangian is familiar[‡7]. The spontaneous breakdown of chiral symmetry makes the σ field vibrate around the potential minimum at $\sigma_0 = -M_0/\sqrt{\tfrac{2}{3}}\gamma$. At the tree level, there are light pseudoscalar mesons with

$$m_\pi^2 = 4M_0 \delta M(1 + SM/2M_0)Z_\pi \tag{9}$$

and heavy scalar mesons of mass

$$m_\sigma^2 = 4M_0^2(1 + 3\delta M/M_0 + 3(\delta M)^2/2M_0^2). \tag{10}$$

Here Z_π is a normalization constant with little SU(3) splittings (for pions $Z_\pi = m_{A_1}^2/m_\rho^2 \approx 2$).

It is induced by the $A^\mu \partial_\mu \pi$ mixing term which also splits A_1 and ρ mass by $m_{A_1}^2 = m_\rho^2 + (3/2)(M_\mathrm{u} + M_\mathrm{d})^2$. Inserting the experimental fact $m_{A_1}^2 \approx 2m_\rho^2$ we find the non-strange quark masses

$$(M_\mathrm{u} + M_\mathrm{d})/2 \approx m_\rho/\sqrt{6} \approx 312\ \mathrm{MeV}, \tag{11}$$

which is in good agreement with other estimates.

Together with $m_\pi \approx 135$ MeV and (9), this gives

$$(\delta M_\mathrm{u} + \delta M_\mathrm{d})/2 \approx 7\ \mathrm{MeV}. \tag{12}$$

The mass of the non-strange scalar excitation becomes $m_\sigma \approx 2M_0 \approx 600$ MeV as observed experimentally. The vector mass itself is given by $m_\rho^2 \approx 3\mu^2/2g^2L_0$. It can be inserted into eq. (5), together with (11), yielding

‡5 In loopwise summation [3,5,6], Q_0 would, in fact, be cut off independent and $Q_0 = 1/4g^2$.

‡6 This equation is split into Lorentz invariants by using, on the r.h.s., the Fierz identity $\gamma^\nu \times \gamma_\nu \to 1 \times 1 + \mathrm{i}\gamma_5 \times \mathrm{i}\gamma_5 - \frac{1}{2}\gamma^\nu \times \gamma_\nu - \frac{1}{2}\gamma^\nu\gamma_5 \times \gamma_\nu\gamma_5$. For $\mu \to \infty$ the propagator $D(\mu; z)$ approaches a δ-function: $\mathrm{i}\int d^4k \exp(-kz)/(2\pi)^4 (k^2 - \mu^2) \to -\mathrm{i}\delta^{(4)}(z)/\mu^2$.

‡7 As usual $D_\mu\sigma \equiv \partial_\mu\sigma - 2\gamma[V_\mu\sigma] - \gamma\{A_\mu\pi\}$; $D_\mu\pi \equiv \partial_\mu\pi - 2\gamma[V_\mu\pi] + \gamma\{A_\mu\sigma\}$. For a review see ref. [9]. We also have changed the field normalization $(\hat{\sigma}, \hat{\pi}) \to -\sqrt{2/3}\gamma(\sigma, \pi)$; $(\hat{V}, \hat{A}) \to -\gamma(V, A)$. Notice that, by construction, all tree diagrams in $\mathcal{L}_\mathrm{h}$ necessarily preserve Zweig's rule while loop corrections bring violations.

$$\mathcal{M} \approx 2\delta M(1 + 3\delta M/2M_0). \quad (13)$$

Using (12) we find extremely light non-strange bare masses:

$$(\mathcal{M}_u + \mathcal{M}_d)/2 \approx 14 \text{ MeV}. \quad (14)$$

The value for $\mathcal{M}_s$ can be estimated from the SU(3) splitting among pseudoscalar masses. The standard discussion [9, 10] gives $c \equiv \mathcal{M}^8/\mathcal{M}^0 \approx -1.29$ such that $2\mathcal{M}_s/(\mathcal{M}_u + \mathcal{M}_d) \approx 31$ and hence $\mathcal{M}_s \approx 435$ MeV, $M_s \approx 432$ MeV. Notice that our values of the bare masses $\mathcal{M}$ are larger than others [11] obtained from a discussion of SU(6) wave functions.

The pseudoscalar decay constants are determined by the symmetry breaking term in $\mathcal{L}_h$ ‡8:

$$f_\pi = \sqrt{3/2}\, 2M/\gamma\sqrt{Z_\pi}. \quad (15)$$

Inserting eq. (11) reveals the KSFR relation ($f_\pi \approx m_\rho/\gamma\sqrt{2}$).

The absolute value of γ can be determined either from the Bethe–Salpeter wave function or by considering the vector projection of (7) for $\mu \to \infty$:

$$\hat{V}^\mu(x) \approx \frac{1}{2}\frac{g^2}{\mu^2} \langle 0|\bar{\psi}(x)\gamma^\mu\psi(x)|0\rangle. \quad (16)$$

The isovector current is $\langle 0|\bar{\psi}\gamma_\mu(\lambda^a/2)\psi|0\rangle = -2(\mu^2\gamma/g^2)V^a_\mu$ which fixes γ via the conventional definition $-(m_\rho^2/\gamma)V^a_\mu$ as‡9:

$$\gamma = m_\rho g/\sqrt{2}\mu = \sqrt{3/2L_0}. \quad (17)$$

A similar consideration determines the absolute magnitude of the vacuum expectations of scalar densities. From the scalar projection of (7)

$$\hat{\sigma}(x) \approx -\frac{g^2}{\mu^2} \langle 0|\bar{\psi}(x)\psi(x)|0\rangle, \quad (18)$$

we find

$$\langle 0|u^a|0\rangle \equiv \langle 0|\bar{\psi}(\lambda^a/2)\psi|0\rangle \approx \frac{1}{2}\frac{\mu^2}{g^2}\,\mathrm{Tr}(\lambda^a M) \approx f_\pi^2 M^a \quad (19)$$

such that

$$\langle 0|u^0|0\rangle \approx f_\pi^2\sqrt{2/3}(M_u + M_d + M_s) \approx 8 \times 10^{-3}\ \text{GeV}^3. \quad (20)$$

The SU(3) breaking of the vacuum becomes

$$c' \equiv \langle u^8\rangle/\langle u^0\rangle \approx M^8/M^0 = (M_u + M_d - 2M_s)/\sqrt{2}(M_u + M_d + M_s) \approx -16\%. \quad (21)$$

It goes without saying that our model satisfies all the algebraic relations following from the $(3, \bar{3})$ character of symmetry breaking [10].

The couplings of the hadron fields to external quarks are obtained similarly. One writes

$$\hat{V}^\mu \approx \frac{1}{2}\frac{g^2}{\mu^2}(\langle 0|\bar{\psi}\gamma^\mu\psi|0\rangle + :\bar{\psi}\gamma^\mu\psi:) \quad (22)$$

etc. for π, σ, A_1 and finds

$$g_{VQQ} = g_{AQQ} = \gamma, \quad g_{\pi QQ}Z_\pi^{-1/2} = g_{\sigma QQ} = 1/2\sqrt{L_0}, \quad (23)$$

in accordance with vector meson dominance and the Goldberger Treiman relation $f_\pi g_{\pi QQ} = (M_u + M_d)/2$. With our quark mass and $Z_\pi \sim 2$ this gives numerically

$$g^2_{VQQ}/4\pi = g^2_{AQQ}/4\pi \approx 2.6,$$
$$\tfrac{1}{2}g^2_{\pi QQ}/4\pi = g^2_{\sigma QQ}/4\pi \approx 0.43 \qquad (\approx g^2_{\pi NN}/4\pi \times \frac{1}{34}\ !). \quad (24)$$

Note that π and σ couplings to quarks are comfortably small.

Inside and in the neighbourhood of any hadron wave function $|h\rangle$ composed of three quarks or quark-antiquark there will be meson fields which can be calculated from these couplings. These fields determine the admixture of coherent pair states by applying to $|h\rangle$ the mixing operator [2]

$$U = T\exp\left(-\mathrm{i}\int_{-\infty}^{0} \mathrm{d}x\,\bar{\psi}(x)m(x)\psi(x)\right). \quad (25)$$

In terms of the properly renormalized fields, $m(x)$ can be written as

$$m(x) = -2g_{\sigma QQ}\sigma' - 2g_{\pi QQ}Z_\pi^{-1/2}\pi - g_{VQQ}V^\mu\gamma_\mu - g_{AQQ}A^\mu\gamma_\mu\gamma_5 \quad (26)$$

Notice that the pion field having the smallest mass contributes most for small q^2 which in the structure function of the hadron $|h\rangle$ corresponds to the region $\xi \equiv -q^2/2pq \approx 0$ [2]. Strange (and even more, charmed)

‡8 By definition of f_π, the last term in (8) is: $-f_\pi m_\pi^2 Z_\pi^{-1/2}\sigma$. Inserting (9) and (5) gives (15).

‡9 Hence $f_\pi = (M_u + M_d)\sqrt{L_0/Z_\pi}$. Also this is found directly in a Bethe Salpeter wave function. Notice that $L_0 \approx 0.046$ such that $\log \mu^2/M_0^2 \approx 7.35$ and $\mu \approx 12.5$ GeV. Hence our approximation of $\mu \gg M_0$ is self-consistent.

quarks, on the other hand, spread out evenly in q^2 and should not be peaked so strongly at small ξ.

It is hoped that a similar "hadronization" of the Lagrangian (1) will render a dual theory of strongly interacting particles.

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