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HARD MESON CALCULATION OF $\sigma\pi\pi$, $A\sigma\pi$ AND σAA VERTICES

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Abstract: The low-energy theorems on the $\theta A_\mu A_\nu$ vertex following from current algebra and from the dimensional properties of the various components of the axial vector current A_μ and of its divergence $\partial^\mu A_\mu$ are extended to finite momenta assuming σ-dominace of the trace of the energy momentum tensor θ and pion and A_1 dominance of A_μ. Smooth vertex functions are determined for the $\sigma\pi\pi$, $\sigma A\pi$ and σAA couplings. The role of Schwinger terms in the extrapolation from the low-energy points is discussed.

1. INTRODUCTION

In addition to the well-known current algebra commutators of vector and current densities, another set of commutators has recently been of considerable interest [1-5].

In fact, when one attempts to impose upon observable local fields an operator condition which serves to fix the asymptotic behaviour of their renormalized propagators, one finds that this can be done most simply by introducing the current of infinitesimal dilatations

$$D_\mu(x) = x^\nu \theta_{\mu\nu}(x) , \tag{1.1}$$

where $\theta_{\mu\nu}(x)$ is the local, finite energy momentum tensor ‡, and by postulating that under the action of $D_0(x)$ the local field of interest, $\phi(x)$, transforms according to

$$i[D_0(x), \phi(y)]_{x_0=y_0} = (x^\mu \partial_\mu + d)\, \phi(x)\, \delta^3(\boldsymbol{x}-\boldsymbol{y}) + \mathrm{ST} . \tag{1.2}$$

Then the field $\phi(x)$ is said to have dimensions d, and it is possible to argue that asymptotically the ϕ-propagator, $\Delta(q)$ will behave as $(q^2)^{d-2}$ for large spacelike q^2 ‡‡. In realistic field theories, where the Lagrangian is not

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‡ In a field theory, the improved energy momentum tensor has to be constructed according to the guide lines of ref. [6].
‡‡ For this argument, see sect. II-3 of ref. [5].

invariant under (1.2), (due for example to the presence of mass terms) the dimension as defined in (1.2) is only an approximate concept. In general there will appear additional logarithmic singularities on the right-hand side of (1.2) which manifest themselves in powers of $(\log q^2)$ in the asymptotic behaviour of $\Delta(q)$. However, deep inelastic scattering of electrons on protons mildly suggests that the dimensions of all components of the electromagnetic currents are equal to three, and no logarithmic terms have been resolved [7]. Guided by this observation, one conjectures that also other observable local operators may have definite dimensions.

For any such operator, the commutator (1.2) leads to a low-energy theorem (LET) connecting the matrix element of an $(n+1)$ point function containing the divergence

$$\partial^{\mu} D_{\mu}(x) = \theta(x) \,, \tag{1.3}$$

at zero four-momentum q, with an n-point function not containing $D_{\mu}(x)$. Such LET's can be exploited physically if one assumes $\theta(x)$ to contain a low-lying resonance * whose pole in the variable q^2 dominates the whole region**. In fact, one usually assumes that the low-energy behaviour of θ is dominated by the $J^P = 0^+$, $I = 0$ particle $\sigma(700)$ of width $\Gamma \approx 400\,\mathrm{MeV}$. Then the LET yields conclusions on the ratios of σ couplings to the other fields appearing.

Unfortunately if one assumes unsubtraced form factors for θ, this procedure leads to incorrect results for the simplest couplings which may be tested experimentally. One obtains from the LET***

$$g_{\sigma\pi\pi}/g_{\sigma\mathrm{NN}} = 2m_{\pi}^2/m_{\sigma}\, m_{\mathrm{N}} \,. \tag{1.4}$$

Experimentally, one fields from the mass and width given above

$$g_{\sigma\pi\pi} \approx 4.6 \approx m_{\sigma}/\sqrt{2} f_{\pi} \,, \tag{1.5}$$

while we obtain $g_{\sigma\mathrm{NN}} \approx 15 \approx g_{\pi\mathrm{NN}}$ from the rough estimate

$$g_{\sigma\mathrm{NN}}\, g_{\sigma\pi\pi} \approx 69 \pm 4 \,, \tag{1.6}$$

indicated by backward πN scattering [8].

Thus, one has to conclude that the low-energy region up to the first pole cannot be described by a pole term alone, but that the higher-lying resonances extend with a considerable piece of the amplitude down to low-energy region. We shall call this piece of amplitude the tail of the higher resonances. It is the philosophy of the so-called hard-meson approach that,

* Or a few resonances.

** In the sense of an unsubtracted dispersion relation.

*** Where the couplings are defined by the effective Lagrangian

$$\mathcal{L} = \tfrac{1}{2} g_{\sigma\pi\pi}\, m_{\sigma}\, \sigma\pi^2 + g_{\sigma\mathrm{NN}}\, \sigma\bar{\mathrm{N}}\mathrm{N} \,,$$

such that

$$\Gamma_{\sigma\to\pi\pi} = \frac{3}{4}\frac{g_{\sigma\pi\pi}^2}{4\pi}\, p \,.$$

below the first pole, this tail can be very well approximated by a constant, or at most a few powers of q^2 to account for its curvature [9]. If the higher singularities do not lie too close to the first pole, such an approximation is indeed quite good.

Notice, however, that such a parametrization of the low-energy region does not imply that the amplitude itself necessarily obeys a once or more subtracted dispersion relation in q^2. In the hard-meson method, Ward identities should not be expected to supply information at energies beyond that of the pole which is used for the parametrization ‡. The drawback in the introduction of such additional polynomials lies in the fact that new constants are introduced which must be determined by additional assumptions. These assumptions are usually inferred by generalizing the experience gained in dealing with field theoretic models.

The particular model which suggested all of the general assumptions to be described below is the simple σ-model [11]. This model yields the amazingly good values (considering its simplicity)

$$g_{\sigma\pi\pi} = \frac{m_\sigma}{f_\pi}\left(1 - \frac{m_\pi^2}{m_\sigma^2}\right) , \tag{1.7}$$

$$g_{\sigma \mathrm{NN}} = g_{\pi \mathrm{NN}} = m_{\mathrm{N}}/f_\pi , \tag{1.8}$$

which is definitely an improvement on (1.4). Indeed, in the σ-model, although in the tree approximation, the vertex θNN is unsubtracted ($q = p' - p$)

$$\langle \mathrm{N}(p')|\theta(0)|\mathrm{N}(p)\rangle = \frac{m_\sigma^2 f_\pi g_{\sigma\mathrm{NN}}}{m_\sigma^2 - q^2}\bar{u}(p')u(p) , \tag{1.9}$$

the $\theta\pi\pi$ vertex is once subtracted in q^2 ‡‡

$$\langle\pi(p')|\theta(0)|\pi(p)\rangle = 3m_\pi^2 - m_\sigma^2 + \frac{m_\sigma^3 f_\pi g_{\sigma\pi\pi}}{m_\sigma^2 - q^2} . \tag{1.10}$$

Guided by instructive models, such as the σ-model and the gluon model, the energy density of hadrons has been conjectured to have the following simple properties with respect to the chiral group of vector and axial vector charges of electromagnetic and weak interactions as well as with respect to dilatations [2, 12].

Only Lorentz scalar local operators break either symmetry. Presently one prefers to write

$$\theta_{00} = \overset{*}{\theta}_{00} + \delta + u , \tag{1.11}$$

‡ If one does not make use of additional information like unitarity. See ref. [10].
‡‡ Indeed in sect. 4, we will see that for a high-mass σ ($m_\sigma^2 > 3m_\pi^2$) the low-energy theorems on the $\theta\pi\pi$ vertex force "a subtraction" in this vertex.

where the complicated part (in the Lorentz sense) θ^*_{00} containing kinetic terms, etc., behaves trivially under the chiral and dilatation groups ‡, while the Lorentz scalars δ and u break dilatation symmetry by having dimensions $d_\delta \neq 4$ and $d \neq 4$ respectively, and u also breaks SU(3) × SU(3)‡‡. At the present time, there is still hope that θ_{00} may need only a c-number δ-term ($d_\delta = 0$), and that such a simplicity assumption will not produce conflict with experiment [13].

For our purposes, the full extent of these popular assumptions on θ_{00} will not be needed. We shall only make the following, more general hypotheses.

(i) The SU(3) × SU(3) symmetry breaking term $u(x)$ in θ_{00} is a Lorentz scalar of dimension d. Then, one can prove [14] immediately that ‡‡‡ †

$$i[Q_5(x_0),\ \theta_{00}(x)] = -\partial^\mu A_\mu(x)\ , \tag{1.12}$$

and hence that

$$i[\varphi_5(x_0),\ u(x)] = -\partial^\mu A_\mu(x)\ . \tag{1.13}$$

(ii) The time and also the space components of the vector and axial vector currents have definite dimensions d^V_t, d^V_s, d^A_t, $d^A_s \equiv d_s$ respectively.

The dimensions d^V_s and d_s are in principle arbitrary. However, as mentioned above the fact that the present data on the ratios of transverse to longitudinal cross sections in deep inelastic electron scattering is consistent with quark model commutators [7], and secondly the very phenomenon of scaling in deep inelastic lepton scattering and its explanation via operator product expansions [12] suggest $d^V_s = d_s = 3$.

The dimensions d^V_t and d^A_t of the time components are, however, fixed by current algebra to be equal to 3. Then from (1.13) we can see that $\partial^\mu A_\mu$ also has a fixed dimension which is equal to the dimension of the symmetry breaking term u.

In sect. 2, assumptions (i) and (ii) will be used to prove

$$i[A_0(x),\ \theta(y)] = (d-4)\,\partial^\mu A_\mu(x)\,\delta^3(\boldsymbol{x}-\boldsymbol{y}) + \mathrm{ST}\ , \tag{1.14}$$

where the first derivative Schwinger terms will be well determined.

We shall make no assumptions on the chiral content of $u(x)$. Defining the

‡ I.e., it is a U(3) × U(3) singlet of dimension four.

‡‡ Hopefully transforming like a simple tensor operator, perhaps like $(3,\overline{3})$, $(\overline{3},3)$ (ref. [14]).

‡‡‡ If all terms in θ_{00} breaking dilatation symmetry were Lorentz scalars $\Sigma_n \omega_n$ of definite dimension $d_n \neq 4$, one would find the analogue of (1.13)

$$i[D(x_0),\theta_{00}(x)] = (x\partial + 4)\,\theta_{00}(x) - \partial^\mu D_\mu(x)\ ,$$

which leads to the "virial theorem" [2]

$$\theta = \sum_n (4-d_n)\,\omega_n\ .$$

However, we shall not use this equation.

† Note in this paper we will always omit isospin indices in places where they are obvious. And all commutators will be taken at equal times.

sigma commutator ‡

$$i[A_0^a(x),\ \partial^\mu A_\mu^b(y)] \equiv \Sigma^{ab}(x,\ y)\ , \tag{1.15}$$

we shall, however, make the additional assumption:

(iii) Schwinger terms are absent in this Σ-commutator, i.e., (no sum over a)

$$\Sigma^{aa}(x,y) = \Sigma(x)\,\delta^3(\boldsymbol{x}-\boldsymbol{y})\ . \tag{1.16}$$

Notice first that as a consequence of the previous assumptions $\Sigma(x)$ has again the same dimension as $u(x)$.

We have now all the commutators we require to derive our Ward identities and low-energy theorems. It only remains to comment here that it is usually an assumption that the commutator $[A_0^a(x),\ A_\mu^a(y)]$ vanishes; however, this property can be shown to be true under assumption (iii) ‡‡.

In this paper we want to present a detailed study of the physical implications of our above assumptions on the $A_1\sigma\pi$ system. To do this we will perform the usual hard-meson technique [9] which will consist of deriving Ward identities and low-energy theorems for the associated vertices $\theta A_\mu A_\nu$, $\theta A_\mu \partial^\nu A_\nu$, and $\theta\partial^\mu A_\mu \partial^\nu A_\nu$ and dominating the low-energy region of the various fields with A_1, σ, π poles along with a smooth background.

It is well known that, when dealing with Ward identities away from the low-energy point, Schwinger terms in local equal-time commutators become visible in the equations: their presence is manifested in the appearance of non-covariant pieces in time-ordered products. Therefore, in order to represent off-shell amplitudes, covariant objects, usually called T^* products, are conveniently defined by adding the so-called sea-gull terms [15].

Since, at present, no general information exists on the Schwinger terms occurring in commutators involving $D_0(x)$, we shall use them only in LET's. For the commutators containing $A_0(x)$, however, the dimensional properties of A_0, A_i and $\partial^\mu A_\mu$ are sufficient to completely determine the first Schwinger term in the commutator $[A_0(x),\ \theta(y)]$ as is shown in sect. 2. We shall assume all higher Schwinger terms to be absent. As far as the Schwinger terms in the remaining commutator ‡‡‡, $[A_0(x),\ A_i(y)]$, are concerned, no such general deduction is possible. We can only refer to models for clues to their structure.

Now, even if all the Schwinger terms were known, the covariantization procedure is still largely an arbitrary one in the choice of different sea-gulls, and the off-shell covariant Ward identities will depend upon this choice. Since nothing is known about the high-energy off-shell behaviour of the vertices under considerations, there is no physical argument known to us for preferring one sea-gull over another. Furthermore, different

‡ $\Sigma^{ab}(x,y) = \delta^{ab}\Sigma(x,y)$ would mean that $(u(x), \partial^\mu A_\mu)$ transform according to a $(\frac{1}{2},\frac{1}{2})$ representation of SU(3) × SU(3).

‡‡ See theorem II of sect. 2.

‡‡‡ We have already made assumption (iii) on the absence of Schwinger terms in the sigma commutator.

models are known to exhibit different covariantization. For example, while the propagator of the local electric current in quantum electrodynamics (QED), calculated by Feynman rules, is a strictly conserved quantity [16] to any order in perturbation theory, one finds, for the same propagator in gauge field theories, no conservation (and photons must be exchanged in the Landau gauge [17] to eliminate the spin-zero part of the "photon" in all Feynman graphs).

We shall choose our covariantization procedure analogously to those of gauge field theories. This choice has the advantage of permitting an easy comparison with such Lagrangian models. The choice of sea-gulls preferred by QED has been applied before [18] and will not be discussed here.

The paper is arranged as follows. Sect. 2 is in the form of a review in which give a detailed discussion of the Schwinger terms appearing in the commutators used in deriving the Ward identities. These Ward identities together with LET's for two and three-point functions are derived in sects. 3 and 4 and their application to the $\sigma A \pi$ system is discussed.

2. GENERAL THEOREMS ON SCHWINGER TERMS

We shall investigate here the general structure of the Schwinger terms expected to appear in our Ward identities ‡. The first theorem deals with the Schwinger term in the commutator of $\theta(x)$ with any vector current whose time-like and space-like components and whose divergence have definite dimensions d_t, d_s and d, respectively. Because of our assumptions (i) and (ii), the axial current will satisfy these theorems with $d_t = 3$.

Theorem I: The equal-time commutator of $\theta(x)$ with the time component $J_0(x)$ of an arbitrary vector operator $J_\mu(x)$ having the dimensional properties listed above is

$$i[\theta(x), J_0(y)] = (d_t + 1 - d)\,\partial^\mu J_\mu(x)\,\delta^3(\boldsymbol{x}-\boldsymbol{y}) + (d_t - d_s) J_i(x)\,\partial^i_{(x)}\,\delta^3(\boldsymbol{x}-\boldsymbol{y})$$
$$+ \sum_{n=2}^{N} \sigma_{k_1 \ldots k_n}(y)\,\partial^{k_1}_{(x)} \ldots \partial^{k_n}_{(x)}\,\delta^3(\boldsymbol{x}-\boldsymbol{y})\,, \qquad (2.1)$$

where the operators $\sigma_k \ldots$ are undetermined and $N < \infty$.

The proof is performed in two steps. One first observes that the dilatation charge, $D(0) = \int d^3x\,D_0(x) = \int d^3x\,x^\mu\,\theta_{0\mu}(x)$, commutes with the generators M_{0i} of the Lorentz group according to

$$i[D(0), M_{0i}] = \int d^3x\,x_i\,\theta(0, \boldsymbol{x})\,. \qquad (2.2)$$

Then one uses the vector character of J_0,

$$i[M_{\mu\nu}, J_0(0)] = g_{\mu 0} J_\nu(0) - g_{\nu 0} J_\mu(0)\,, \qquad (2.3)$$

‡ Earlier derivations under slightly more restrictive assumptions are given in refs. [19, 20].

and the Jacobi identity to derive the result

$$i[\int d^3x\, x_i\, \theta(0, \boldsymbol{x}),\ J_0(0)] = i[D(0),\ J_i(0)] - d_t\, J_i(0) \ ,$$
$$= (d_s - d_t)\, J_i(0) \ , \quad (2.4)$$

so that the lowest-order Schwinger terms is as given in (2.1). Second, one writes the Heisenberg equation

$$i[D(0),\ H(0)] = \frac{\partial D(0)}{\partial t} - \frac{dD(0)}{dt} = H(0) - \int d^3x\, \theta(0, \boldsymbol{x}) \ , \quad (2.5)$$

commutes this equation with $J_0(0)$, and uses again the Jacobi identities to obtain

$$i[\int d^3x\, \theta(0, \boldsymbol{x}),\ J_0(0)] = (d_t + 1 - d)\, \partial^\mu\, J_\mu(0) - (d_t - d_s)\, \partial^i J_i(0) \ , \quad (2.6)$$

which proves the theorem.

Therefore, whenever $d_s \neq d_t$, a well determined lowest order Schwinger term exists in the commutator (2.1). Notice that such a Schwinger term must arise because of the presence of a term in $\theta(x)$ which has dimension $(4 + d_s - d_t)$, and that therefore a δ-term is necessary in θ_{00} whenever $d_s \neq d_t$ and $4 + d_s - d_t \neq d$.

The second theorem deals with the connection between the commutator $[A_0(x), A_k(y)]$ and the Schwinger term in the Σ-commutator (1.15) (ref. [20]). It is usually assumed that

$$-i[A_0^a(x),\ A_k^b(y)] = \epsilon^{abc}\, V_k^c(x)\, \delta^3(\boldsymbol{x}-\boldsymbol{y}) - S^{ab}(y)\, \partial_k^{(x)}\, \delta^3(\boldsymbol{x}-\boldsymbol{y})$$
$$- \sum_{n=2}^{N} \sigma^{ab}_{k, k_1 \dots k_n}(y)\, \partial^{k_1}_{(x)} \dots \partial^{k_n}_{(x)}\, \delta^3(\boldsymbol{x}-\boldsymbol{y}) \ . \quad (2.7)$$

This is not necessarily true for non-conserved currents, however. One rather finds:

Theorem II ‡: Let

$$T_k^{ab}(x) \equiv \int d^3y (x_k - y_k) [A_0^b(x),\ \partial^\mu A_\mu^a(y)] \ , \quad (2.8)$$

then

$$[A_0^a(x),\ A_k^b(y)] = (i\epsilon^{abc}\, V_k^c(x) - T_k^{ab}(x))\, \delta^3(\boldsymbol{x}-\boldsymbol{y}) + \mathrm{ST} \ . \quad (2.9)$$

The proof follows upon commuting (2.8) with M_{0i}, and using the following commutators of M_{0i}

$$i[M_{0i},\ A_0(x)] = (x_0\, \partial_i - x_i\, \partial_0) A_0(x) - A_i(x) \ , \quad (2.10)$$

$$i[M_{0i},\ \partial^\mu A_\mu(x)] = (x_0\, \partial_i - x_i\, \partial_0)\, \partial^\mu A_\mu(x) \ , \quad (2.11)$$

‡ Clearly, the same theorem also holds for vector currents.

$$i[M_{0i}, Q_5(x_0)] = -\int d^3x\, x_i\, \partial^\mu A_\mu(x)\ , \qquad (2.12)$$

along with the Jacobi identity.

Thus, from our assumption (iii), we find indeed that $T_k^{ab} = 0$ and therefore (2.7). Notice that (2.8) implies that the dimension of T_k^{ab} is $d-1$, while according to (2.9) it is d_S. Therefore T_k^{ab} cannot exist unless $d_S = d-1$ (which is itself not possible unless $d \geqslant 2$ since $d_S \geqslant 1$‡).

Unfortunately, the Schwinger terms in (2.7) are completely model dependent. All that we may conclude is that the dimension of $S^{ab}(y)$ is $d_S - 1$, so that a c-number $S^{ab}(y)$ implies $d_S = 1$, as is the case for the algebra of fields# [21].

3. WARD IDENTITIES FOR THE TWO-POINT FUNCTIONS

The commutators derived in the last section can be used to obtain LET's and WI's involving the following two-point functions

$$\Delta(p) \equiv -i \int dx\, e^{-ipx} \langle 0|\, T(\partial^\mu A_\mu(x)\, \partial^\nu A_\nu(0))\, |0\rangle\ , \qquad (3.1)$$

$$\Delta_\mu(p) \equiv \int dx\, e^{-ipx} \langle 0|\, T(A_\mu(x)\, \partial^\nu A_\nu(0))\, |0\rangle$$
$$\equiv p_\mu\, \hat{\Delta}(p)\ , \qquad (3.2)$$

$$\Delta_{\mu\nu}(p) \equiv -i \int dx\, e^{-ipx} \langle 0|\, T(A_\mu(x) A_\nu(0))\, |0\rangle\ , \qquad (3.3)$$

$$\Delta_{\theta\Sigma}(q) \equiv -i \int dx\, e^{-iqx} \langle 0|\, T(\theta(x)\, \Sigma(0))\, |0\rangle\ , \qquad (3.4)$$

$$\Delta_{\mu D\Sigma}(q) \equiv \int dx\, e^{-iqx} \langle 0|\, T(D_\mu(x)\, \Sigma(0))\, |0\rangle\ , \qquad (3.5)$$

and $\Delta_{\theta S}(q)$, $\Delta_{\mu DS}(q)$, defined in complere analogy to (3.4) and (3.5), where $S(x) \equiv S^{aa}(x)$ is defined in eq. (2.7). Using the commutation relations for A_0 with A_i and with $\partial^\mu A_\mu$, and also the fact that Σ and S have dimensions and $(d_S - 1)$, respectively, we obtain the following WI's:

$$p^\mu\, \Delta_\mu(p) \equiv p^2\, \hat{\Delta}(p) = \Delta(p) - \Sigma\ , \qquad (3.6)$$

$$p^\mu\, \Delta_{\mu\nu}(p) = \Delta_\nu(p) - p^k g_{k\nu} S\ , \qquad (3.7)$$

$$q^\mu\, \Delta_{\mu D\Sigma}(q) = \Delta_{\theta\Sigma}(q) - d\Sigma + \mathrm{ST}\ , \qquad (3.8)$$

$$q^\mu\, \Delta_{\mu DS}(q) = \Delta_{\theta S}(q) - (d_S - 1) S + \mathrm{ST}\ , \qquad (3.9)$$

where we have introduced the notation $\Sigma \equiv \langle 0|\, \Sigma(0)\, |0\rangle$ and $S \equiv \langle 0|\, S(0)\, |0\rangle$. As usual, the Schwinger term in the commutator (2.7) causes the time-

‡ For a discussion, see ref. [20].

Dimensional theorems involving c-numbers are circumvented if the c-numbers are divergent, as in the quark model for example. An infinite c-number has no definite dimension.

ordered product in (3.3), and subsequently the WI (3.7), to be non-covariant. As we discussed in sect. 1, we introduce a covariant propagator by modifying the time-time component of the propagator (3.3) by

$$\Delta^*_{\mu\nu}(p) \equiv \Delta_{\mu\nu}(p) - g_{\mu\nu} g_{\nu 0} S \,. \tag{3.10}$$

This covariant propagator satisfies the covariant WI

$$p^\mu \Delta^*_{\mu\nu}(p) = \Delta_\nu(p) - p_\nu S \,. \tag{3.7'}$$

Note that we could just as well introduce another covariant propagator

$$\Delta^{**}_{\mu\nu}(p) \equiv \Delta_{\mu\nu}(p) + (g_{\mu\nu} - g_{\mu 0} g_{\nu 0}) S \,, \tag{3.11}$$

which satisfies the simpler WI

$$p^\mu \Delta^{**}_{\mu\nu}(p) = \Delta_\nu(p) \,. \tag{3.7"}$$

Working with the first choice of covariant is suggested by a field theoretic model - the Feynman prescription for any Yang-Mills type Lagrangian containing the current field identity [17], while the second choice is preferred by quantum electrodynamics where, as is well known, all current vertices in the latter theory are taken to be conserved [16]; this behaviour corresponding to the WI (3.7") with $\partial^\mu A_\mu$ zero. Since our smoothness assumptions are suggested by Yang-Mills type effective Lagrangians, we shall work with the first type of covariant.

Eqs. (3.6), (3.8) and (3.9) lead to the following LET's

$$\Delta(0) = \Sigma \,, \tag{3.12}$$

$$\Delta_{\theta\Sigma}(0) = d\Sigma \,, \tag{3.13}$$

$$\Delta_{\theta S}(0) = (d_S - 1)\, S \,, \tag{3.14}$$

such that, from (3.6)

$$p^2 \hat{\Delta}(p) = \Delta(p) - \Delta(0) \,. \tag{3.15}$$

If we assume $\Delta(p)$ to satisfy an unsubtracted dispersion relation ‡

$$\Delta(p) = \int \frac{\rho_\pi(\mu^2)\, d\mu^2}{p^2 - \mu^2} \,, \tag{3.16}$$

then

$$\hat{\Delta}(p) = \int \frac{\rho_\pi(\mu^2)\, \mu^{-2}}{p^2 - \mu^2}\, d\mu^2 \,. \tag{3.17}$$

In particular, pion dominance for low p^2 gives

‡ The pion with $\langle 0 | A^a_\mu(0) | \pi^b p \rangle \equiv i p_\mu f_\pi \delta^{ab}$ contributes $\rho_\pi(\mu^2) = f_\pi^2 m_\pi^4 \delta(\mu^2 - m_\pi^2)$.

$$\Delta(p) = \frac{f_\pi^2 m_\pi^4}{p^2 - m_\pi^2}\,, \qquad \hat{\Delta}(p) = \frac{f_\pi^2 m_\pi^2}{p^2 - m_\pi^2}\,. \tag{3.18}$$

Similarly, we write for the vector propagator the usual unsubtracted dispersion relations

$$\Delta^*_{\mu\nu}(p) = \Delta^{(A)}_{\mu\nu}(p) + p_\mu p_\nu \int \frac{\rho_{(p)}(\mu^2)\,\mu^{-4}}{p^2 - \mu^2}\,d\mu^2\,, \tag{3.19}$$

$$\Delta^{(A)}_{\mu\nu}(p) = \int \frac{(p_\mu p_\nu - \mu^2 g_{\mu\nu})}{p - \mu^2}\,\rho_{(A)}(\mu^2)\,\mu^{-4}\,d\mu^2\,. \tag{3.20}$$

Then the WI (3.7') leads to ‡

$$\int (\rho_{(A)}(\mu^2) + \rho_{(p)}(\mu^2))\,\mu^{-4}\,d\mu^2 + S = \int \frac{(\rho_\pi(\mu^2) - \rho_{(p)}(\mu^2)\,\mu^{-2}\,d\mu^2}{p^2 - \mu^2}\,. \tag{3.21}$$

We conclude that

$$\rho_{(p)}(\mu^2) = \rho_\pi(\mu^2)\,, \tag{3.22}$$

and (3.21) becomes

$$S = -(C_A + C_\pi)\,, \tag{3.23}$$

where

$$C_{\substack{A\\ \pi}} \equiv \int \rho_{\substack{(A)\\ \pi}}(\mu^2)\,\mu^{-4}\,d\mu^2\,, \tag{3.24}$$

A_1 dominance of C_A and pion dominance of C_π gives ‡‡

$$S = -\frac{m_A^2}{\gamma_A^2} - f_\pi^2\,. \tag{3.25}$$

We shall also assume σ-dominance of the spectral functions $\Delta_{\theta\Sigma_S}(q^2)$ in supposing these propagators to be well approximated by unsubtracted forms for small q^2, viz.

$$\Delta_{\theta\Sigma_S}(q^2) = -\Delta_{\theta\Sigma_S}(0)\,m_\sigma^2/(q^2 - m_\sigma^2)\,, \tag{3.26}$$

with $\Delta_{\theta\Sigma_S}(0)$ given by (3.13) and (3.14).

‡ The A_1 meson contributes to the spectral function $\rho_A(\mu^2) = (m_A^6)/(\gamma_A^2)\,\delta(\mu^2 - m_A^2)$ where $\langle 0|A_\mu|A_1(q,\lambda)\rangle = (m_A^2)/(\gamma_A)\epsilon_\mu(q,\lambda)$.

‡‡ We remind the reader that in the case of vector currents in the pole approximation $S_V = -m_\rho^2/\gamma_\rho^2$ and then the assumption that $S = S_V$ leads to Weinberg's first sum rule [22].

4. WARD IDENTITIES FOR THE THREE-POINT FUNCTIONS INVOLVING θ, A_μ, A_ν

We are now ready to discuss the full consequences upon the complex σ, π, A of our chiral and dimensional assumptions. We shall obtain the vertices $\sigma\pi\pi$, $\sigma\pi$A, σAA. Define the following three-point functions:

$$T(q, p) \equiv \int \mathrm{d}x\,\mathrm{d}y\, \mathrm{e}^{-i(qx-py)} \langle 0 | T(\theta(x)\partial^\mu A_\mu(y)\partial^\nu A_\nu(0)|0\rangle , \qquad (4.1)$$

$$T_\mu(q, p) \equiv i \int \mathrm{d}x\,\mathrm{d}y\, \mathrm{e}^{-i(qx-py)} \langle 0 | T(\theta(x) A_\mu(y)\partial^\nu A_\nu(0))| \ \rangle , \qquad (4.2)$$

$$T_{\mu\nu}(q, p) \equiv - \int \mathrm{d}x\,\mathrm{d}y\, \mathrm{e}^{-i(qx-py)} \langle 0 | T(\theta(x) A_\mu(y) A_\nu(0))|0\rangle , \qquad (4.3)$$

as well as

$$\sigma_\lambda(q, p) \equiv i \int \mathrm{d}x\,\mathrm{d}y\, \mathrm{e}^{-i(qx-py)} \langle 0 | T(D_\lambda(x)\partial^\mu A_\mu(y)\partial^\nu A_\nu(0))|0\rangle , \qquad (4.4)$$

$$\sigma_{\lambda\mu}(q, p) \equiv - \int \mathrm{d}x\,\mathrm{d}y\, \mathrm{e}^{-i(qx-py)} \langle 0 | T(D_\lambda(x) A_\mu(y)\partial^\nu A_\nu(0))|0\rangle , \qquad (4.5)$$

$$\sigma_{\lambda\mu\nu}(q, p) \equiv i \int \mathrm{d}x\,\mathrm{d}y\, \mathrm{e}^{-i(qx-py)} \langle 0 | T(D_\lambda(x) A_\mu(y) A_\nu(0))|0\rangle . \qquad (4.6)$$

As a consequence of our assumptions stated in sect. 1, the functions T are found to obey the WI's ($k \equiv q - p$):

$$p^\mu T_\mu(q, p) = - T(q, p) + (4-d)\,\Delta(k) - \Delta_{\theta\Sigma}(q) - (d_\mathrm{S} - 3) p^i \Delta_i(k) , \qquad (4.7)$$

$$T_{\mu\nu}(q, p) k^\nu = - T_\mu(q, p) - (4-d)\,\Delta_\mu(p) - g_{\mu i}\, k^i \Delta_{\theta S}(q) + (d_\mathrm{S} - 3) k^i \Delta_{i\mu}(p) . \qquad (4.8)$$

The analogous WI for the functions σ will again be used only at the low energy point $q = 0$, where the contribution of the Schwinger terms vanishes. These LET's read ‡:

$$T(0, p) = - \left(2d - 4 - p \frac{\partial}{\partial p}\right) \Delta(p) , \qquad (4.9)$$

$$T_\nu(0, p) = \left(d + d_\mathrm{S} - 4 - p \frac{\partial}{\partial p}\right) \Delta_\nu(p) - (d_\mathrm{S} - 3)\, g_{\mathrm{o}\nu} \Delta_\mathrm{o}(p) , \qquad (4.10)$$

$$T_{\mu\nu}(0, p) = \left(2d_\mathrm{S} - 4 - p \frac{\partial}{\partial p}\right) \Delta_{\mu\nu}(p) - (d_\mathrm{S} - 3)\, g_{\mathrm{o}\mu} \Delta_{\mathrm{o}\nu}(p) - (d_\mathrm{S} - 3)\, g_{\mathrm{o}\nu} \Delta_{\mathrm{o}\mu}(p) . \qquad (4.11)$$

We must now choose T^* products order to obtain covariant equations. As discussed in the previous section, we add sea-gulls which preserve the spatial components of the non-covariant time ordered products in order to construct the covariants for which we shall make smoothness assumptions. These T^* functions are given by

‡ We will ignore possible anomalies - for a discussion of these, see ref. [5]. We have no good physical reasons for doing this - however, their inclusion would leave us with no predictions.

$$T^*_\nu(q,\,p) = T_\nu(q,\,p) - (d_s - 3)\,\Delta_o(k)\,g_{o\nu}\ , \tag{4.12}$$

$$\begin{aligned} T^*_{\mu\nu}(q,\,p) &= T_{\mu\nu}(q,\,p) - g_{o\mu}\,g_{o\nu}\,\Delta_{\theta S}(q) \\ &\quad + (d_s - 3)\,[g_{\nu o}\,\Delta^*_{\mu o}(p) + g_{\mu o}\,\Delta^*_{\nu o}(k) - (C_A + C_\pi)\,g_{\mu o}\,g_{\nu o}]\ . \end{aligned} \tag{4.13}$$

A derivation of these sea-gulls is presented in appendix A.

Inserting (4.12) and (4.13) into (4.7)-(4.11), and using the covariant propagator $\Delta^*_{\mu\nu}$ of (3.10), we obtain the following covariant WI's and LET's involving the covariant T^* functions:

$$p^\mu\,T^*_\mu(q,\,p) = -\,T(q,\,p) + (4-d)\,\Delta(k) - \Delta_{\theta\Sigma}(q) - (d_s - 3)\,p^\mu\,\Delta_\mu(k)\ , \tag{4.14}$$

$$T^*_{\mu\nu}(q,\,p)\,k^\nu = -\,T^*_\mu(q,\,p) - (4-d)\,\Delta_\mu(p) - k_\mu\,\Delta_{\theta S}(q) + (d_s - 3)\,k^\nu\,\Delta^*_{\mu\nu}(p)\ , \tag{4.15}$$

$$T^*_\mu(0,\,p) = \left(\mathrm{d} + d_s - 4 - p\,\frac{\partial}{\partial p}\right)\Delta_\mu(p)\ , \tag{4.16}$$

$$T^*_{\mu\nu}(0,\,p) = \left(2d_s - 4 - p\,\frac{\partial}{\partial p}\right)\Delta^*_{\mu\nu}(p)\ . \tag{4.17}$$

The (already) covariant LET (4.9) may be carried over unchanged.

Notice that the LET (4.16) carries the information that the timelike and spacelike components of A_μ have dimensions 3 and d_S, respectively. These dimensions have already been used, however, in the construction of (4.12). It should not then seem surprising that (4.16) is redundant, and can be obtained from (4.9) and (4.14), the latter equation at $q = 0$ ‡. In eq. (4.17), a similar thing occurs only with respect to the contraction of $T^*_{\mu\nu}$ with p^μ. This information is completely contained in the previous WI's. However, the other possible independent cotraction of $T^*_{\mu\nu}$, i.e., with $g^{\mu\nu}$, yields a LET which cannot be obtained from the WI's at $q = 0$. Therefore, in conjunction with (4.9), (4.14) and (4.15), the remaining two eqs. (4.16) and (4.17) may be replaced by the single equation

$$T^{*\mu}_\mu(0,\,p) = \left(2d_s - 4 - p\,\frac{\partial}{\partial p}\right)\Delta^{*\mu}_\mu(p)\ . \tag{4.18}$$

In order to extract physical consequences from these WI's and LET's, we follow the procedure of Weinberg and Schnitzer [9] in defining reduced vertex functions Γ, Γ_μ and $\Gamma_{\mu\nu}$ for the $\sigma\pi\pi$, $\sigma A\pi$ and σAA couplings:

$$T(q,\,p) \equiv -\,\Gamma(q,\,p)\,\Delta(p)\,\Delta(k)\ , \tag{4.19}$$

‡ Note that $T^*_\mu(0,\,p)$ may be written $T^*_\mu(0,\,p) = p_\mu A(p^2)$. Hence contracting (4.16) with p^μ does not suppress any information.

$$T^*_\mu(q,p) \equiv \Gamma(q,p)\Delta_\mu(p)\Delta(k) + \Gamma^\nu(q,p)\Delta^{(A)}_{\mu\nu}(p)\Delta(k)C_A^{-1} , \tag{4.20}$$

$$T^*_{\mu\nu}(q,p) \equiv -\Gamma(q,p)\Delta_\mu(p)\Delta_\nu(k) - \Gamma^\lambda(q,k)\Delta^{(A)}_{\lambda\nu}(k)\Delta_\mu(p)C_A^{-1}$$
$$-\Gamma^\lambda(q,p)\Delta^{(A)}_{\lambda\mu}(p)\Delta_\nu(k)C_A^{-1}$$
$$-\Gamma^{\lambda\rho}(q,p)\Delta^{(A)}_{\lambda\mu}(p)\Delta^{(A)}_{\rho\nu}(k)C_A^{-2} , \tag{4.21}$$

where $\Gamma_{\mu\nu}$ has the crossing property

$$\Gamma_{\mu\nu}(q,p) = \Gamma_{\nu\mu}(q,k) . \tag{4.22}$$

The WI's (4.14) and (4.15) may be written in terms of these reduced vertex functions as

$$-\Sigma\Gamma(q,p) + p^\nu\Gamma_\nu(q,p) - (4-d) + \Delta^{-1}(k)\Delta_{\theta\Sigma}(q)$$
$$+(d_S-3)pk\hat{\Delta}(k)\Delta^{-1}(k) = 0 , \tag{4.23}$$

$$\{\Sigma\Gamma^\lambda(q,p) - k_\rho\Gamma^{\lambda\rho}(q,p)\}\Delta^{(A)}_{\lambda\mu}(p)C_A^{-1} + k_\mu\Delta_{\theta S}(q)$$
$$+\Delta_\mu(p)\Delta^{-1}(p)\Delta_{\theta\Sigma}(q) + (d_s-3)\{pk\hat{\Delta}(p)\Delta^{-1}(p) - k^\lambda\Delta^{(A)}_{\mu\lambda}(p)\} = 0 , \tag{4.24}$$

while the LET's (4.9) and (4.17) yield

$$\Gamma(0,p) = \Delta^{-2}(p)\left(2d - 4 - p\frac{\partial}{\partial p}\right)\Delta(p) , \tag{4.25}$$

$$C_A^{-2}\Gamma^{\lambda\rho}(0,p)\Delta^{(A)}_{\lambda\mu}(p)\Delta^{(A)}_{\rho\nu}(p)$$
$$= \left(2_s - 4 - p\frac{\partial}{\partial p}\right)\{\Delta_\mu(p)\Delta_\nu(p)\Delta^{-1}(p) - \Delta^{(A)}_{\mu\nu}(p)\} , \tag{4.26}$$

where only one of the tensor parts, the coefficient of $g_{\mu\nu}$ or coefficient of $p_\mu p_\nu$ or the trace in (4.30) yields independent information.

Within the hard meson philosophy, we now parametrize the propagators by single poles as discussed in sect. 3, taking advantage of all the WI's involving the two-point functions. We also assume that the low-energy regions of the vertex functions Γ, Γ_μ and $\Gamma_{\mu\nu}$ can be described by a σ-pole in the form

$$\Gamma(q,p) = \frac{m_\sigma^4}{f_\pi^2 m_\pi^4}\frac{1}{q^2 - m_\sigma^2}G , \tag{4.27}$$

$$\Gamma_\mu(q,p) = \frac{m_\sigma^2}{m_\pi^2}\frac{1}{q^2 - m_\sigma^2}(Ap_\mu + 2Bq_\mu) , \tag{4.28}$$

$$\Gamma_{\mu\nu}(q,p) = m_\sigma^2 f_\pi^2 \frac{1}{q^2 - m^2} (C g_{\mu\nu} + m_\sigma^{-2} [2E k_\mu p_\nu + 2E' k_\nu p_\mu + (H+H') k_\mu k_\nu + (H-H') p_\mu p_\nu]) , \tag{4.29}$$

where the invariant functions G, A, B, C, E, E', H and H' are low-order polynomials in the variables p^2, q^2, k^2 with the proper crossing properties ‡ and serve to describe the effect of the tails of the higher resonances in all three variables.

We shall use polynomials in p^2, q^2, k^2 of sufficient degree to be able to include the results of Lagrangian models of gauge fields. Accordingly, we allow G to vary quadratically in these variables. Then an inspection of the WI's shows that for consistency we should include only linear variation in the function A, B and C, while setting E, E', H and H' constant. Since H' is antisymmetric under crossing, it is zero to this order.

Using the described parametrization, we solve the WI's (4.23), (4.24) and the LET's (4.25), (4.26) to find that the invariant functions can be expressed in terms of four free parameters, γ, γ_1, γ_2 and E, not counting the dimensions d and d_S. The complete result is presented in appendix B. Here we just state some of the more relevant consequences.

(i) Taking all particles onto their mass shells, we find the following coupling constants ‡‡

$$g_{\sigma\pi\pi} = \gamma \left[(1-\gamma_1) + (d - 2 + 2\gamma_1 + 2\gamma_2) \frac{m_\pi^2}{m_\sigma^2} \right] , \tag{4.30}$$

$$g_{\sigma A\pi} = \frac{-2 f_\pi \gamma \gamma_A}{m_\sigma} \left[1 - \gamma_1 + \gamma_2 \frac{m_\pi^2}{m_\sigma^2} - \tfrac{1}{2}(d_S - 1) + \frac{m_A^2}{2 m_\sigma^2} E \right] , \tag{4.31}$$

$$g_{\sigma AA} = \frac{f_\pi^2 \gamma m_A}{C_A m_\sigma} \left[(d_S - 3) + (1 - d_S) \frac{C_A}{f_\pi^2} + 2\gamma_1 - E \right] , \tag{4.32}$$ ‡‡‡

$$h_{\sigma AA} = E \frac{f_\pi^2 \gamma m_A}{C_A m_\sigma} . \tag{4.33}$$

‡ From (4.22) we see that C, E, E' and H are symmetric under exchange of p^2 and k^2, and that H' is antisymmetric.

‡‡ These coupling constants are defined by the effective Lagrangian

$$\mathcal{L} = \tfrac{1}{2} m_\sigma g_{\sigma\pi\pi} \sigma \pi^2 + g_{\sigma A\pi} A_\mu \pi \partial^\mu \sigma + m_A g_{\sigma AA} \sigma A_\mu A^\mu - \frac{m_A}{m_\sigma^2} h_{\sigma AA} \sigma \partial_\mu A_\nu \partial^\nu A^\mu .$$

‡‡‡ In the expression for $g_{\sigma AA}$ we may use the KSFR relation $C_A = f_\pi^2$ to obtain

$$g_{\sigma AA} = \frac{m_A \gamma}{m_\sigma} [-2 + 2\gamma_1 - E] .$$

Here γ is the coupling strength of σ to the energy momentum tensor $\langle 0|\theta(0)|\sigma\rangle = m_\sigma^3\gamma^{-1}$ ‡. Thus, an appropriate specification of the four free parameters allows us to fit any set of these four coupling constants consistently with the WI's. Predictions can only arise if we make additional smoothness assumptions. Consider the behaviour of the following form factors with respect to the momentum transfer $q \equiv p' - p$:

$$\theta(q^2) \equiv \langle\pi(p')|\theta(0)|\pi(p)\rangle = \gamma_1 q^2 + 2m_\pi^2 - \frac{m_\sigma^2\gamma^{-1} g_{\sigma\pi\pi} q^2}{q^2 - m_\sigma^2}$$

$$-i\langle\sigma(p')|A_\mu(0)|\pi(p)\rangle \equiv F_+(q^2)(p'+p)_\mu + F_-(q^2)q_\mu , \tag{4.34}$$

where

$$F_+(q^2) = \frac{m_A^2 f_\pi\gamma}{m_\sigma^3}\xi - \tfrac{1}{2}\frac{m_A^2\gamma_A^{-1} g_{A\sigma\pi}}{q^2 - m_A^2} ,$$

$$\xi = \tfrac{1}{2}\left[E - (d_S - 1)\frac{m_\sigma^2}{m_A^2}\right] , \tag{4.35}$$

$$F_-(q^2) = -\gamma_2\gamma\frac{m_\pi^2}{m_\sigma^2} + \frac{m_\sigma f_\pi g_{\sigma\pi\pi}}{q^2 - m_\pi^2} + \tfrac{1}{2}\frac{(m_\sigma^2 - m_\pi^2) g_{A\sigma\pi}\gamma_A^{-1}}{q^2 - m_A^2} , \tag{4.36}$$

$$f(q^2) \equiv \langle\sigma(p')|\partial^\mu A_\mu(0)|\pi(p)\rangle = F_+(q^2)(m_\sigma^2 - m_\pi^2) + F_-(q^2)q^2 . \tag{4.37}$$

We now list a possible set of smoothness assumptions and their consequences for the couplings.

(ii) If $\theta(q^2)$ and $f(q^2)$ are only once subtracted, then $\gamma_1 = \gamma_2 = 0$. This was the smoothness assumption of ref. [16]. Note that this condition is not fulfilled in the case of the gauge field Lagrangian of Gasiorowicz and Geffen (GG) [23]. There one finds $\gamma_1 = (1 - 1/Z)$, where $Z = m_A^2/m_\rho^2 \approx 2$. If we modify the energy momentum tensor of the GG model by adding the correction term [6] $-\frac{1}{6}d(\partial_\mu\partial_\nu - \Box g_{\mu\nu})(\sigma^2 + \pi^2)$ to the Belinfante tensor, thus assigning dimension d to the σ and π fields [18]: we obtain instead $\gamma_1 = (1 - d/Z)$. Thus we see that, in this model, a once subtraced $\theta(q^2)$ can only be obtained with $d = Z$ (≈ 2).

(iii) If (ii) holds and $F_+(q^2)$ is unsubtracted ‡‡ then we obtain, in addition to

‡ In Lagrangian models [2, 13] where dimensions are adjusted by inclusion of factors $e^{b\sigma}$, the parameter b is connected with γ via $b = -\gamma/m_\sigma$.

‡‡ We should mention that in this case the dimensional information provides that the subtraction in $f(q^2)$ is of order m_π^2/m_σ^2. Carruthers [24] used the assumption of unsubtractedness of these form factors to obtain $(g_{\sigma A\pi})/(g_{\sigma\pi\pi}) = -2(f_\pi\gamma_A)/(m_\sigma) \times (1 - m_\pi^2/m_\sigma^2)$ which is the above result for $d = 1$.

$$g_{\sigma\pi\pi} = \gamma\left[1 + (d-2)\frac{m_\pi^2}{m_\sigma^2}\right] , \tag{4.38}$$

also

$$g_{\sigma A\pi} = -2\gamma\frac{f_\pi \gamma_A}{m_\gamma} , \tag{4.39}$$

$$g_{\sigma AA} = \frac{\gamma f_\pi^2 m_A}{C_A m_\sigma}\left[-2 + (d_S - 1)\left(1 - \frac{C_A}{f_\pi^2} - \frac{m_\sigma^2}{m_A^2}\right)\right] , \tag{4.40}$$

$$h_{\sigma AA} = \frac{\gamma f_\pi^2 m_\sigma}{C_A m_A}(d_S - 1) , \tag{4.41}$$

i.e., the d-wave coupling constant vanishes if $d_S = 1$. The value for $g_{A\sigma\pi}$ in (4.39) was the result obtained in ref. [18]‡.

Within our general assumptions the coupling strength is an arbitrary parameter. However, it can be related to the dimensional content of the Hamiltonian density and the chiral content of the breaking term u by a spectral function sum rule [2, 4]. In the GG Lagrangian with $d_S = 1$ and σ and π of dimension d one has

$$\gamma = -\frac{m_\sigma}{Z^{3/2} f_\pi d} . \tag{4.42}$$

In the same GG model, assumption (iii) on the form factors is not satisfied because of the presence of the κ- term [necessary to include d-wave coupling in $A\rho\pi$ ♯ and causing a subtraction in the matrix element $\langle\pi|\rho|\pi\rangle$ proportional to $(C_A\kappa - 1)$]. In fact this Lagrangian model yields the following value for the parameter E ♯♯

‡ For a comparison of this value of $g_{\sigma A\pi}$ to those obtained by other methods, see ref. [18].

♯ See eq. (5.29) of ref. [21]. In the $A\rho\pi$ coupling constant, defined by $\mathcal{L} = g_{A\rho\pi}\,\rho_\mu\cdot(A^\mu\times\pi) + h_{A\rho\pi}\,\rho_\mu\cdot(\partial^\mu A^\nu\times\partial_\nu\pi)$, κ occurs as $g_{A\rho\pi} = (m_\rho^2/2f_\pi)(2+\delta)$, $h_{A\rho\pi} = (1/2f_\pi)\delta$ where $\delta \equiv (m_A^2/\gamma_A)\kappa$. In $g_{\rho\pi\pi}$ one finds δ (ref. [9]) as $g_{\rho\pi\pi} = (m_\rho/4\sqrt{2}f_\pi)(3-\delta)$.

♯♯ With $\gamma_1 = (1 - d/Z)$ and E given by (4.43), our equations yield

$$g_{\sigma\pi\pi} = -\frac{m_\sigma}{f_\pi Z^{3/2}}\left[1 + (Z-2)\frac{m_\pi^2}{m_\sigma^2}\right],$$

$$g_{A\sigma\pi} = Z^{-3/2}\gamma_A(2 - Z\delta) ,$$

$$g_{\sigma AA} = \gamma_A^2 f_\pi m_\sigma^2 m_A^{-3} Z^{-1/2}\delta ,$$

in agreement with the GG model.

$$E = - m_\sigma^2 d \frac{\kappa}{\gamma_A} \equiv - \frac{m_\sigma^2}{m_A^2} d\delta , \tag{4.43}$$

where we have introduced the familiar δ-parameter frequently used in hard pion calculations [9].

(iv) If we assume that the d-wave coupling vanishes, i.e., $E = 0$, [instead of assumption (iii)], then $g_{A\sigma\pi}$ vanishes for $d_s = 3$ ‡. Note that in this case, $F_+(q^2)$ is a constant in the low energy region up to the A_1 pole.

(v) In the Lagrangian models that we have studied, which contain no spin 2 f-meson, we always find that $\gamma_2 = 0$. However, we argue that γ_2 becomes non-zero in general on introducing the f-meson ‡‡, since we can show (see appendix C) that in our model

$$\frac{\partial}{\partial q^2} G_1(q^2)\Big|_{q^2=0} = - \gamma_2 / m_\sigma^2 , \tag{4.44}$$

where G_1 is the tensor form factor defined in

$$\langle \pi(p')|\theta_{\mu\nu}(0)|\pi(p)\rangle = \tfrac{1}{2} [\Sigma_\mu \Sigma_\nu G_1(q^2) + (q^2 g_{\mu\nu} - q_\mu q_\nu) G_2(q^2)] , \tag{4.45}$$

where $q = p' - p$, $\Sigma = p' + p$. Thus if we assume $G_1(q^2)$ to be at most of once subtracted form for low q^2 we have

$$G_1(q^2) = 1 + \frac{\gamma_2 q^2 m_f^2 / m_\sigma^2}{q^2 - m_f^2} . \tag{4.46}$$

Requiring $G_1(q^2)$ to be unsubtracted [26] would then require $\gamma_2 = - m_\sigma^2 / m_f^2$ ‡‡‡. If γ_2 has such a typical order of magnitude i.e., 0(1), we would be justified in neglecting it in considering the coupling constants (4.30) and (4.31) since γ_2 always appears in these formulae with an associated factor m_π^2 / m_σ^2.

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‡ We thank Professor B. Zumino for pointing this out to us.

‡‡ Professor B. Zumino has informed us that in Lagrangian models [25] incorporating f- and σ-dominance of the stress energy momentum tensor, extra higher-derivative $\sigma\pi\pi$ couplings are necessarily introduced which cause higher off-shell momentum dependence which only disappears in the limit $m_f^2 \to \infty$. In the simplest model with minimal coupling, however, the $\sigma\pi\pi$ coupling constant is still independent of m_f^2 and given by (4.38). The appearance of just two arbitrary parameters γ_1, γ_2 seems to be correlated to the fact that one can introduce into this pion Lagrangian [25] (at the lowest-derivative level) just two independent non-minimal couplings to the f- and σ-mesons.

‡‡‡ Note then in this case the condition for G_2 to be unsubtraced also in our model is $\gamma_1 = 0$.

APPENDIX A

Derivation of sea-gulls

We shall present a short construction of the T^* products (4.12) and (4.13). Consider the non-covariant WI's (4.7) and (4.8). These WI's show immediately which sea-gull terms are required for covariance of T_μ and $T_{\mu\nu}$. As we explain in the text, we choose sea-gull terms preserving the spatial components of T_μ and $T_{\mu\nu}$.

Obviously,

$$T^*_\mu(q,p) = T_\mu(q,p) - (d_s - 3)\,\Delta_o(k)\,g_{o\mu}\,, \tag{A.1}$$

is a correct choice. Inserting this relation into (4.8), we note that certainly the choice

$$T^*_{\mu\nu}(q,p) = T_{\mu\nu}(q,p) + g_{o\mu}\,g_{o\nu}(d_s - 3)\,\hat{\Delta}(k^2) - g_{o\mu}\,g_{o\nu}\,\Delta_{\theta S}(q)$$
$$+ (d_s - 3)\,g_{o\nu}\,\Delta^*_{o\mu}(p) + S_{\mu\nu}(p,k)\,, \tag{A.2}$$

will make (4.8) covariant if $S_{\mu\nu}(p,k)k^\nu = 0$. The spatial components of $T_{\mu\nu}$ are unchanged if $S_{ij}(p,k) = 0$. Finally, we have to enforce the correct crossing property for $T^*_{\mu\nu}(q,p)$ (see eq. (4.22)).

From these conditions, we find

$$S_{\mu\nu}(p,k) = g_{\mu o}(g_{\nu o}k^2 - k_o k_\nu)\,\Delta_1(k^2)\,, \tag{A.3}$$

where $\Delta_1(k)$ is one of the invariants occuring in the covariant decomposition of the propagator

$$\Delta^*_{\mu\nu}(k) = k_\mu k_\nu \Delta_1(k^2) - g_{\mu\nu}\Delta_2(k^2)\,. \tag{A.4}$$

Inserting (A.3) and (A.4) into (A.2) and using the spectral function (3.19), we indeed recover the T^* product (4.13).

APPENDIX B

Parametrization of vertex functions

The invariant functions of the vertices Γ, Γ_μ, and $\Gamma_{\mu\nu}$ in (4.27)-(4.29) are found from the WI's (4.23)-(4.26) in terms of the three parameters γ_1, γ_2 and ξ to be

$$G = 2\,\frac{m_\pi^2}{m_\sigma^2}(d-2) + \frac{1}{m_\sigma^2}(p^2+k^2)(1-d) - \frac{q^2}{m_\sigma^2}\left[1 + (d-4)\,\frac{m_\pi^2}{m_\sigma^2}\right]$$
$$+ \frac{\gamma_1}{m_\sigma^4}\,q^2(q^2 - p^2 - k^2) + \frac{\gamma_2}{m_\sigma^4}\left[(k^2-p^2)^2 - q^2(p^2+k^2)\right]\,,$$

$$A = (d - d_S + 1) + \frac{1}{m_\sigma^2} \{(2\gamma_1 + \gamma_2 + d_S - 3 - \xi)\, q^2 - (\gamma_2 + \xi)(p^2 - k^2)\} ,$$

$$2B = (d_S - 1) + \frac{1}{m_\sigma^2} \{(3 - d_S - 2\gamma_1)\, q^2 + 2\xi p^2 + 2\gamma_2 k^2\} ,$$

$$C = 2\frac{C_A}{f_\pi^2}(d_S - 2) + \frac{1}{m_\sigma^2}\left\{\left(2\gamma_1 - 2\xi - [d_S - 1]\frac{m_\sigma^2}{m_A^2} - (d_S - 3)\left(\frac{C_A}{f_\pi^2} - 1\right)\right) q^2 - (d_S - 1)\frac{C_A}{f_\pi^2}\frac{m_\sigma^2}{m_A^2}\, p^2\right\} ,$$

$$E' = -2\gamma_2 ,$$

$$H = 2\xi - 2\gamma_2 + \frac{m_\sigma^2}{m_A^2}\left(\frac{C_A}{f_\pi^2} + 1\right)(d_S - 1) ,$$

$$E = 2\xi + (d_S - 1)\frac{m_\sigma^2}{m_A^2} .$$

APPENDIX C

In this Appendix we show that in our model

$$\frac{\partial}{\partial q^2} G_1(q^2, p^2, k^2)|_{q^2=0} = -\gamma_2/m_\sigma^2 , \tag{C.1}$$

consider the covariant reduced vertex [3]

$$\Gamma_{\mu\nu}(q, p) \equiv -\Delta^{-1}(p)\Delta^{-1}(k)\int \mathrm{d}x\,\mathrm{d}y\, \mathrm{e}^{-i(qx-py)} \times \langle 0|T^*(\theta_{\mu\nu}(x)\,\partial^\rho A_\rho(y)\,\partial^\lambda A_\lambda(0))|0\rangle$$

$$\equiv \tfrac{1}{2}(f_\pi m_\pi^2)^{-2}\{\Sigma_\mu \Sigma_\nu G_1 + (q^2 g_{\mu\nu} - q_\mu q_\nu) G_2 + 2m_\pi^2 g_{\mu\nu} G_3 + (\Sigma_\mu q_\nu + \Sigma_\nu q_\mu) G_4\} , \tag{C.2}$$

where $\Sigma = p - k$, $q = p + k$. The form factors $G_i = G_i(q^2, p^2, k^2)$ can be considered to characterize the matter distribution of the pion. The conservation Ward identity on $\Gamma_{\mu\nu}$ reads:

$$q_\mu \Gamma^{\mu\nu}(q,p) = -k^\nu \Delta^{-1}(p) - p^\nu \Delta^{-1}(k) , \tag{C.3}$$

which in terms of the form factors G_i reads

$$\Sigma q G_1 + q^2 G_4 = f_\pi^2 m_\pi^4 (\Delta^{-1}(p) - \Delta^{-1}(k)) , \tag{C.4}$$

$$2m_\pi^2 G_3 + \Sigma q G_4 = -f_\pi^2 m_\pi^4 (\Delta^{-1}(p) + \Delta^{-1}(k)) . \tag{C.5}$$

Thus the Ward identity (C.2) determines the off-shell form factors G_3, G_4 in terms of the on-shell form factor G_1 and also gives information on G_1 at $q^2 = 0$, e.g., $G_1(0, m_\pi, m_\pi^2) = 1$. We now define the scalar form factor

$$2m_\pi^2 G_s = f_\pi^2 m_\pi^4 \Gamma_\mu^\mu = \tfrac{1}{2} \Sigma^2 G_1 + \tfrac{3}{2} q^2 G_2 + 4m_\pi^2 G_3 + \Sigma q G_4 . \tag{C.6}$$

Then it follows from (C.3) and (C.4) that

$$2m_\pi^2 G_s(0, p^2, k^2) = f_\pi^2 m_\pi^4 \left\{ \frac{p^2+k^2}{p^2-k^2} (\Delta^{-1}(p) - \Delta^{-1}(k)) - 2(\Delta^{-1}(p) + \Delta^{-1}(k)) \right\}$$
$$+ (p^2-k^2)^2 \frac{\partial}{\partial q^2} G_1(q^2, p^2, k^2)\Big|_{q^2=0} . \tag{C.7}$$

In the pole approximation

$$\Delta^{-1}(p) = f_\pi^{-2} m_\pi^{-4} (p^2 - m_\pi^2) ,$$

we have therefore

$$2m_\pi^2 G_s(0, p^2, k^2) = 4m_\pi^2 - (p^2+k^2) + (p^2-k^2)^2 \frac{\partial}{\partial q^2} G_1(q^2, p^2, k^2)\Big|_{q^2=0} . \tag{C.8}$$

But since Γ_μ^μ is related to Γ through the trace identity

$$\Gamma_\mu^\mu(q,p) = \Gamma(q,p) - d(\Delta^{-1}(p) + \Delta^{-1}(k)) , \tag{C.9}$$

we have with our parametrization given in (4.27) and Appendix B

$$2m_\pi^2 G_s(0, p^2, k^2) = 4m_\pi^4 - (p^2+k^2) - \frac{\gamma_2}{m_\sigma^2} (p^2-k^2)^2 , \tag{C.10}$$

thus we conclude that, in our model, (C.1) holds.

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