Critical Exponents without β-Function

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We point out that the recently developed strong-coupling theory enables us to calculate the three main critical exponents ν, η, ω, from the knowledge of only the two renormalization constants Z_m of mass and Z_ϕ of wave function. The renormalization constant of the coupling strength is superfluous, and with it also the β-function, the crucial quantity of the renormalization group approach to critical phenomena.

1. Some time ago we have shown [1–4] that there exists a simple way of extracting the strong-coupling properties of a ϕ^4-theory from perturbation expansions. In particular, we were able to find the power behavior of the renormalization constants in the limit of large couplings, and from this all critical exponents of the system. By using the known expansion coefficients of the renormalization constants in three dimensions up to six loops we derived extremely accurate values for the critical exponents.

This is possible because of the fact that ϕ^4-theory displays experimentally observed scaling behavior as the bare mass m_B goes to zero and fluctuations become large. For dimensional reasons, the zero bare mass limit is in $4-\epsilon$ dimensions equivalent to an infinite bare coupling constant limit $g_B \to \infty$. The existence of a scaling behavior implies therefore that the perturbation expansions of the renormalized coupling constant and all critical exponents in powers of g_B have all a strong-coupling behavior [5] of the form

$$f(g_B) = f^* - \frac{\text{const}}{g_B^{\omega/\epsilon}} + \dots \, . \tag{1}$$

The number f^* is the critical coupling or exponent, and the power ω is the exponent of the approach to scaling. The parameter $\epsilon = 4 - D$ denotes as usual the deviation of the space dimension from the naively scale-invariant dimension 4.

Apart from ω, there are two independent critical exponents, for instance ν which rules the divergence of the coherence length $\xi \propto |T - T_c|^{-\nu}$ as the temperature T approaches the critical temperature T_c, and the exponent γ which does the same thing for the magnetic susceptibility. The purpose of this paper is to point out that the strong-coupling theory developed in [1,3] allows us to calculate all three critical exponents from the perturbation expansions of only the two renormalization constants Z_m of mass and Z_ϕ of wave function. There is no need to go through the hardest calculation for the renormalization constant of the coupling strength, and we do not have to know the famous β-function of the renormalization group approach to critical phenomena, in which the exponent ω is found from the derivative of the β-function at its zero.

2. Let us briefly recall the relevant formulas. Given the first $N+1$ expansion terms of the critical exponents, $f_N(g_B) = \sum_{n=0}^N a_n g_B^n$, we assume that the strong-coupling behavior (1) continues systematically as an inverse power series in $g_B^{-\omega/\epsilon}$, $f_M(g_B) = \sum_{m=0}^M b_m (g_B^{-2/q})^m$, with some finite convergence radius g_s [6] (for examples see [7–9]). Then the Nth approximation to the value f^* is obtained from the formula

$$f_N^* = \operatorname*{opt}_{\hat g_B} \left[\sum_{j=0}^{N} a_j \hat g_B^j \sum_{k=0}^{N-j} \binom{-qj/2}{k} (-1)^k \right], \tag{2}$$

where the expression in brackets has to be optimized in the variational parameter $\hat g_B$. The optimum is the smoothest of the real extrema. If there are none, the turning points serve the same purpose.

The derivation of this expression is simple: We replace g_B in $f_N(g_B)$ trivially by $\bar g_B \equiv g_B/\kappa^q$ with $\kappa = 1$. Then we rewrite, again trivially, κ^{-q} as $(K^2+\kappa^2-K^2)^{-q/2}$ with an arbitrary parameter K. Each term is now expanded in powers of $r = (\kappa^2 - K^2)/K^2$ assuming r to be of the order g_B. The we take the limit $g_B \to \infty$ at a fixed ratio $\hat g_B \equiv g_B/K^q$, so that $K \to \infty$ like $g_B^{1/q}$ and $r \to -1$, yielding (2). Since the final result to all orders cannot depend on the arbitrary parameter K, we expect the best result to any finite order to be optimal at an extremal value of K, i.e., of $\hat g_B$.

The strong-coupling approach to the limiting value $r = -1 + \kappa^2/K^2 = -1 + O(g_B^{-2/q})$ implies the leading correction to f_N^* to be of the order of $g_B^{-2/q}$. Application of the theory to a function with the strong-coupling behavior (1) requires therefore setting the parameter q equal to $2\epsilon/\omega$ in formula (2).

For $N=2$ and 3, the strong coupling limits (2) are very simple. Defining $\rho \equiv 1 + q/2 = 1 + \epsilon/\omega$, one has for $N = 2$:

$$f_2^* = \operatorname*{opt}_{\hat g_B} \left[a_0 + a_1 \rho \hat g_B + a_2 \hat g_B^2 \right] = a_0 - \frac{1}{4} \frac{a_1^2}{a_2} \rho^2, \tag{3}$$

and for $N = 3$:

$$f_3^* = \operatorname*{opt}_{\hat g_B} \left[a_0 + \tfrac{1}{2} a_1 \rho(\rho+1) \hat g_B + a_2 (2\rho - 1) \hat g_B^2 + a_3 \hat g_B^3 \right]$$

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$$= a_0 - \frac{1}{3}\frac{\bar a_1 \bar a_2}{a_3}\left(1-\frac{2}{3}r\right) + \frac{2}{27}\frac{\bar a_2^3}{a_3^2}\left(1-r\right), \quad (4)$$

where $r \equiv \sqrt{1-3\bar a_1 a_3/\bar a_2^2}$ and $\bar a_1 \equiv \frac{1}{2}a_1\rho(\rho+1)$ and $\bar a_2 \equiv a_2(2\rho-1)$. The positive square root must be taken to connect g_3^* smoothly to g_2^* for small g_B. If the square root is imaginary, the optimum is given by the unique turning point, leading once more to (4), but with $r=0$.

Before we can apply formula (2), we must find the exponent ω describing the approach to scaling. We do this by studying the strong-coupling limit of the logarithmic derivative $s(g_B) \equiv g_B f'(g_B)/f(g_B)$ of any critical exponent $f(g_B)$, again via formula (2). Since $f(g_B)$ approaches a constant f^* like (1), its logarithmic derivative $s(g_B)$ must have the same type of behavior with $s^*=0$. This equation determines ω. For an expansion of the critical exponents up to order g_B^3, it is easy to find explicit results. Let us denote the generic expansion of the exponents for a moment by $F(g_B) = A_1 g_B + A_2 g_B^2 + \ldots$. If the expansions start with a nonzero value at $g_B=0$, this may be subtracted. The logarithmic derivative of $F(g_B)$ is

$$s(g_B) = 1 + \hat A_2 g_B + (2\hat A_3 - \hat A_2^2)g_B^2 + (\hat A_2^3 - 3\hat A_2\hat A_3 + 3\hat A_4)g_B^3 + \ldots , \quad (5)$$

where $\hat A_i = A_i/A_1$. The expansion coefficients on the right-hand sides serve as coefficients a_0, a_1 a_2 in formulas (3) or (4), with the left-hand being set equal to zero to ensure that $s(g_B) \to s^* = 0$ and thus $f(g_B) \to f^* =$const for $g_B \to \infty$.

Another way to determine ω uses the fact that if $F(g_B)$ approaches F^* as in (1), the function

$$h(g_B) \equiv g_B \frac{F''(g_B)}{F'(g_B)} = 2\hat A_2 g_B + (-4\hat A_2^2 + 6\hat A_3)g_B^2 + (8\hat A_2^3 - 18\hat A_2\hat A_3 + 12\hat A_4)g_B^3 + \ldots \quad (6)$$

has the strong-coupling limit

$$h(g_B) \to h^* = -\frac{\omega}{\epsilon} - 1. \quad (7)$$

3. These formulas can now be applied directly to the power series of the renormalization constants of mass and wave functions. Their power behavior for $g_B \to \infty$ is of the form

$$\frac{m^2}{m_B^2} \propto g_B^{-\eta_m/\epsilon} \propto m^{\eta_m}, \qquad \frac{\phi^2}{\phi_B^2} \propto g_B^{\eta/\epsilon} \propto m^{-\eta}. \quad (8)$$

The powers functions η_m and η can therefore be calculated from the strong-coupling limits of the logarithmic derivatives

$$\eta_m(g_B) = -\frac{\epsilon\, d}{d\log g_B}\log\frac{m^2}{m_B^2}, \; \eta(g_B) = \frac{\epsilon\, d}{d\log g_B}\log\frac{\phi^2}{\phi_B^2}. \quad (9)$$

These functions have the perturbation expansions up to the order g_B^3

$$\eta_m(g_B) = \frac{n+2}{3}g_B - \frac{n+2}{18}\left(5 + 2\frac{n+8}{\epsilon}\right)g_B^2 \quad (10)$$

$$+\frac{2+n}{108}\left[3\,(37+5n) + \frac{244+38n}{\epsilon} + \frac{4\,(8+n)^2}{\epsilon^2}\right] g_B^3, \quad (11)$$

$$\eta(g_B) = \frac{n+2}{18}g_B^2 - \left(16 + 10\,n + n^2\right)\frac{(\epsilon+8)}{216\,\epsilon}g_B^3. \quad (12)$$

The critical exponent ν is obtained from the strong-coupling limit of the function $\nu(g_B) = 1/[2-\eta_m(g_B)]$, whereas the exponent γ is the same limit of the function $\gamma(g_B) = \nu(g_B)[2-\eta(g_B)]$. These functions have the expansions up tp order g_B^3:

$$\nu(g_B) = \frac{1}{2} + \frac{n+2}{12}g_B + \frac{n+2}{72}\left(n-3-2\frac{n+8}{\epsilon}\right)g_B^2$$

$$+\frac{2+n}{432\epsilon^2}\left[4(8+n)^2 + \epsilon^2\left(95+9n+n^2\right) - 2\epsilon\left(-90+n+2n^2\right)\right]g_B^3, \quad (13)$$

$$\gamma(g_B) = 1 + \frac{n+2}{6}g_B + \frac{n+2}{36}\left(n-4-2\frac{n+8}{\epsilon}\right)g_B^2 \quad (14)$$

$$+\frac{2+n}{432\epsilon^2}\left[8(8+n)^2 + 4\epsilon\left(106+n-2n^2\right) + \epsilon^2\left(194+17n+2n^2\right)\right]g_B^3.$$

There is no need to give more expansion coefficients, which can all be downloaded from the internet [10].

4. We begin by calculating the critical exponent ω from the requirement that the expansions $\nu(g_B)$ has a constant strong-coupling limit. Then also the subtracted function $F(g_B) = \nu(g_B) - \nu(0)$ has a constant limit. From (13) and (5) we find the expansion coefficients of the logarithmic derivative

$$a_0 = 1,\; a_1 = \frac{1}{6}\left(n-3-2\frac{n+8}{\epsilon}\right), \quad (15)$$

$$a_3 = \frac{1}{36\epsilon^2}\left[4(8+n)^2 - 4\epsilon\left(-66-4n+n^2\right) + \epsilon^2\left(181+24n+n^2\right)\right].$$

Inserted into (3), the condition of a zero strong-coupling limit yields

$$\rho = 2\frac{\sqrt{4(8+n)^2 - 4\epsilon\,(-66-4n+n^2) + \epsilon^2\,(181+24n+n^2)}}{2(8+n)+\epsilon(n-3)}. \quad (16)$$

Let us also derive ω from the vanishing of the logarithmic derivative of the critical exponent γ. Here we find

$$\rho = 2\frac{\sqrt{4(8+n)^2 - 4\epsilon\,(-74-5n+n^2) + \epsilon^2\,(178+25n+n^2)}}{2(8+n)+\epsilon(n-3)}.. \quad (17)$$

The resulting exponents $\omega = \epsilon/(\rho-1)$ are plotted against ϵ in Fig. 1, together with the plots derived in Ref. [3] from the expansion of the renormalized coupling constant $g(g_B)$ in powers of g_B. The first two terms of the ϵ-expansion of ω are, in all expressions

$$\omega = \epsilon - 3\frac{3n+14}{(n+8)^2}\epsilon^2 + \ldots . \quad (18)$$

Let us also calculate ω from the auxiliary function $h(g_B)$ of Eq. (6) by solving Eq. (7) for ω, which reads more explicitly:

$$-\frac{\omega}{\epsilon} - 1 = -\frac{\rho}{\rho-1} = -\frac{1}{2}\frac{\hat A_2^2\,\rho^2}{3\hat A_3 - 2\hat A_2^2}. \quad (19)$$

yielding for $\rho = 1 + \epsilon/\omega$:

$$\rho = 1 + \frac{\epsilon}{\omega} = \frac{1}{2} + \sqrt{\frac{6\hat{A}_3}{\hat{A}_2^2} - \frac{15}{4}}. \tag{20}$$

From $F(g_B) = \nu(g_B) - \nu(0)$ we obtain

$$\rho = \frac{1}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{12\,(8+n)^2 - 12\,\epsilon\,(-80 - 7\,n + n^2) + \epsilon^2\,(715 + 102\,n + 3\,n^2)}}{2(8+n)+\epsilon(n-3)}, \tag{21}$$

while $F(g_B) = \gamma(g_B) - \gamma(0)$ yields

$$\rho = \frac{1}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{4\,(8+n)^2 + \epsilon\,(352 + 32\,n - 4\,n^2) + \epsilon^2\,(232 + 36\,n + n^2)}}{2(8+n)+\epsilon(n-3)}. \tag{22}$$

The resulting $\omega = \epsilon/(\rho - 1)$ are also plotted in Fig. 1. The ϵ-expansions are again the same as in (18).

5. With ω being determined, the critical exponents ν and γ are calculated as before in Ref. [3] by inserting their power series coefficients into the strong-coupling equation (3). Depending on the different expressions for the resummed $\omega(\epsilon)$ we obtain from the limiting values f^* various resummed functions $\nu(\epsilon)$ and $\gamma(\epsilon)$. Their common ϵ-expansions up to ϵ^2 are

$$\nu = \frac{1}{2} + \frac{1}{4}\frac{n+2}{n+8}\epsilon + \frac{(n+2)(n+3)(n+20)}{8(n+8)^3}\epsilon^2, \tag{23}$$

$$\gamma = 1 + \frac{1}{2}\frac{n+2}{n+8}\epsilon + \frac{1}{4}\frac{(n+2)(n^2+22n+52)}{(n+8)^3}\epsilon^2. \tag{24}$$

Thus we have shown that the ϵ-expansions of all critical exponents including ω can be obtained from only two renormalization constants ϕ^2/ϕ_B^2 and m^2/m_B^2. Thus there is no need to calculate the renormalization constant of the coupling strength.

6. In three dimensions, the two renormalization constants ϕ^2/ϕ_B^2 and m^2/m_B^2 have been calculated up to seven loops [11], and the principal critical exponents have been derived via conventional resummation methods in [11], [12], and via variational perturbation theory by the present author in [2,4]. The the latter works, ω was derived from the expansion of the renormalized coupling constant in powers of g_B, which is only known up to six loops. In the spirit of the present discussion, we would like to recalculate ω from one of the two expansions of the two renormalization constants known up to seven loops. As an example we take the expansion for $\bar{\eta} \equiv \eta - \eta_m$. The reason for this choice is that the theoretical large-order behavior can be fitted extremely well to the known seven expansion coefficients, so that higher-order coefficients can be predicted quite reliably. This was done in [2], and the extrapolated expansion can be found in Table V of that paper. From this we may determine ω via the condition that $s(g_B) = d\log\bar{\eta}(g_B)/d\log g_B(g_B)$ vanishes in the strong-coupling limit, i.e., we the optimum of Eq. (2) for s^* should be zero. We do this only for the universality class of the superfluid transition of helium, proceeding as follows: For a various ω-values we calculate the strong-coupling values s_N^* for increasing orders N and extrapolate them to infinite order by a procedure explained in Ref. [2]. From the results we find the ω-value at which s_∞^* is zero to be $\omega = 0.790$. The extrapolation of s_N^* for this is shown in Fig. 2.

The successive strong-coupling approximtions converge exponentially fast against the final result [1], faster than any Padé-type of approximation. The reason is the presence of the power sequence $g_B^{-n\omega/\epsilon}$ in the strong-coupling expansion at each order.

7. The reader may wonder why we can so easily discard the renormalized coupling strength $g(g_B)$. The answer is simple: Instead of $g(g_B)$, we can just as well parametrize the coupling strength by the parameter $g_\nu(g_B) \equiv [\nu(g_B) - \nu(0)]/\nu'(0)$ or $g_\gamma(g_B) \equiv [\gamma(g_B) - \gamma(0)]/\nu'(0)$ which both start out like

$$g_{\nu,\gamma} = g_B - \frac{1}{3}\left[\frac{n+8}{\epsilon} + (2 - n/2)\right] g_B^2 + O(g_B^3),$$

but continue differently. The renormalized coupling constant $g(g_B)$ has the same first terms except for the last parentheses. For either of these expansions we can define β-type of functions

$$\beta_{\nu,\gamma}(g_B) = -\epsilon g_{\nu,\gamma}(g_B)\frac{d\log g_{\nu,\gamma}(g_B)}{d\log g_B}, \tag{25}$$

and express these in terms of $g_{\nu,\gamma}$. Up to the order $g_{\nu,\gamma}^2$, this yields

$$\beta_\nu(g_\nu) = -\epsilon g_\nu + \left[\frac{8+n}{3\epsilon} + \frac{3-n}{6}\right] g_\nu^2 + \frac{1}{36\epsilon^2}\left[4(8+n)^2 + \epsilon^2\left(-77-21n+n^2\right) - 2\epsilon\left(-6+19n+2n^2\right)\right] g_\nu^3, \tag{26}$$

$$\beta_\gamma(g_\gamma) = -\epsilon g_\gamma + \left[\frac{8+n}{3\epsilon} + \frac{4-n}{6}\right] g_\gamma^2 + \frac{1}{72\epsilon^2}\left[8(8+n)^2 + \epsilon^2\left(-130-49n+2n^2\right) - 4\epsilon\left(-22+17n+2n^2\right)\right] g_\gamma^3. \tag{27}$$

whose zeros lie at

$$g_\nu = \frac{3\epsilon}{n+8} + \frac{3\,(20+n)\,(3+n)}{2\,(n+8)^3}\epsilon^2, \tag{28}$$

$$g_\gamma = \frac{3\epsilon}{8+n} + \frac{3\,\left(52 + 22\,n + n^2\right)}{2\,(8+n)^3}\epsilon^2, \tag{29}$$

thus yielding once more the critical exponents ν, γ of Eqs. (23) and (24). The slopes of these β-like functions are universally equal to ω of Eq. (18), as follows directly from the definitions (25) and the strong-coupling behavior (1) where $g_{\nu,\gamma}$ are close to the zeros (28), (29).

8. Summarizing we see that no β-function is needed to calculate the principal critical exponents of ϕ^4-theories, ν, γ, and ω, which can all be obtained from the two renormalization constants of mass and wave function, and thus from the perturbation expansion of the self-energy $\Sigma(\mathbf{p}^2)$ in powers of the bare coupling constant. The two renormalization constants extracted from $\Sigma(0)$ and $\Sigma'(0)$.

It should be noted that the order in ϵ to which our β-function-free approach yields the critical exponent ω is one unit lower than the one obtained from the β-function calculated with the same number of loops diagrams. However, the calculation of the coupling constant renormalization constant involves many more Feynman diagrams than that of the others, which is the reason why its seven loop expansion in three dimensions is still unknown, while the others are. Thus, the above observation may eventually be of practical use, in particular for possible future automated computer calculations of the critical exponents.

Finally let us remark that the above strong-coupling theory of critical exponents can easily be extended to allow for higher confluent singularities in the approach to scaling with the help of the recently developed Jasch-Kleinert resummation algorithm [13].

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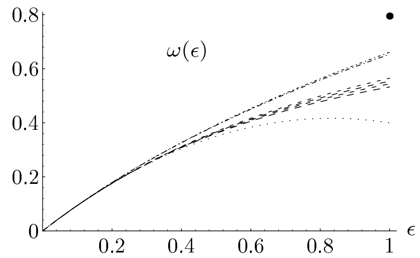
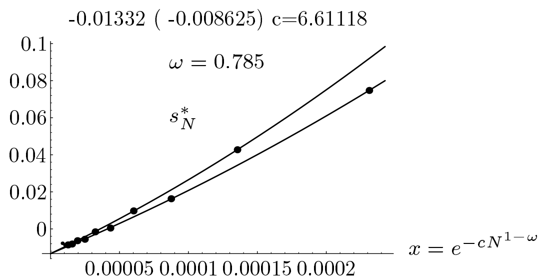


FIG. 1. Plots of the various solutions for $\omega(\epsilon)$ obtained from the perturbation expansions of $\nu(g_B)$, $\gamma(g_B)$. The dotted curve is the universal ϵ-expansion up to ϵ^2 of Eq. (18). The dashed curves are ordered with increasing dash lengths, from (22), (21), (17), (16). The dashed-dotted curves were calculated in Ref. [3] from the perturbation expansion of $g(g_B)$, once as in (17), (16), and once as in (22), (21). The dot shows the accurate value derived below from a seven-loop expansion in three dimensions.



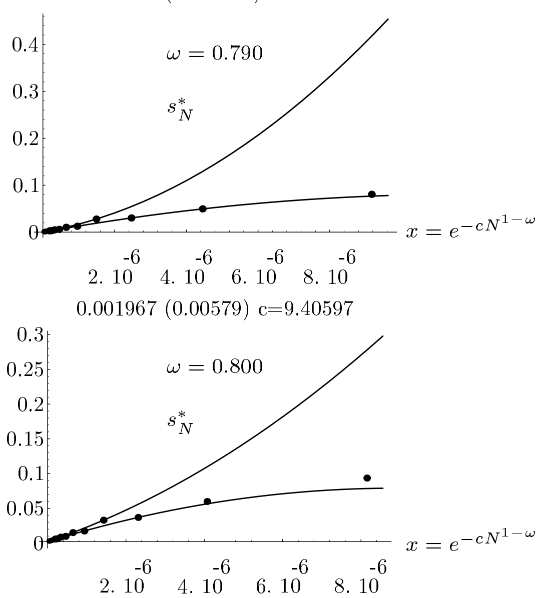


FIG. 2. Successive approximations to the strong-coupling of the logarithmic derivative $s(g_B) = d\log\bar{\eta}(g_B)/d\log g_B(g_B)$ and their extrapolation to infinite order for the critical exponents $\omega = 0.785$, $\omega = 0.790$, and $\omega = 0.800$, showing that the best value is $\omega = 0.790$. The numbers on top specify the limiting value s^*_∞, the last calculated value of order 11, and the parameter c in the theoretically expected large-order behavior which should be nearly linear in $x = e^{-cN^{1-\omega}}$, plotted as curves. The calculated values s^*_N are fitted best separately for even and odd orders N, and the conditions that the two curves intercept at $x = 0$ fixes c.